DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Friday, May 7, 2004, 1–4 p.m.

All problems have equal weight.

1. Prove or find a counterexample for each of the following statements. Here \( \mu \) denotes the Lebesgue measure on the real line.

(a) Let \( \{\phi_n\} \) denote a sequence of functions on \([0, 1]\) such that

\[
\int_0^1 |\phi_n| \, d\mu \leq C
\]

for some constant \( C \), and

\[
\lim_{n \to \infty} \int_E \phi_n \, d\mu = 0 \quad \text{for each measurable subset } E \text{ of } [0,1].
\]

Then, \( \lim_{n \to \infty} \int_0^1 f \phi_n \, d\mu = 0 \) for each bounded and measurable function \( f \) on \([0, 1]\).

(b) If \( \{f_n\} \subset L^2([0,1]) \) and \( \lim_{n \to \infty} \int_0^1 f_n^2 \, d\mu = 0 \), then \( \lim_{n \to \infty} f_n(x) = 0 \) for almost all \( x \in [0, 1] \).

2. Let \( \mu \) denote a finite Borel measure on \( R \). Evaluate the following limits:

\[
\lim_{t \to \infty} \int_R f(tx) \, d\mu(x)
\]

\[
\lim_{t \to 0} \int_R f(tx) \, d\mu(x),
\]

where \( f \) denotes a continuous function on \( R \) with compact support. Justify your answers.
3. Let $H$ denote a real Hilbert space with an inner product $<x, y>$. Consider the function $Q(x, y) = <x, Ay>$ on $H \times H$ for some linear transformation $A$ on $H$.

(a) Show that $Q$ is continuous if and only if $A$ is continuous.

(b) If $Q$ is continuous show that

$$
\sup \frac{|Q(x, y)|}{\|x\|\|y\|} = \sup \frac{|A(x)|}{\|x\|}, \quad x \neq 0, y \neq 0.
$$

4. Let $M$ denote a closed linear subspace of a Banach space $X$ and let $x$ be any point of $X$ not in $M$.

(a) Show that $\inf \{\|x - y\| : y \in M\} = \delta > 0$

(b) Show that there exists a bounded linear functional $f$ on $X$ such that $f(x) = \|x\|$ and $f(y) = 0$ for all $y \in M$.

5. (a) Define normal family of analytic functions.

(b) Let $\mathcal{F} = \{f \mid f$ is holomorphic in $|z| < 1$ and $|f^{(n)}(0)| \leq n!$ for $n = 0, 1, 2, \ldots\}$. Prove that $\mathcal{F}$ is a normal family.

6. (a) State Schwarz’s Lemma.

(b) Let $f$ be a 1–1 analytic mapping from the unit disc $\Delta = \{z \mid |z| < 1\}$ onto itself such that $f(0) = 0$. Prove that $f$ is of the form $f(z) = e^{i\theta}z$ for some $\theta$. 