Transfer relations in essentially tame local Langlands correspondence

by

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Abstract

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Let F be a non-Archimedean local field and G be the general linear group GL_n over F. Bushnell and Henniart described the essentially tame local Langlands correspondence of $G(F)$ using rectifiers, which are certain characters defined on tamely ramified elliptic maximal tori of $G(F)$. They obtained such result by studying the automorphic induction character formula. We relate this formula with the spectral transfer character formula, based on the theory of twisted endoscopy of Kottwitz, Langlands and Shelstad. The two main results in this article are

(i) to show that the automorphic induction character formula is equal to the spectral transfer character formula under the same Whittaker normalization and

(ii) to express the essentially tame local Langlands correspondence using the admissible embeddings constructed by Langlands-Shelstad χ-data, and to express the rectifiers of Bushnell-Henniart by certain endoscopic transfer factors.

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Contents

Chapter 1

Introduction

1.1 Historical background

Let F be a non-Archimedean local field with residue field $\mathbf{k}_F$ of order q a power of p. Let G be GL_n the general linear group of invertible $n \times n$ matrices over F. We know that the supercuspidal spectrum of $G(F)$ is bijective to the collection of irreducible n-dimensional smooth complex representations of the Weil group W_F of F. This is a consequence of the local Langlands correspondence for GL_n proved independently in [10], [11] in the characteristic 0 case, and in [23] in the positive characteristic case.

Bushnell and Henniart described in [6] such correspondence when restricted to the essentially tame case, which is briefly summarized as follows. Let $\mathcal{G}_n^{\mathrm{et}}(F)$ be the set of equivalence classes of essentially tame irreducible representations of W_F of degree n and $\mathcal{A}_n^{\mathrm{et}}(F)$ be the set of isomorphism classes of essentially tame irreducible supercuspidals. Bushnell and Henniart proved that there exists a unique bijection

$$\mathcal{L}_n = {}_F\mathcal{L}_n^{\mathrm{et}} : \mathcal{G}_n^{\mathrm{et}}(F) \to \mathcal{A}_n^{\mathrm{et}}(F), \sigma \mapsto \pi \tag{1.1.1}$$

which satisfies certain canonical conditions together with the compatibilities of automorphic induction [12], [13] and base-change [1], [14] for cyclic extensions (see the precise statement in Proposition 3.2 of [6]). We call the map $\mathcal{L}_n$ the *essentially tame local*

Langlands Correspondence. In most literature, σ is called the Langlands parameter of π.

To describe $\mathcal{L}_n$ we need to introduce the third set $P_n(F)$ of equivalence classes (E, ξ) of admissible characters ξ of $E^\times$ over F, where E goes through tamely ramified extensions over F of degree n. By [6] we know that $P_n(F)$ bijectively parameterizes both $\mathcal{G}_n^{\text{et}}(F)$ and $\mathcal{A}_n^{\text{et}}(F)$ simultaneously. We denote the bijections by

$$\Sigma_n : P_n(F) \to \mathcal{G}_n^{\text{et}}(F),\ (E, \xi) \mapsto \sigma_\xi \tag{1.1.2}$$

and

$$\Pi_n : P_n(F) \to \mathcal{A}_n^{\text{et}}(F),\ (E, \xi) \mapsto \pi_\xi. \tag{1.1.3}$$

We describe these bijections in simple words.

(i) The correspondence Σ_n is simply the induction of representation $\sigma_\xi = \mathrm{Ind}_{W_E}^{W_F} \xi$ if we regard ξ as a character of W_E by class field theory [30]. According to the compatibility condition characterizing $\mathcal{L}_n$, the composition $\mathcal{L}_n \circ \Sigma_n : P_n(F) \to \mathcal{A}_n^{\text{et}}(F)$ consists of successive lifts of representations by automorphic induction.

