University of Toronto

Faculty of Arts and Science

Final Examinations, April–May 2008

MAT454H1S/1001HS — Complex Analysis

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Duration — 3 hours

No aids allowed.
Total marks for this paper: 100.
All questions are equal in value.

1.(a) Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ has a zero.

(b) Prove the Theorem of Weierstrass: If 0 is an essential isolated singularity of a holomorphic function, then the image of any punctured neighbourhood of 0 is dense in $\mathbb{C}$.

2.(a) Prove that every entire function which is one-to-one must be linear.

(b) Let $f(z) = z - z^2 + z^3 - z^4 + \ldots$, where $|z| < 1$. Find the largest domain to which $f(z)$ can be extended analytically.

3. Let $\mathcal{A}$ denote the set of all holomorphic functions $f(z)$ on the open unit disk $D = \{|z| < 1\}$ such that $f(0) = 1$ and $\operatorname{Re} f > 0$.

(a) Show that, if $f \in \mathcal{A}$, then

$$\frac{1 - |z|}{1 + |z|} \le |f(z)| \le \frac{1 + |z|}{1 - |z|}.$$

(b) Prove that $\mathcal{A}$ is a normal family.

(c) How large can $|f'(0)|$ be?

4. Show that

$$\int_0^1 x^{n-2} \sqrt{x(1-x)}\, dx = -\pi c_n, \quad n = 2, 3, 4, \ldots,$$

where c_n is the coefficient of x^n in the binomial expansion of $\sqrt{1-x}$, $|x| < 1$.

5.(a) Show that the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

represents an entire function with simple zeros at the negative integers.

(b) Define $H(z)$ by

$$\frac{1}{H(z)} = ze^{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$

Prove that

$$\frac{d}{dz}\left(\frac{H'(z)}{H(z)}\right) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}.$$

6.(a) Prove that the following statement is equivalent to Picard's Little Theorem: *If $f(z)$ and $g(z)$ are entire and $e^f + e^g = 1$, then f, g are constant.*

(b) Suppose that f, g and h are entire. Prove that if $h = e^f + e^g$, then h has either no zeros or infinitely many zeros in $\mathbb{C}$.

(c) Use Picard's Big Theorem to prove that if f is entire and h is a nonconstant polynomial, then he^f takes every value.