Department of Mathematics
University of Toronto

FINAL EXAMINATIONS, April-May 2008

MAT1000Y
Real Analysis

Examiner: A. del Junco
Duration: 3 hours

NO AIDS ALLOWED. **Total: 100 marks**

You may not use calculators, cell phones, or PDAs during the exam. Partial credit will be given for partially correct work. Please read through the entire test before starting, and take note of how many points each question is worth. There are five problems on this exam.

1. [20 marks] Suppose $(X, \mathcal{B}, \mu)$ is a sigma-finite measure space, $f \in L^\infty(\mu)$, $g \in L^1(\mu)$ and $g \geq 0$. Show that
$$\int fg d\mu = \int_0^\infty \int_{\{g>t\}} f d\mu \, dt.$$
Here $\{g > t\}$ denotes $\{x \in X : g(x) > t\}$. (Suggestion: Fubini.)

2. [15 marks] Suppose $f_1, f_2, \ldots$ is a sequence of measurable functions on $[0,1]$ taking values in $\mathbb{C}$. Show that there is a sequence $c_1, c_2, \ldots$ of (strictly) positive constants such that $\sum_n c_n f_n(x)$ is absolutely convergent for almost all x.

3. [15 marks] Suppose μ_n is a sequence of finite measures on $[0,1]$ such that
$$\lim_{n\to\infty} \int_0^1 x^{10k} d\mu_n(x) = a_k$$
exists for each $k = 0, 1, 2, \ldots$ Show that there is a finite measure μ on $[0,1]$ such that $\int_0^1 x^{10k} d\mu(x) = a_k$ and that
$$\int_0^1 x^k d\mu_n(x) \to \int_0^1 x^k d\mu(x)$$
for each $k = 0, 1, 2, \ldots$

4. [30 marks total]
   (a) [10 marks] Suppose φ is an unbounded linear functional on a normed vector space $(X, \|\cdot\|)$. Show that $\ker \varphi$ is dense in X.
   (b) [10 marks] Suppose E is a dense subspace of a separable Hilbert space $\mathcal{H}$. Show that E contains an ortho-normal basis of $\mathcal{H}$.

(c) [10 marks] Suppose that $f \in L^1[0,1]$ but $f \notin L^2[0,1]$. Show that $L^2[0,1]$ has an orthonormal basis $\{e_n\}$ such that e_n is continuous and $\int_0^1 f(x)e_n(x)dx = 0$ for all n. Suggestion: apply ~~(b)~~ (a) above to an appropriate functional on $(C[0,1], \|\cdot\|_2)$

5. [20 marks total]

(a) [10 marks] Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \simeq [0, 2\pi)$, endowed with its normalized Lebesgue measure. Suppose that $f \in L^1 = L^1(\mathbb{T})$, α is an irrational number and $f(x) = f(x + 2\pi\alpha)$ a.e. Show that f is a.e. equal to a constant. Suggestion: consider the Fourier co-efficients of $f(x) - f(x + 2\pi\alpha)$.

(b) [10 marks] Show that the set of functions of the form

$$f(x) - f(x + 2\pi\alpha) + c1_{\mathbb{T}},\ f \in L^1, c \in \mathbb{C},$$

is dense in L^1. Suggestion: use the Hahn-Banach theorem and part (a).