(ii) The correspondence Π_n is a compact induction

$$\pi_\xi = \mathrm{cInd}_{\mathbf{J}_\xi}^{G(F)} \Lambda_\xi \tag{1.1.4}$$

of certain representation Λ_ξ of a compact-mod-center subgroup $\mathbf{J}_\xi$ of $G(F)$, which is called an extended maximal type associated to ξ in the sense of [3]. We remark that the construction of $(\mathbf{J}_\xi, \Lambda_\xi)$ from (E, ξ) is completely local and representation theoretic.

In the case $p \nmid n$, traditionally known as the tame case, any irreducible supercuspidals of $\mathrm{GL}_n(F)$ and any n-dimensional irreducible complex representations of W_F are essentially tame, and $P_n(F)$ consists of all (E, ξ) for E goes through separable extensions over F of degree n. In this case the correspondence (1.1.3) was constructed in [16], and based on such construction the description of $\mathcal{L}_n$ using $P_n(F)$ as parameters was studied in [24] and [26]. In particular when $n = 2$, the book [2] contains a more extensive treatment in this theory, based on certain technical computations in [4] among the others.

With the setup above, we can describe $\mathcal{L}_n$ as follows. The composition of the bijections (1.1.3), (3.1) and the inverse of (3.2)

$$\mu : P_n(F) \xrightarrow{\Sigma_n} \mathcal{G}_n^{\mathrm{et}}(F) \xrightarrow{\mathcal{L}_n} \mathcal{A}_n^{\mathrm{et}}(F) \xrightarrow{\Pi_n^{-1}} P_n(F)$$

does not give the identity map on $P_n(F)$. Bushnell and Henniart proved in [9] that for any admissible character ξ of $E^\times$, there is a unique tamely ramified character ${}_F\mu_\xi$ of $E^\times$, depending on the restriction $\xi|_{U_E^1}$ of ξ on the 1-unit group U_E^1 of $E^\times$, such that ${}_F\mu_\xi \cdot \xi$ is also admissible and

$$\mu(E, \xi) = (E, {}_F\mu_\xi \cdot \xi).$$

We call ${}_F\mu_\xi$ the rectifier of ξ.

We briefly explain how to deduce the values of ${}_F\mu_\xi$. We first construct a sequence of subfields

$$F \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_l \subseteq E,$$

which satisfies the following conditions.

(I) K_0/F is the maximal unramified sub-extension of E/F;

(II) $K_l/K_{l-1}, \ldots, K_1/K_0$ are quadratic totally ramified;

(III) E/K_l is totally ramified of odd degree.

Notice that the extensions in (I) and (II) are cyclic. The one in (III) is cyclic if we adjoint to the base field F sufficient amount of roots of unity. This is the standard base-change technique.

Each rectifier ${}_F\mu_\xi$ admits a factorization

$$ {}_F\mu_\xi = ({}_{K_0/F}\mu_\xi)({}_{K_1/K_0}\mu_\xi) \cdots ({}_{K_l/K_{l-1}}\mu_\xi)({}_{K_l}\mu_\xi) \tag{1.1.5}$$

with each factor being tamely ramified. The approach in the series [6], [7], [9] is to deduce each factor on the right side of (1.1.5) through an inductive process as follows. Assume at this moment that all the field extensions in (III) above are cyclic. We reduce to the case when K/F is a cyclic sub-extension of E/F and assume that ${}_K\mu_\xi$ is known when ξ is regarded as an admissible character of $E^\times$ over K. We can deduce a new character ${}_{K/F}\mu_\xi$ of $E^\times$, called a ν-rectifier of ξ, by making use of the automorphic induction formula (in the sense of [12]) for cyclic extension, and comparing this formula with the Mackey induction formula from the compact induction (1.1.4). We then define

$$ {}_F\mu_\xi = ({}_{K/F}\mu_\xi)({}_K\mu_\xi). $$

In general we can apply a base change to assume that the extension in (III) is always cyclic, so that the rectifier ${}_{K_l}\mu_\xi$ is defined (see the proof of Theorem 3.5 and section 4.6-4.9 of [6]).

