(1) (a) Define what is meant by weak convergence of a sequence of vectors \( \{x_n\} \) in a Banach space \( X \).

(b) Suppose \( \{x_n\} \) is a sequence of vectors in the Banach space \( l^1 \) which converges weakly. Prove that it also converges in norm. \((l^1 \text{ is the space of absolutely summable sequences of complex numbers.})\)

(c) Show by example that the result in the previous question fails in \( l^p \) for \( 1 < p < \infty \) and also in \( L^1(m) \), where \( m \) denotes Lebesgue measure on the unit interval.

(d) Can there be a continuous linear map, with a continuous inverse, from \( l^1 \) to \( L^1(m) \)? Justify your answer.

(2) Suppose \( \{e_n, \ n = 1, 2, \ldots \} \) is an ortho-normal basis for an infinite dimensional Hilbert space \( \mathcal{H} \). Let \( \{\delta_n\} \) be a fixed sequence of positive reals and let \( K \) denote the set of vectors \( x \in \mathcal{H} \) such that \( |\langle x, e_n \rangle| \leq \delta_n \) for all \( n \). Show that \( K \) is compact if and only if \( \sum n \delta_n^2 < \infty \).

(3) (a) Suppose \( X \) is a normed vector space. Show that \( X \) is complete if and only if every absolutely convergent series in \( X \) converges in norm. \((\sum x_n \text{ is called absolutely convergent if } \sum \|x_n\| < \infty.\)

(b) Show that \( L^p(X, m) \) is complete for \( 1 < p < \infty \). Here \( (X, m) \) denotes any measure space.

(4) In this problem \( T \) denotes the 1-torus \( \mathbb{R}/2\pi\mathbb{Z} \) equipped with normalized Lebesgue measure, \( f \in L^1(T) \) and \( a_n = \hat{f}(n) \) is the \( n \)-th Fourier co-efficient. \( C^1(T) \) denotes the space of all functions on \( T \) which are continuously differentiable when viewed as \( 2\pi \)-periodic functions on the line.

(a) If \( f \in C^1(T) \) show that \( \sum_{n \in \mathbb{Z}} n^2 |a_n|^2 < \infty \).

(b) If \( \sum_{n \in \mathbb{Z}} |n||a_n| < \infty \) show that \( f \in C^1(T) \).