No aids allowed.
Do as many questions as you can. An expected perfect result is 60 pts or higher.

(1) a) Prove that the quotient of a group $G$ by its center $Z(G)$ is a cyclic group if and only if $G$ is Abelian, i.e. $G = Z(G)$. (5 pts)
   b) Classify all groups of order 8. (5 pts)

(2) a) Prove that for a group $G$ its commutant $H$ is the maximal Abelian quotient of $G$. (5 pts)
   b) Consider the symmetric group $S_n$. Prove that there are exactly 2 different group maps from $S_n$ to the multiplicative group of complex numbers $\mathbb{C}^*$. (5 pts)

(3) Let $\mathbb{F}$ be a finite field with $q$ elements. Consider a finite dimensional vector space $\mathbb{F}^n$ over $\mathbb{F}$. Denote the group of linear automorphisms of $\mathbb{F}^n$ (resp. the group of non-strictly upper triangular automorphisms of $\mathbb{F}^n$) by $G$ (resp. by $B$).
   a) Prove that any element of $B$ is a product of a diagonal element and several elements of the form $A_{ij}$, $i < j$. Here $A_{ij}$ denotes the elementary matrix with units on the diagonal and the only non-zero non-diagonal entry placed in the box $(i, j)$. (5 pts)
   b) Find the number of double cosets of $G$ by $B$. (10 pts)

(4) Consider the set $\mathcal{N}$ of nilpotent square matrices over $\mathbb{C}$ of the size $n$. The group $GL(n)$ acts on $\mathcal{N}$ by conjugation. Classify the orbits. (10 pts)

(5) Let $A$ be a commutative ring. Prove that a polynomial $f(x) \in A[x]$ is invertible in $A[x]$ if and only if its constant term is
invertible in $A$ and the rest of the coefficients are nilpotent in $A$. (15 pts)

(6) Let $V$ be a vector space of dimension $n$. Consider an invertible linear map $A: V \to V$.

a) Give the definition of the exterior powers $\Lambda^k(V)$. (5 pts)

b) Let $\Lambda^{n-1}(A)$ be the corresponding automorphism of $\Lambda^{n-1}(V)$ (it takes $v_1 \wedge \ldots \wedge v_{n-1}$ to $A(v_1) \wedge \ldots \wedge A(v_{n-1})$). Find the determinant of $\Lambda^{n-1}(A)$ as a function of $\det(A)$. (10 pts)

(7) a) Define the $n$-th cyclotomic polynomial $\Phi_n(x)$ over $\mathbb{Q}$. (5 pts)

b) Find explicitly $\Phi_{15}(x)$. (5 pts)

(8) Determine the splitting field and its degree over $\mathbb{Q}$ for the polynomial $x^6 - 4$. (10 pts)

(9) a) Classify the finite Galois extensions of a finite field $\mathbb{F}_p$. State the description of the Galois groups for the extensions. (5 pts)

b) Let $q = p^n$. Prove that the multiplicative group of the field $\mathbb{F}_q$ is cyclic. (5 pts)

c) Prove that for any $m$ there exists an irreducible polynomial of degree $m$ over $\mathbb{F}_p$. (5 pts)