DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

January 1995

1. \( G \) is a non-abelian group of order 21.
   (a) Prove that \( G \) is not simple.
   (b) Describe a composition series for \( G \).
   (c) Must \( G \) be solvable? (Justify your answer.)
   (d) Must \( G \) be nilpotent? (Justify your answer.)

2. (a) \((A, +)\) is an abelian group, and \( S \subset A \). For each abelian \( M \) and map \( \varphi: S \to M \), there is a homomorphism \( \Phi: A \to M \) whose restriction to \( S \) is \( \varphi \). Does it follow that \( A \) is free abelian?
   (b) How many (isomorphism classes of) abelian groups of order 168 are there?
   (c) Prove that \( \mathbb{Z}[x, y]/(x^2 - y^2) \) is a Noetherian ring.
   (d) Prove that \( \mathbb{Z}_2[x]/(x^3 - 1) \cong \mathbb{Z}_2 \times F_4 \), where \( F_4 \) is a field of order 4.

3. \( R \) is a principal ideal domain, and \( M \) is a finitely generated \( R \)-module.
   (a) Explain briefly the meaning of each of the following:
      (a.1) \( M \) has free rank \( r \).
      (a.2) \( M \) has invariant factors \( d_1, \ldots, d_k \).
   (b) If \( M = A_1 \oplus \cdots \oplus A_s \), where the \( A_i \) are non-zero cyclic \( R \)-modules, show that the number \( k \) of invariant factors of \( M \) is no more than \( s \).

4. \( U \) is a vector space (over the field \( F \)) with basis \( B = \alpha_1, \ldots, \alpha_r, \beta_{r+1}, \ldots, \beta_n \), and \( f: U \to U \) is a linear transformation whose kernel is the span of \( \beta_{r+1}, \ldots, \beta_n \).
   (a) Describe a basis of \( U \otimes F \) in terms of \( B \).
   (b) Describe the map \( f \otimes f: U \otimes F \to U \otimes F \).
   (c) Find a basis of the kernel of \( f \otimes f \).
5. (a) Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean domain.
   (b) State the factorization of 5 as a product of primes in $\mathbb{Z}[i]$.
   (c) Explain why $F = \mathbb{Z}[i]/(1 + 2i)$ is a finite field, and find $|F|$.

6. (a) If $K$ is the splitting field over $\mathbb{Q}$ of an irreducible polynomial of degree $n$, and $G$ is the Galois group of $K$ over $\mathbb{Q}$, explain why $G$ is (isomorphic to) a subgroup of the symmetric group $S_n$.
   (b) If $K$ is the splitting field of $x^3 - 4x^2 - 6$ over $\mathbb{Q}$, find the Galois group of $K$ over $\mathbb{Q}$.

7. Let $R_n$ ($n = 1, 2, \ldots$) be rings, and put $S_k = R_1 \times \cdots \times R_k$ and $S_\infty = \prod_{n=1}^{\infty} R_n$. Denote by $\varepsilon_i$ the function $R_i \to S_k$ ($1 \leq i \leq k \leq \infty$) such that $\varepsilon_i(x)$ has $i$th coordinate $x$ and all other coordinates zero.
   (a) If $J$ is a minimal left ideal of some $R_i$ ($1 \leq i \leq k$), show that $\varepsilon_i(J)$ is a minimal left ideal of $S_k$, and that every minimal left ideal of $S_k$ is of this form.
   (b) If each $R_i$ is semisimple, show the $S_k$ are semisimple ($1 \leq k < \infty$).
   (c) Under what conditions is $S_\infty$ semisimple? (Justify your answer.)

N.B. Each ring $R$ is required to have an identity element, and modules and ring homomorphisms are required to be unitary.