DEPARTMENT OF MATHEMATICS  
University of Toronto 

Analysis Exam (3 hours) 

May 1998 

No aids. 
Do all questions. 

1. (a) If $f$ is analytic and $|f(z)| \leq M$ for $|z| \leq R$, find an upper bound for $|f^{(n)}(z)|$ when $|z| \leq R/2$. 

(b) Suppose $f$ is analytic in a region $\Omega$. Suppose $\gamma$ is a piecewise smooth closed curve in $\Omega$ (not necessarily simple) which does not meet any of the zeros of $f$. Let $p$ be a nonnegative integer. What is the value of 

$$\int_{\gamma} \frac{f'(z)}{f(z)} z^p \, dz$$  

in terms of the zeros of $f$?

2. (a) Let $f$ and $g$ be 1-1 analytic mappings from an open connected set $\Omega \subset \mathbb{C}$ onto the unit open disc $\Delta$. Suppose for some point $z_0 \in \Omega$, $f(z_0) = g(z_0) = 0$. What is the relation between $f$ and $g$?

(b) Let $f$ be a 1-1 analytic map of the unit disc $\Delta$ onto the unit square with center 0, satisfying $f(0) = 0$. Show that $f(iz) = if(z)$.

3. Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $h \in L_1(X, \mathcal{F}, \mu)$ be a complex valued function. Show that $\nu(F) = \int_F h \, d\mu$, $F \in \mathcal{F}$ defines a complex measure on $(X, \mathcal{F})$. Also show that $f \in L_1(X, \mathcal{F}, |\nu|)$, where $|\nu|$ is the total variation of $\nu$, if and only if $fh \in L_1(X, \mathcal{F}, \mu)$. 

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4. Let $c$ be the set of all convergent real sequences $x = (x_n)$ with the supremum norm $\|x\| = \sup_n |x_n|$. Let $c_0$ be the subspace of sequences converging to zero.

(a) Show that $\ell_1$ is the dual space of $c_0$.

(b) Find a representation for the dual space of $c$.

5. Let $\mathbb{T}$ be the unit circle with the Lebesgue measure and let $\mathbb{Z}$ be the set of all integers. Let $A$ be the space of functions $f$ in $L_1(\mathbb{T})$ for which $\|f\|_A = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|$ is finite, where $\hat{f}(k)$ is the $k$-th Fourier coefficient for $f$. Show that each function in $A$ is continuous. If $f_n$ is a sequence in $A$ such that $f_n$ converges to a function $f$ uniformly in $\mathbb{T}$ and if $\|f_n\|_A \leq 1$ for all $n$, then show that $f \in A$ and $\|f\|_A \leq 1$.

6. (a) State the closed graph theorem.

(b) Let $Y$ be a normed subspace of $L^1(\mathbb{R})$ with

$$Y = \left\{ f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} |xf(x)| \, dx < \infty \right\}.$$

Let $T : Y \to Y$ be an operator defined as $L(f)(x) = xf(x)$. Show that $T$ is discontinuous but has a closed graph.