University of Toronto

Department of Mathematics

Analysis Examination

Tuesday, September 2, 2008, 1–4 p.m.

Duration 3 hours

(1) (a) Define what is meant by weak convergence of a sequence of vectors $\{x_n\}$ in a Banach space X.

(b) Suppose $\{x_n\}$ is a sequence of vectors in the Banach space l^1 which converges weakly. Prove that it also converges in norm. (l^1 is the space of absolutely summable sequences of complex numbers.)

(c) Show by example that the result in the previous question fails in l^p for $1 < p < \infty$ and also in $L^1(m)$, where m denotes Lebesgue measure on the unit interval.

(d) Can there be a continuous linear map, with a continuous inverse, from l^1 to $L^1(m)$? Justify your answer.

(2) Suppose $\{e_n,\ n = 1, 2, \ldots\}$ is an ortho-normal basis for an infinite dimensional Hilbert space $\mathcal{H}$. Let $\{\delta_n\}$ be a fixed sequence of positive reals and let K denote the set of vectors $x \in \mathcal{H}$ such that $|\langle x, e_n\rangle| \leq \delta_n$ for all n. Show that K is compact if and only if $\sum_n \delta_n^2 < \infty$.

(3) (a) Suppose X is a normed vector space. Show that X is complete if and only if every absolutely convergent series in X converges in norm. ($\sum x_n$ is called absolutely convergent if $\sum \|x_n\| < \infty$.)

(b) Show that $L^p(X, m)$ is complete for $1 \leq p < \infty$. Here (X, m) denotes any measure space.

(4) In this problem $\mathbb{T}$ denotes the 1-torus $\mathbb{R}/2\pi\mathbb{Z}$ equipped with normalized Lebesgue measure, $f \in L^1(\mathbb{T})$ and $a_n = \hat{f}(n)$ is the n-th Fourier co-efficient. $C^1(\mathbb{T})$ denotes the space of all functions on $\mathbb{T}$ which are continuously differentiable when viewed as 2π-periodic functions on the line.

(a) If $f \in C^1(\mathbb{T})$ show that $\sum_{n\in\mathbb{Z}} n^2|a_n|^2 < \infty$.

(b) If $\sum_{n\in\mathbb{Z}} |n||a_n| < \infty$ show that $f \in C^1(\mathbb{T})$.

(5) Let f be an entire function such that $\mathrm{Re}(f(z)) \geq -2$ for all $z \in \mathbb{C}$. Show that f is a constant.

(6) Let f be an analytic function in the unit disc $D = \{|z| < 1\}$. Suppose that $|f(z)| \leq 1$ in D. Prove that if f has at least two fixed points z_1, z_2 (that is, $f(z_i) = z_i$ for $i = 1, 2$, $z_1 \neq z_2$), then $f(z) = z$ for all $z \in D$.