DEPARTMENT OF MATHEMATICS
University of Toronto

Complex Analysis Exam (1 1/2 hours)

May 1998

No aids.
Do all questions.

1. (a) If $f$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ when $|z| \leq R/2$.

(b) Suppose $f$ is analytic in a region $\Omega$. Suppose $\gamma$ is a piecewise smooth closed curve in $\Omega$ (not necessarily simple) which does not meet any of the zeros of $f$. Let $p$ be a nonnegative integer. What is the value of

$$\int_{\gamma} \frac{f'(z)}{f(z)} z^p \, dz$$

in terms of the zeros of $f$?

2. (a) Let $f$ and $g$ be 1-1 analytic mappings from an open connected set $\Omega \subset \mathbb{C}$ onto the unit open disc $\Delta$. Suppose for some point $z_0 \in \Omega$, $f(z_0) = g(z_0) = 0$. What is the relation between $f$ and $g$?

(b) Let $f$ be a 1-1 analytic map of the unit disc $\Delta$ onto the unit square with center 0, satisfying $f(0) = 0$. Show that $f(i z) = i f(z)$.

3. (Harnack’s Theorem). Let $\Omega$ be a connected open subset of $\mathbb{C}$. Let $\{v_n(z)\}_{n=1,2,\ldots}$ be a sequence of real harmonic functions on $\Omega$. Suppose that for all $z \in \Omega$ $v_n(z) \leq v_{n+1}(z)$ for $n = 1, 2, \ldots$ and let

$$v(z) = \lim_{n \to \infty} v_n(z).$$

Show that either $v \equiv +\infty$ on $\Omega$ or $v(z)$ is harmonic on $\Omega$. 

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