University of Toronto  
Department of Mathematics  
Real Analysis Examination  
Tuesday, September 6, 2005, 1–3 p.m.  
Duration 2 hours  

No aids allowed.  
All questions are equal in value.  

1. Let \( \mu \) denote Lebesgue measure on the unit interval \([0, 1]\) and \( \mathcal{B} \) the \( \sigma \)-algebra of Lebesgue measurable subsets of \([0, 1]\). Given sets \( A, B \in \mathcal{B} \) define \( d(A, B) = \mu(A \Delta B) \). \( (A \Delta B \) denotes the symmetric difference.) Show that \( d \) is a pseudo-metric on \( \mathcal{B} \) and hence defines a metric on the space \( \mathcal{B} \) of equivalence classes of sets in \( \mathcal{B} \) modulo null sets. Show that this metric is complete. Give an example to show that it is not compact.  

2. Suppose \( x_n \) is a sequence of vectors in a Hilbert space \( \mathcal{H} \) which converges weakly to a limit \( x \).  
   (a) Show that \( \|x_n\| \) is bounded.  
   (b) Show that there is a subsequence \( \{x_{n_i}\} \) such that \( \frac{1}{N} \sum_{i=1}^{N} x_{n_i} \) converges in norm to \( x \). Suggestion: show that without loss of generality one may take \( x = 0 \). After \( x_{n_j} \) has been chosen for \( j < i \) choose \( x_{n_i} \) so that \( \langle x_{n_i}, x_{n_j} \rangle < 2^{-i} \) for all \( j < i \). Use this to estimate \( \|\sum_{i=1}^{N} x_{n_i}\| \).  
   (c) Use part (b) to show that any convex norm-closed subset of \( \mathcal{H} \) is weakly closed.  

3. Suppose \( \alpha \) is any irrational number and define a transformation \( T \) from the 1-torus \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \cong [0, 1) \) to itself by \( Tx = x + \alpha \). The addition is in \( \mathbb{R}/\mathbb{Z} \), that is \( Tx \) is the fractional part of \( x + \alpha \) as a transformation of \([0, 1)\). Let \( \mu \) denote Lebesgue measure on \( \mathbb{T} \). If \( f \in L_2(\mu) \) and \( f = f \circ T \) show that \( f \) is (almost everywhere) equal to a constant function. Hint: expand \( f \) in a Fourier series.  

4. Show that any norm-closed subspace of a normed vector space is weakly closed. By considering \( c_0(\mathbb{N}) \subset l_\infty(\mathbb{N}) \) show that a norm-closed subspace of the dual of a normed vector space need not be closed in the weak-* topology.