1. Find the real values of $a, b$ which minimize $\int_1^\infty |x^{-2} - ax^{-3} - bx^{-4}|^2 \, dx$. Hint: Work in an appropriate Hilbert space.

2. Suppose $F$ is a closed subset of $(0, 1)$, $G = (0, 1) \setminus F$ and let

$$d(x, F) = \inf_{y \in F} |x - y|$$

denote the distance from $x$ to $F$. Let

$$M(x) = \int_0^1 \frac{d(y, F)}{|x - y|^2} \, dy.$$ 

(a) Show that $M(x) = \infty$ for all $x \in G$.
(b) Show that $M(x) < \infty$ for almost all $x \in F$ by showing that $\int_F M(x) \, dx < 2\mu(G)$, where $\mu$ denotes Lebesgue measure on $(0, 1)$. Hint: The integral defining $M(x)$ may be restricted to elements $y$ in $G$. Reverse the order of integration and then bound the inner integral by observing that for each $y \in G$ one has $F \subseteq \{x : |x - y| \geq d(y, F)\}$.

3. Suppose $X$ is a Banach space. A projection on $X$ is a linear map $P : X \to X$ such that $P^2 = P$.

(a) Show that $I - P$ is also a projection whose kernel is the range of $P$. ($I$ denotes the identity map on $X$).
(b) Show that the range and the kernel of $P$ span $X$.
(c) Show that $P$ is continuous if and only if the range and kernel are closed subspaces of $X$. (One direction is easy and for the other you can use a non-trivial theorem about the continuity of linear maps on Banach spaces.)

4. In this problem $L_p$ denotes the $L_p$-space of $[0, 2\pi]$ endowed with Lebesgue measure. For $f \in L_1$, $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt$ denotes the $n$th Fourier coefficient of $f$. Suppose $f \in L_1$, $\hat{f}(0) = 0$ and let $F(t) = \int_0^t f(s) \, ds$.

(a) Show that $\hat{F}(n) = \frac{1}{in} \hat{f}(n)$.
(b) Suppose that $f \in L_2$. Show that $\sum_{n \in \mathbb{Z}} |\hat{F}(n)| < \infty$. Hint: Use part (a) and the Cauchy-Schwartz inequality.