The automorphic induction is subsumed in the Langlands-Shelstad transfer principle [22], which belongs to the theory of twisted endoscopy [18]. We provide a brief introduction as follows. We call a connected F-quasi-split reductive group H a twisted endoscopic group of G if the dual of H is a twisted centralizer of a semi-simple element in the dual of G. This applies to the case when K/F is cyclic and $(G, H) = (\mathrm{GL}_n, \mathrm{Res}_{K/F}\mathrm{GL}_{n/|K/F|})$. In this article, we replace the word 'twisted endoscopy' by simply 'endoscopy' for convenience. We remark that the transfer principle is conditional on the Fundamental Lemma, which is conjectured in [22], formulated in the twisted case in [18], and proved by Waldspurger in [31] for the cases we concern in this article and by [25] for the ordinary endoscopy in general.

The transfer principle conjectures that for each irreducible tempered representation ρ of $H(F)$ there is an irreducible tempered representation π of $G(F)$ which satisfies certain character relations in terms of ρ. In particular if ρ is essentially tame supercuspidal and satisfies the regular condition by the action of the Galois group $\Gamma_{K/F}$, then π is also essentially tame supercuspidal. The character relations between π and ρ, which we shall call the spectral transfer character relation, is known to be the same as the automorphic induction character relation up to a constant.

1.2 Main results of the author

The two main results of the author and their proofs comprise chapter 8 and 9 respectively. Below we just introduce the main idea of the statements. For the readers who know the background theory well, we encourage them to take a look at the interlude in chapter 5. There we briefly describe the idea in proving the two main results in more detail, since then we have enough information to clarify the notions and terminologies.

1.2.1 The first result

The first main result is to clarify the 'up to a constant' relation between the automorphic induction character relation and the spectral transfer character relation stated in the last paragraph of 1.1.

Suppose π is an essentially tame supercuspidal of $G(F)$ automorphically induced from an essentially tame supercuspidal ρ of $H(F)$. A necessary condition is that π is isomorphic to its twisted by the character κ of $F^\times$ of the cyclic extension K/F by local class field theory, i.e. there is an $G(F)$-intertwining operator

$$\Psi : \kappa\pi := (\kappa \circ \det) \otimes \pi \to \pi$$

defined up to a constant. Let Θ_ρ be the character of ρ and Θ^κ_π be the twisted character of π depending on the intertwining operator Ψ. A very rough form of the automorphic induction character relation looks like

$$\Theta^\kappa_\pi(\gamma) = \Delta_{\mathrm{HH}}(\gamma) \sum_{g \in \Gamma_{K/F}} \Theta_{\rho^g}(\gamma) \tag{1.2.1}$$

for γ lies in the subset of elliptic semi-simple elements of $H(F)$ and is regular in $G(F)$, a subset large enough to characterize the formula. Here Δ_{HH} is a transfer factor defined by

Henniart-Herb [12], depending on the choices of several auxiliary objects which are hidden in the background for the moment. Varying these choices only changes the transfer factor Δ_{HH}, and so the right side of (1.2.1), only by a constant independent of h. The point is that, in order to compare (1.2.1) with the Mackey induction formula from (1.1.4), we have to choose the correct normalization of Ψ depending on the underlying representations.

On the other hand, Langlands-Shelstad defines a transfer factor Δ_{LS} which can be easily shown to be equal to Δ_{HH} again up to a constant. To get a canonical normalization of Δ_{LS}, we make use of the quasi-split property of G, i.e. G contains a Borel subgroup B defined over F, called the standard one for instance. We then normalize Δ_{LS} as in section 5.3 of [18], in the sense that it depends only on the datum related to the standard Borel B, called a standard Whittaker datum, and does not depend on other auxiliary data. The (rough form of the) sum

$$\Delta_{\mathrm{LS}}(\gamma) \sum_{g\in\Gamma_{K/F}} \Theta_{\rho^g}(\gamma) \tag{1.2.2}$$

should also determine π and differ from (1.2.1) by a constant.

