Instructions: Answer all questions. Unless noted otherwise, explanation and justification of your answers is expected.

1. (32 marks)  
   a) Show that every group of order 200 has a nontrivial normal subgroup.
   b) Make a list (up to isomorphism) of all abelian groups of order 200.
   c) Let $T$ be a $4 \times 4$ matrix with entries in $\mathbb{C}$ whose minimum polynomial is $(\lambda - 2)(\lambda - 3)$. Make a list of all possibilities for the Jordan normal form of $T$.
   d) Find all the ideals of the ring $\mathbb{Z}[x]/(2, x^3 + 26)$.

2. (12 marks)  
   a) Let $T$ belong to $\text{GL}_k(\mathbb{C})$ such that $T^n = I$ (where $I$ denotes the identity matrix). Show that $T$ is diagonalizable.
   b) Let $G$ be a finite group and let $\rho : G \to \text{GL}_k(\mathbb{C})$ be a representation. Show that $\chi_\rho(g^{-1}) = \bar{\chi_\rho(g)}$ for all $g \in G$. ($\bar{a}$ denotes the complex conjugate of $a$.)

3. (16 marks)  
   a) Define (or give a condition equivalent to) Noetherian ring.
   b) Show that a Principal Ideal Domain is Noetherian.
   c) Show that in a Principal Ideal Domain, every prime ideal is maximal.
   d) Let $R$ be an integral domain.
      (i) Show that if $x \in R$ is prime, then $x$ is irreducible.
      (ii) Give an example to show that $x$ can be irreducible but not prime.

4. (16 marks)  
   a) Define the semidirect product of groups.
   b) Find groups $H$ and $K$ such that $A_4 \cong H \rtimes K$. ($A_4$ denotes the alternating group.)
   c) Show that $A_4$ is solvable.
   d) Find elements $x$ and $y$ in $A_4$ such $x$ and $y$ are conjugate in $S_4$ but $x$ and $y$ are not conjugate in $A_4$.
   e) Find the character table of $A_4$. 

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5. (10 marks) Let $f(x)$ be an irreducible cubic in $\mathbb{Q}[x]$.
   
   a) What are the possibilities for the Galois group of $f$?
   
   b) Given some particular $f(x)$, describe how you would determine which one is the Galois group of $f$.

6. (14 marks) Let $F \subset K$ be an extension of fields.
   
   a) Define what it means to say that an element $x \in K$ is algebraic over $F$.
   
   b) Suppose that $a, b \in K$ are such that $F(a)$ and $F(b)$ are normal separable extensions of $F$ with $[F(a) : F]$ relatively prime to $[F(b) : F]$.
      
      (i) Show that $F(a) \cap F(b) = F$.
      
      (ii) Show that $F(a, b) = F(a + b)$. 