University of Toronto
Department of Mathematics
Algebra Examination
Thursday, September 4, 2008, 1-4 p.m.
Duration: 3 hours

1. Let G be a finite group and p a prime. Show that if H is a p-subgroup of G, then $[N_G(H) : H] \equiv [G : H] (\bmod p)$. (*idea of solution*: consider orbits of left multiplication action of H on the coset space G/H.)

2. Let R be a commutative ring with 1.
   a) Prove that if R is finite, then every prime ideal of R is maximal.
   b) Suppose that for each $a \in R$, there exists an integer $n \geq 2$ (n depending on a) such that $a^n = a$. Prove that every prime ideal of R is maximal.

3. Prove that for any finite abelian group G, there exists a Galois extension K such that $Gal(K/\mathbb{Q}) \simeq G$. [You may need a special case of Dirichlet's theorem on arithmetic progression, namely, given any integer $m > 1$, there are infinitely many primes p with $p \equiv 1 \pmod m$.]

4. Prove that any commuting set of diagonalizable linear transformations is simultaneously diagonalizable. More precisely, let V be a finite dimensional vector space over a field k and let $\{T_\alpha\}_{\alpha \in A}$ be a set of commuting linear transformations on V which are diagonalizable. Then there exists a basis of V such that the matrices of T_α with respect to that basis are all diagonal matrices for all $\alpha \in A$.

5. Let G be a finite group. Let $n \in \mathbb{N}$. Define $\theta_n : G \to \mathbb{N}$ by
$$\theta_n(g) = \#\{\, h \in G \mid h^n = g \,\}, \qquad g \in G.$$
   Let χ_i, $1 \leq i \leq r$ be the distinct irreducible (complex) characters of G. Set
$$\nu_n(\chi_i) = |G|^{-1} \sum_{g \in G} \chi_i(g^n).$$
   Prove that $\theta_n = \sum_{i=1}^{r} \nu_n(\chi_i)\, \chi_i$. (*Idea of solution*: Use orthogonality relations.)

6. Let p be a prime and let $\mathbb{F}_p$ be the finite field with p elements. Suppose that L is a Galois extension of fields such that $\mathrm{Gal}(L/K) = GL_2(\mathbb{F}_p)$. Let L_1 and L_2 be the subfields of L containing K that correspond to the subgroups $H_1 = SL_2(\mathbb{F}_p)$ and
$$H_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ a, c \in \mathbb{F}_p^\times,\ b \in \mathbb{F}_p \right\}$$
   of $\mathrm{Gal}(L/K)$.
   a) Compute the degrees $[L_1 : K]$, $[L_2 : K]$ and $[L_1 \cap L_2 : K]$.
   b) Let L_1L_2 be the composite of L_1 and L_2. Prove that L_1L_2 is a Galois extension of L_2, but L_1L_2 is not a Galois extension of K.
   c) Compute $\mathrm{Gal}(L/L_1L_2)$ and $\mathrm{Gal}(L_1L_2/L_2)$.