We give some idea on how the two transfer factors look like. The Henniart-Herb transfer factor Δ_{HH} depends on the discriminant of the element γ. More precisely, if γ lies in $E^\times$ as an elliptic torus of $H(F)$ and is regular in $G(F)$, then roughly speaking $\Delta_{\mathrm{HH}}(\gamma)$ depends on

$$\prod_{\substack{\Gamma_F/\Gamma_E=\{g_1,\dots,g_n\}\\ i<j}} ({}^{g_i}\gamma - {}^{g_j}\gamma) \tag{1.2.3}$$

which makes use of the symmetry of the cosets of the Galois group. The Langlands-Shelstad transfer factor Δ_{LS} is a product of several other factors, each of them depends on the root system $\Phi=\Phi(G,T)$ in G of the elliptic torus T whose F-points is $E^\times$. The roots in Φ are of the form

$$\gamma \mapsto {}^{g_i}\gamma({}^{g_j}\gamma)^{-1},\ g_i,g_j\in\Gamma_F/\Gamma_E,\ g_i\neq g_j. \tag{1.2.4}$$

The similarity between (1.2.3) and (1.2.4) gives the relation between the two transfer factors, see Proposition 4.19 for instance. We would make use of the interplay between the Galois groups and the root system: there is a easy bijection in Proposition 2.1 between the Galois orbits of the root system and the double cosets of the Galois group.

To compare (1.2.1) and (1.2.2) we apply the theory of Whittaker model. The intertwining operator Ψ is actually normalized by another Whittaker datum depending on the internal structure of the supercuspidal π. If π comes from an admissible character ξ via Π_n in (1.1.3), then the internal structure of π comes from the corresponding internal structure of ξ called the jump data. We now recall that any two Whittaker data are conjugate under $G(F)$. Therefore we would compute a constant, temporarily denoted by $\kappa(x)$, depending on the element $x \in G(F)$ which conjugate the standard Whittaker datum and the one related on ξ, such that

$$\kappa(x)\Delta_{\mathrm{HH}}(\gamma) \sum_{g \in \Gamma_{K/F}} \Theta_{\rho^g}(\gamma) \tag{1.2.5}$$

is standard Whittaker normalized. In other words, the factor $\kappa(x)\Delta_{\mathrm{HH}}(\gamma)$ is independent of the representation π (or ρ). The main task of the first result is to compute $\kappa(x)$ and show that

Theorem 1.1 (Theorem 8.5). *If K/F is one of the extensions in (I)-(III) and is moreover cyclic, then*

$$\Delta_{\mathrm{LS}} = \kappa(x)\Delta_{\mathrm{HH}}$$

and so the normalized automorphic induction character (1.2.5) and the spectral transfer formula (1.2.2) are equal.

We remark that in a preprint of Hiraga and Ichino [15] they prove that (1.2.5) and (1.2.2) are equal for arbitrary cyclic extension K/F not necessarily tamely ramified. Their method is to make use of a global argument to reduce the pair (G, H) to a tame toral case, i.e. H is an elliptic torus of G which splits over a tamely ramified cyclic extension of F. Our method is to directly compute the transfer factors and the normalization constants such that the computation (if we take the existence of automorphic induction for granted) is completely local.

1.2.2 The second result

The second main result is to express the Bushnell-Henniart's rectifiers in terms of Langlands-Shelstad transfer factors, and hence express the essentially tame local Langlands correspondence in terms of admissible embeddings of L-groups. Recall in [22] that Langlands and Shelstad introduced a collection of characters, called χ-data, to construct certain admissible embeddings of L-groups ${}^LT \hookrightarrow {}^LG$ for a pair (G,T) of a F-quasi-split reductive group G containing T as a maximal F-torus. In the case $(G,T) = (\mathrm{GL}_n, \mathrm{Res}_{E/F}\mathbb{G}_m)$, a set of χ-data consists of certain characters $\{\chi_\lambda\}_{\lambda \in W_F\backslash\Phi}$. Here λ runs through a suitable subset, denoted by $W_F\backslash\Phi$ at this moment, of representatives of the W_F-orbits of the root system $\Phi = \Phi(G,T)$, such that for each $\lambda \in W_F\backslash\Phi$ the character χ_λ is defined on the multiplicative group of a field extension E_λ containing E. The precise description from a set of χ-data to an admissible embedding

