

University of Toronto
Faculty of Arts and Science
Final Examinations, April–May 2008
MAT454H1S/1001HS — Complex Analysis
Instructor: Edward Bierstone
Duration — 3 hours

No aids allowed.

Total marks for this paper: 100.

All questions are equal in value.

- 1.(a) Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ has a zero.
- (b) Prove the Theorem of Weierstrass: If 0 is an essential isolated singularity of a holomorphic function, then the image of any punctured neighbourhood of 0 is dense in \mathbb{C} .
- 2.(a) Prove that every entire function which is one-to-one must be linear.
- (b) Let $f(z) = z - z^2 + z^3 - z^4 + \dots$, where $|z| < 1$. Find the largest domain to which $f(z)$ can be extended analytically.
3. Let \mathcal{A} denote the set of all holomorphic functions $f(z)$ on the open unit disk $D = \{|z| < 1\}$ such that $f(0) = 1$ and $\operatorname{Re} f > 0$.
- (a) Show that, if $f \in \mathcal{A}$, then

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

- (b) Prove that \mathcal{A} is a normal family.
- (c) How large can $|f'(0)|$ be?

4. Show that

$$\int_0^1 x^{n-2} \sqrt{x(1-x)} dx = -\pi c_n, \quad n = 2, 3, 4, \dots,$$

where c_n is the coefficient of x^n in the binomial expansion of $\sqrt{1-x}$, $|x| < 1$.

5.(a) Show that the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

represents an entire function with simple zeros at the negative integers.

(b) Define $H(z)$ by

$$\frac{1}{H(z)} = ze^z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Prove that

$$\frac{d}{dz} \left(\frac{H'(z)}{H(z)} \right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}.$$

- 6.(a) Prove that the following statement is equivalent to Picard's Little Theorem:
If $f(z)$ and $g(z)$ are entire and $e^f + e^g = 1$, then f, g are constant.
- (b) Suppose that f, g and h are entire. Prove that if $h = e^f + e^g$, then h has either no zeros or infinitely many zeros in \mathbb{C} .
- (c) Use Picard's Big Theorem to prove that if f is entire and h is a nonconstant polynomial, then he^f takes every value.