Department of Mathematics
University of Toronto

Real Analysis Comprehensive Exam
September 5, 2012

Please be brief but justify your answers, citing relevant theorems.

1. (a) Given two functions \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n) \), with \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), consider their convolution

\[
 f \ast g(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy .
\]

Prove that the integral is well-defined for each \( x \in \mathbb{R}^n \), and that \( f \ast g \) is bounded.

(b) Furthermore, \( f \ast g \) is continuous, and \( \lim_{|x| \to \infty} f \ast g(x) = 0 \).

(c) If, instead, \( f \) and \( g \) are integrable functions, prove that \( f \ast g(x) \) is well-defined for a.e. \( x \), and agrees almost everywhere with an integrable function.

2. (a) Let \( \ell \) be a bounded linear functional on \( L^2(\mathbb{R}) \). Prove (directly from the definition) that the function \( F(x) = \ell(\mathcal{X}_{[0,x]}) \) is absolutely continuous on \([0,1]\). (Here \( \mathcal{X}_{[0,x]} \) is the function defined by \( \mathcal{X}_{[0,x]}(t) = 1 \) if \( t \in [0, x] \) and \( \mathcal{X}_{[0,x]}(t) = 0 \) if \( t \notin [0, x] \).)

(b) Use the Riesz representation theorem to find a formula for the derivative \( F'(x) \) for a.e. \( x \).

3. (a) Consider the Banach space \( L^p([0,1]) \) where \( 1 < p < \infty \). What is the norm of this space? What is the dual of this space? (Just state—no proof needed).

(b) Let \( (X, ||\cdot||) \) be a Banach space. Define what it means for a sequence \( f_n \) to converge weakly to an element \( g \in X \).

(c) Consider a sequence \( f_n \in L^p([0,1]) \) with \( 1 < p < \infty \) and assume that \( ||f_n||_{L^p} \leq 1 \) and \( f_n(x) \to 0 \) for almost every \( x \in [0, 1] \). Prove that \( f_n \) converges weakly to 0. (Hint: Use Egorov’s theorem.)

(d) Give an example of a sequence \( f_n \in L^1([0,1]) \) with \( ||f_n||_{L^1} = 1 \) for all \( n \) and such that \( f_n(x) \to 0 \) for almost every \( x \in [0, 1] \) yet \( f_n \) does not converge weakly to 0.

4. (a) Consider the space \( L^1([0,2\pi]) \). Define the Fourier transform on this space. Define the Fourier transform on \( L^2([0,2\pi]) \).

(b) Prove that if \( f \in L^1([0,2\pi]) \) and \( \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 < \infty \) then \( f \in L^2([0,2\pi]) \).

(c) Prove that if \( \sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty \) then \( f \) agrees almost everywhere with a continuous function.