Instructions: Do all questions. You may use a previous part of a question to solve a subsequent part even if you have not done the previous part. Do not be concerned if you can’t do all the problems. Good luck

1. [25 marks] Quickies
   (a) Prove that if $T$ is a linear transformation of a Hilbert space $\mathcal{H}$ into itself such that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$ then $T$ is bounded.
   (b) Suppose $C$ is a convex subset with non-empty interior in a normed vector space $X$. Show that the interior of $C$ is dense in $C$. (Hint: draw a picture.)
   (c) Suppose $\{a_i\}$ is a sequence in $l_2$ and $a_i$ is never zero. Show that there is a $\{b_i\} \in l_1$ such that $\{\frac{b_i}{a_i}\}$ is not in $l_2$. (Hint: restate this as a question about a linear operator between Banach spaces.)

2. [15 marks]
   (a) Suppose $f, g \in L^2(\mathbb{R})$. Show that the convolution $f * g$ is an everywhere defined continuous function.
   (b) Suppose $f \in L^1(\mathbb{R})$ and is not equal a.e. to the zero function. Show that $f * f$ is not identically zero. (Hint: use properties of the Fourier transform.)
   (c) Deduce from (b) that if $A, B \subset \mathbb{R}$ are subsets of positive measure then the set $A + B$ contains a non-empty open interval. (Hint: it is easy to reduce to the case when $A = B$ and $A$ has finite measure.)

3. [15 marks] Suppose $1 \leq p < \infty \ f_n, f \in L_p[0,1], \ f_n \to f$ almost everywhere and $\|f_n\|_p \to \|f\|_p$. Prove or disprove: $\|f_n - f\|_p \to 0$

4. [15 marks] Let $m$ denote Lebesgue measure on $\mathbb{R}^2$ and suppose that $A \subset \mathbb{R}^2$ with $m(A) < \infty$.
   (a) Prove that $m(A \Delta (A+x)) \to 0$ as $x \to 0$ in $\mathbb{R}^2$. Here $\Delta$ denotes the symmetric difference of set. (Hint: this is clear if $A = I \times J$ where $I$ and $J$ are bounded intervals in $\mathbb{R}$. Proceed by an approximation argument.)
   (b) Suppose $m(A) > 0$. Use part (a) to show that there is an $\epsilon > 0$ such that for each $\delta < \epsilon$, $A$ contains the vertices of some square of side $\delta$. (Hint: show that the intersection of four suitable translates of $A$ has positive measure, and hence is non-empty.)