$$\{\chi_\lambda\}_{\lambda \in W_F\backslash\Phi} \mapsto (\chi_{\{\chi_\lambda,\xi\}} : {}^LT \to {}^LG)$$

can be found in (2.5) of [22].

The main result is that particular choices of χ-data, depending on a fixed admissible character ξ of $T(F) = E^\times$, yield the rectifier of ξ in a simple manner.

Theorem 1.2 (Theorem 9.1). *For each admissible character ξ, its rectifier ${}_F\mu_\xi$ has a factorization of the form*

$${}_F\mu_\xi = \prod_{\lambda \in W_F\backslash\Phi} \chi_{\lambda,\xi}|_{E^\times}$$

for canonical choices of tamely ramified χ-data $\{\chi_{\lambda,\xi}\}_\lambda$ depending on ξ.

We can compare this product to the original factorization (1.1.5) provided by Bushnell and Henniart. The new factorization in Theorem 1.2 is a finer one, in the sense that for every intermediate field extension L/K appears on the right side of (1.1.5), we have a factorization for the ν-rectifier

$$ {}_{L/K}\mu_\xi = \prod_{\substack{\lambda\in W_F\backslash\Phi \\ \lambda|_K\equiv 1,\, \lambda|_L\neq 1}} \chi_{\lambda,\xi}|_{E^\times}. $$

This leads to expressing the ν-rectifiers by certain transfer factors denoted by Δ_{III_2}, a fact stated as Corollary 9.9. We would also ask the reader to compare this fact to Corollary 4.13 of Theorem 4.10.

We can therefore interpret the essentially tame local Langlands correspondence using admissible embeddings as follows. Via the local Langlands correspondence for the torus T, i.e.

$$ \mathrm{Hom}(E^\times, \mathbb{C}^\times) \cong H^1(W_F, \hat{T}), $$

we let $\tilde{\xi} : W_F \to {}^LT$ be a 1-cocycle whose class corresponds to the admissible character $\xi : E^\times \to \mathbb{C}^\times$. Moreover given the χ-data as in Theorem 1.2, we consider the inverse collection $\{\chi_{\lambda,\xi}^{-1}\}_\lambda$ which is also a χ-data by definition.

Theorem 1.3. *The natural projection*

$$ \chi_{\{\chi_{\lambda,\xi}^{-1}\}} \circ \tilde{\xi} : W_F \to {}^LT \to {}^LG $$

onto $\mathrm{GL}_n(\mathbb{C})$ *is isomorphic to* $\sigma_{{}_F\mu_\xi^{-1}\xi} = \mathrm{Ind}_{W_E}^{W_F}({}_F\mu_\xi^{-1}\xi)$ *as representations of* W_F*, hence is the Langlands parameter of the supercuspidal* π_ξ*.*

1.3 Outline of the article

The first three chapters contain mainly classical theory. Chapter 2 describes the rudiment objects for the whole theory, although we immediately bring out a few technical results which will be used frequently in later chapters. In chapter 3 we state the main statement of essentially tame local Langlands correspondence and give the notion of automorphic induction. In chapter 4 we introduce the Langlands-Shelstad endoscopic theory in brief detail. We would compute explicitly those transfer factors of our concern and compare them to those provided by Henniart-Herb.

Before going into non-classical matters we give in chapter 5 an interlude which outline the main results and sketch some technical details of the proofs. Then in chapter 6 and 7 we construct the essentially tame supercuspidals from admissible characters and certain finite modules arising from the constructions. At the end of chapter 7 we give the explicit values of the rectifiers in terms of certain invariants, called the t-factors, of the finite modules. In the last two chapters 8 and 9 we state and prove the main results described in section 1.2.

