DEPARTMENT OF MATHEMATICS
University of Toronto

Analysis Exam (3 hours)

Monday, May 5, 2003, 1–4 p.m.

No aids.
Do all questions.
Questions will be weighted equally.

1. Let \((E, B, \mu)\) be a measure space. The measure \(\mu\) is \(\sigma\)-finite if there exists \(\{E_n\} \subset B\) such that
\[
E = \bigcup_{n=1}^\infty E_n \quad \text{and} \quad \mu(E_n) < \infty \forall n.
\]
Prove that \(\mu\) is \(\sigma\)-finite if and only if there exists \(f > 0, f \in L^1(\mu)\).

2. Let \(H\) be a Hilbert space with orthonormal basis \(\{\phi_k\}\). \(\mathcal{L}(H, H)\) is the Banach space of bounded linear operators from \(H\) to \(H\) with the norm on \(\mathcal{L}(H, H)\) being induced by the norm on \(H\):
\[
\|A\|_{\mathcal{L}(H, H)} = \sup_{x \neq 0} \frac{\|Ax\|_H}{\|x\|_H}
\]

\(A \in \mathcal{L}(H, H)\) is a compact operator if

\[
\overline{AU}
\]
is compact whenever \(U\) is a bounded subset of \(H\). That is, the closure of the image of a bounded set is compact.

a) Let
\[
\psi_n \in \text{span}\{\phi_1, \ldots, \phi_n\}^\perp \quad \text{and} \quad \|\psi_n\| = 1.
\]
Prove that the sequence \(\psi_n\) converges weakly to 0.

b) Assume that the sequence \(\{x_n\} \subset H\) converges weakly to \(x\). Assume \(A\) is a compact operator. Prove that the sequence \(\{Ax_n\}\) converges strongly to \(Ax\) (i.e., \(\|Ax_n - Ax\| \to 0\)).

c) Assume \(A\) is a compact operator. Construct a sequence of finite-rank operators \(\{A_n\}\) such that
\[
\lim_{n \to \infty} \|A_n - A\|_{\mathcal{L}(H, H)} = 0.
\]
3. Let $f : [0,1] \to [0,1]$ be continuously differentiable and satisfy $f(0) = 0$ and $f(1) = 1$.

   a) Let
   \[ A_n = \{ x \in [0,1] \mid |f'(x)| < 1/n \} \quad \text{and} \quad B_n = f(A_n). \]
   Prove that $\mu(B_n) \leq 1/n$ where $\mu$ is Lebesgue measure.

   b) A point $x_0$ is a critical point of $f$ if $f'(x_0) = 0$. The image of a critical point, $f(x_0)$ is called a critical value. Prove that the set of critical values of $f$ has Lebesgue measure zero.

   c) Prove there exists at least one horizontal line $y = y_0 \in [0,1]$ which is nowhere tangent to the graph of $f$ in $\mathbb{R}^2$. (Recall that the graph of $f$ is the set of points $\{(x, f(x))\}$.)

4. Let $X$ be a complex Banach space. Let $I$ denote the identity operator. We say that $B \in \mathcal{L}(X, X)$ is invertible if $B$ is injective, onto, and $B^{-1} \in \mathcal{L}(X, X)$.

   a) Let $S, T \in \mathcal{L}(X, X)$. Prove that $I - ST$ is invertible if and only if $I - TS$ is invertible. (*Hint: do some formal manipulations using geometric series to try and write one inverse in terms of the other.)

   b) Let $S, T \in \mathcal{L}(X, X)$. Prove that
   \[ \text{spectrum}(ST) - \{0\} = \text{spectrum}(TS) - \{0\}. \]

   c) Let $S, T \in \mathcal{L}(X, X)$. Prove that $ST - TS \neq I$. (*Hint: assume $ST - TS = I$ and then prove the spectrum of $ST$ must be unbounded and then ... *)

5. Let $f(z) = \frac{g(z)}{h(z)}$ where $g$, $h$ are analytic in a neighbourhood of $z_0$, $g(z_0) \neq 0$ and $h(z_0) = h'(z_0) = 0$. Show that the residue of $f$ at $z_0$ is given by
   \[ \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0)h'''(z_0)}{(h''(z_0))^2}. \]

6. Let $\mathcal{F}$ denote the family of circles and lines in the plane.

   a) Prove that any linear fractional transformation maps members of $\mathcal{F}$ to members of $\mathcal{F}$.

   b) What is the image of the circle with center 0 and radius 2 under the mapping $z \to \frac{z - i}{z + i}$?