DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Tuesday, September 2, 2003, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. (a) Let $E$ be a normed real vector space, $x_0 \in E$. Prove there exists a linear functional $\phi$ on $E$ so that $\phi(\alpha x_0) = -3\alpha$ for all $\alpha \in \mathbb{R}$.

(b) Assume $E$ has an inner product. Use the inner product to write your $\phi$ in an explicit manner.

2. Let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{|\xi| \to \infty} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx = 0.$$ 

3. Let $X$ be a Banach space and $A : X \to X$ a bounded operator. Recall that $\mathcal{L}(X, X)$ is the Banach space of bounded linear operators from $X$ to $X$ with the norm induced by the norm on $X$.

a) Fix $t \in \mathbb{R}$. Construct

$$e^{tA} \in \mathcal{L}(X, X).$$

(That is, define an operator $B$ that is the most sensible definition of $e^{tA}$ that you can think of and prove that $B \in \mathcal{L}(X, X)$.)

b) Given $x_0 \in X$ we define a path $x(t) \in X$ for $t \in \mathbb{R}$ by

$$x(t) = e^{tA}x_0.$$ 

Prove that

$$\lim_{h \to 0} \frac{x(t + h) - x(t)}{h}$$
exists in $X$ and call this limit “$dx/dt$ at time $t$”. Prove that at each time $t$

$$
\frac{dx}{dt}(t) = Ae^{tA}x_0 = Ax(t).
$$

4. Let $H$ be a Hilbert space and $A : H \to H$ be a bounded linear operator. The point
spectrum of $A$ is:

$$
\sigma(A) := \{ \lambda \in \mathbb{C} \mid Ax = \lambda x, \text{ for some } x \in H, \ x \neq 0 \}
$$

Prove or disprove:

$$
\sup\{ |\lambda| : \lambda \in \sigma(A) \} = \|A\|.
$$

5. (a) State Schwarz’s Lemma.

(b) Prove that every 1 - 1 analytic mapping from $\Delta := \{ z \mid |z| < 1 \}$ onto $\Delta$ is of the
form

$$
f(z) = e^{i\theta}\left(\frac{z - \alpha}{1 - \overline{\alpha}z}\right) \text{ for some } \alpha \in \Delta.
$$

6. (a) Define normal family (of analytic functions) and state a general theorem which
gives a criterion for a family of analytic functions to be normal.

b) Consider $\left\{ f \mid f = \sum_{n=0}^{\infty} a_nz^n \text{ with } |a_n| \leq n \text{ for } n = 1, 2, \ldots \right\}$.

Using (a) above (or otherwise) show that this is a normal family of analytic functions.