1. (a) Let $U, V$ denote domains in $\mathbb{C}$ and let $f : U \to V$ be a holomorphic mapping. Suppose that $f$ is proper (i.e., $f^{-1}(K)$ is compact, for every compact subset $K$ of $V$). Prove that $f(U) = V$.

(b) Is the assertion in (a) true if “holomorphic” is replaced by “continuous”? Explain.

2. (a) Let $f(z)$ denote a holomorphic function in $|z| < R$ such that $|f(z)| \leq M$. Suppose that $f(z_0) = w_0$, where $|z_0| < R$. Show that

$$\frac{|M(f(z) - w_0)|}{M^2 - w_0 f(z)} \leq \frac{|R(z - z_0)|}{R^2 - z_0 z}.$$

(Hint. First consider the case $f(0) = 0$.)

(b) Show that if $|f(z)| \leq 1$ for $|z| < 1$, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

3. Let $\{f_n\}$ be a sequence of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ which is bounded uniformly on compact subsets of $\Omega$. Let $\{z_k\}$ be a sequence of distinct points in $\Omega$ with $\lim_{k \to \infty} z_k = z_0 \in \Omega$. Assume that $\lim_{n \to \infty} f_n(z_k)$ exists, for all $k$. Prove that $\{f_n\}$ converges uniformly on compact subsets of $\Omega$.

4. Use residues to show that

$$\int_0^1 \frac{dx}{\sqrt{x^2 - x^3}} = \frac{2\pi}{\sqrt{3}}.$$
5. (a) An elliptic function on $\mathbb{C}$ means a doubly-periodic meromorphic function. Show that any even elliptic function $f(z)$ can be written in the form

$$f(z) = c \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

(where $c$ is a constant and $\wp(z)$ denotes the Weierstrass $\wp$-function with the same periods), provided that $0$ is neither a zero nor a pole of $f$.

(b) Show that every even elliptic function $f$ can be written $f = R(\wp)$, where $R$ is a rational function.

(c) Show that every elliptic function $f$ can be written $f = R(\wp, \wp')$, where $R$ is rational.

6. (a) Give an example of a meromorphic function on $\mathbb{C}$ that omits two values.

(b) Prove Picard’s little theorem for meromorphic functions: Every meromorphic function on $\mathbb{C}$ that omits three distinct values is constant.