Chapter 2

Basic setup

In this chapter we setup the basic objects. Throughout we let

(i) F be a non-Archimedean local field,

(ii) $\mathbf{k}_F$ be the residue field of F, with q elements and of characteristic p,

(iii) Γ_F be the Galois group of F, and

(iv) G be GL_n as a reductive group over F.

In section 2.1 we describe elliptic tori in $G(F)$ and the actions of the Galois group on them. We hence have the induced action on the root system of each torus. For explicit computations we need to identify the root system with the non-trivial double cosets of Galois groups. We then state in section 2.2 certain parity results on the double cosets which are essential to the main results of this article. In section 2.3 we introduce the dual groups and the L-groups, which are the main ingredient of the 'Galois side' of the local Langlands correspondence.

2.1 Root systems

Given a field extension E/F of degree n, let T be the induced torus

$$T = \mathrm{Res}_{E/F}\mathbb{G}_m.$$

By identifying E with an n-dimensional F-vector space, we may embed T into G as an elliptic maximal torus. We can express the Galois action on the induced torus as follows. By identifying our maximal torus T as the group of functions $\Gamma_F/\Gamma_E \to \mathbb{G}_m$, i.e.

$$\mathrm{Res}_{E/F}\mathbb{G}_m \cong \mathrm{Fun}(\Gamma_F/\Gamma_E, \mathbb{G}_m),$$

we can write the Γ_F-action on T by

$$({}^g f)(x\Gamma_E) = {}^g(f(g^{-1}x\Gamma_E)), \text{ for all } f \in \mathrm{Fun}(\Gamma_F/\Gamma_E, \mathbb{G}_m),\ g, x \in \Gamma_F.$$

Hence the F-point of T is

$$\mathrm{Fun}(\Gamma_F/\Gamma_E, \mathbb{G}_m)^{\Gamma_F} = \{f : \Gamma_F/\Gamma_E \to \bar{F}^\times | {}^g(f(g^{-1}x\Gamma_E)) = f(x\Gamma_E) \text{ for all } g, x \in \Gamma_F\}$$

which is isomorphic to $E^\times$ by the evaluation at the trivial coset Γ_E

$$\mathrm{Fun}(\Gamma_F/\Gamma_E, \mathbb{G}_m)^{\Gamma_F} \to E^\times,\ f \mapsto f(\Gamma_E), \tag{2.1.1}$$

Notice that constant functions correspond to elements in $F^\times$.

Let $\Phi = \Phi(G,T)$ be the root system of T in G. It generates a free abelian group $X^*(T)$ of rank n, called the root lattice. We are going to assign a basis for $X^*(T)$ and express the Γ_F-action on Φ explicitly. The root lattice can be expressed as

$$\begin{aligned} X^*(T) &= \mathrm{Hom}(T, \mathbb{G}_m) \\ &= \{\phi : \mathrm{Res}_{E/F}\mathbb{G}_m \to \mathbb{G}_m \text{ over } F,\ \phi(fh) = \phi(f)\phi(h) \text{ for all } f, h\}. \end{aligned}$$

Take a set of coset representatives $\{g_1, \ldots, g_n\}$ of Γ_F/Γ_E with $g_1 = 1$ and write

$$gg_i\Gamma_E = g(g, g_i)\Gamma_E \text{ for some } g(g, g_i) \in \{g_1, \ldots, g_n\}.$$

