University of Toronto
Faculty of Arts and Sciences
April 2009
Math 1000/457 Final Exam

Duration: 3 hours.
You may use the attached list of formulas. No other aids are allowed during the exam.

Problems 1-3 are each worth 40 points. Each problem has several parts.

1. (Measure theory)
Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is an \( L^1 \) function with \( \| f \|_{L^1} \leq 1 \). Define \( g : \mathbb{R} \to \mathbb{R} \) by the formula:

\[
g(x) := \int_{-\infty}^{\infty} \frac{1}{1 + |x - y|^2} f(y) \, dy.
\]

a. (10 points) Prove that \( \lim_{x \to +\infty} g(x) = 0 \).

b. (15 points) Prove that \( g(x) \) is continuous.

c. (15 points) Prove that there exists a point \( x \) with \( |x| < 100 \) and \( |g(x)| < 1 \).

2. (Functional analysis)

a. (20 points) Let \( B \) be any Banach space. Let \( v_n \) be a sequence of vectors in \( B \). Suppose that \( v_n \) converges strongly to \( w \), and that \( v_n \) converges weakly to \( u \). Prove that \( u = w \).

b. (10 points) Find the maximum of \( \int_0^1 x^2 g(x) \, dx \) among measurable \( g : [0, 1] \to \mathbb{R} \) with \( \int_0^1 |g(x)|^2 \, dx = 1 \).

c. (10 points) Find the maximum of \( \int_0^1 x^2 g(x) \, dx \) among measurable \( g : [0, 1] \to \mathbb{R} \) with \( \int_0^1 |g(x)|^2 \, dx = 1 \) and \( \int_0^1 g(x) \, dx = 0 \).

3. (Fourier analysis)
Suppose that \( f \) is a Schwartz function on the real line, that \( f \) is supported in the interval \([-1, 1]\), and that \( |f(x)| \) and \( |f'(x)| \) are at most 1 for every \( x \in [-1, 1] \).

a. (10 points) Prove that \( |\hat{f}(\omega)| \leq 100\omega^{-1} \) for every \( \omega \in \mathbb{R} \).

b. (15 points) Prove that

\[
\int_{-\infty}^{\infty} |\omega|^2 |\hat{f}(\omega)|^2 \, d\omega \leq 100.
\]

c. (15 points) Let \( I_N f \) be the partial Fourier integral defined by

\[
I_N f(x) := \int_{-\frac{N}{2}}^{\frac{N}{2}} \hat{f}(\omega) e^{i\omega x} \, d\omega.
\]
\[
I_N f(x) := \int_{-N}^{N} e^{2\pi i \omega x} \hat{f}(\omega) d\omega.
\]

Using part b., prove that \(I_N f\) approximates \(f\) in the sense that

\[|I_N f(x) - f(x)| \leq 10^4 N^{-1/2}\]

for every \(x \in \mathbb{R}\).