DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam
September 6, 2001

Time: 3 hours
No aids allowed

1. Let $G$ be a group of order 56 and let $f, h : G \rightarrow G$ be maps such that
\[ f(x) = x^3, \quad h(x) = x^4. \]

a) Show that $G$ is abelian $\iff f, h$ are homomorphisms.
b) State a generalization of a) for an arbitrary group of order $u \geq 3$.
c) Use b) to show that a non-cyclic group of order 4 is abelian.

2. Let $A$ be a commutative ring with 1 and let $P = (x)$ be a principal ideal of $A$. Consider $I = \bigcap_{n=1}^\infty P^n$.

a) Suppose $P$ is prime. Let $Q$ be a prime ideal of $A$ such that $Q \subseteq P$. Show that $Q \subseteq I$.
b) Assume that $x$ is not a zero divisor in $A$. Show that $I$ is prime and $I = xI$.
c) Assume that $A$ is an integral domain, that $P$ is prime, and that $I$ is finitely generated. Prove that $I = (0)$.

3. Let $A$ be a $n \times n$ matrix with entries in $\mathbb{C}$ (i.e. $A \in M_n(\mathbb{C})$), such that $A^r = I$ for some $r \in \mathbb{N}$.

a) Show that if $A$ has a unique eigenvalue $\zeta$, then $A = \zeta I$.
b) Assume that $A \in M_n(\mathbb{F}_2)$, where $\mathbb{F}_2$ is a finite field with 2 elements. Does a) still hold? If not, produce a counterexample.
c) Let $k$ be a field. Prove that the ring $M_n(k)$ contains an isomorphic copy of every extension of $k$ of degree at most $n$.

4. Let $K$ be a field and let $F = K(a), L = K(b)$ be two extensions of $K$ (both contained in an algebraic closure $\overline{K}$ of $K$).

a) Assume that $F$ and $L$ are normal, separable extensions of $K$ and that the extension degrees $[F : K]$ and $[L : K]$ are coprime. Show that $a + b$ generates the composite extension $FL$.
b) Assume only that $F \cap L = K$. Give an example where $a + b$ does not generate $FL$.

5. Let $K/\mathbb{Q}$ be the splitting field of the cyclotomic polynomial $\phi_{10}(x)$.

a) Describe $K$ and determine the degree of the extension $[K : \mathbb{Q}]$. 

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b) Determine the Galois group $G = \text{Gal}(K/\mathbb{Q})$ as well as the complete relationship between subgroups of $G$ and subfields of $K$. Does $G$ contain any subgroup of order 5?