The characters $\{\phi_{g_1}, \ldots, \phi_{g_n}\}$ where

$$\phi_{g_i} \in X^*(T),\, \phi_{g_i}(f) = f(g_i\Gamma_E)$$

form a basis for $X^*(T)$. Hence any character in $X^*(T)$ can be expressed as

$$\phi_{g_1}^{m_1} \cdots \phi_{g_n}^{m_n}(f) = f(g_1\Gamma_E)^{m_1} \cdots f(g_n\Gamma_E)^{m_n}$$

and so the Γ_F-action on $X^*(T)$ can be expressed as

$$\begin{aligned}({}^g(\phi_{g_1}^{m_1} \cdots \phi_{g_n}^{m_n}))(f) &= {}^g(\phi_{g_1}^{m_1} \cdots \phi_{g_n}^{m_n}({}^{g^{-1}}f)) \\ &= {}^g\left[({}^{g^{-1}}f)(g_1\Gamma_E)^{m_1} \cdots ({}^{g^{-1}}f)(g_n\Gamma_E)^{m_n}\right] \\ &= f(gg_1\Gamma_E)^{m_1} \cdots f(gg_n\Gamma_E)^{m_n}.\end{aligned}$$

Hence we have an action on the dual torus $\hat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$ by extending the action on $\mathbb{C}^\times$ trivially. In coordinates, we can write any element in $\hat{T}$ as

$$\sum_{i=1}^{n} \phi_{g_i} \otimes z_{g_i} \text{ for } z_{g_i} \in \mathbb{C}^\times.$$

Therefore we have the Γ_F-action

$$g : \sum_{i=1}^{n} \phi_{g_i} \otimes z_{g_i} \mapsto \sum_{i=1}^{n} \phi_{g(g,g_i)} \otimes z_{g_i} = \sum_{j=1}^{n} \phi_{g_j} \otimes z_{g(g^{-1},g_i)}. \tag{2.1.2}$$

The roots in Φ are denoted by

$$\left[\begin{smallmatrix} g_i \\ g_j \end{smallmatrix}\right] = \phi_{g_i}\phi_{g_j}^{-1} \text{ for } g_i \neq g_j.$$

By the evaluation isomorphism (2.1.1),

$$\left[\begin{smallmatrix} g_i \\ g_j \end{smallmatrix}\right](t) = {}^{g_i}t({}^{g_j}t)^{-1} \text{ for all } t \in E^\times.$$

The Γ_F-action on Φ is therefore given by

$$\left(g \cdot \left[\begin{smallmatrix} g_i \\ g_j \end{smallmatrix}\right]\right)(f) = f(gg_i\Gamma_E)f(gg_j\Gamma_E)^{-1} = \left[\begin{smallmatrix} gg_i \\ gg_j \end{smallmatrix}\right](f).$$

Notice that such action factors through the action of the Weyl group $\Omega(G,T)$ of T. It is clear that the Γ_F-orbit of a root contains an element of the form

$$\left[\begin{smallmatrix}1\\g\end{smallmatrix}\right] \text{ for some } g \in \{g_2, \dots, g_n\}.$$

For each λ in the root system Φ we denote the stabilizers

$$\Gamma_\lambda = \{g \in \Gamma_F | g\lambda = \lambda\} \quad \text{and} \quad \Gamma_{\pm\lambda} = \{g \in \Gamma_F | g\lambda = \pm\lambda\},$$

and fixed fields

$$E_\lambda = \bar{F}^{\Gamma_\lambda} \quad \text{and} \quad E_{\pm\lambda} = \bar{F}^{\Gamma_{\pm\lambda}}.$$

In general, E_λ is a field extension of some conjugate of E. We call a root λ *symmetric* if $|E_\lambda/E_{\pm\lambda}| = 2$, and *asymmetric* otherwise. By definition this symmetry is preserved by the Γ_F-action. Write $[\lambda]$ be the Γ_F-orbit of λ. Let

(i) $\Gamma_F\backslash\Phi_{\mathrm{sym}}$ be the set of Γ_F-orbits of symmetric roots,

(ii) $\Gamma_F\backslash\Phi_{\mathrm{asym}}$ be the set of Γ_F-orbits of asymmetric roots, and

(iii) $\Gamma_F\backslash\Phi_{\mathrm{asym}/\pm}$ be the set of equivalent classes of asymmetric Γ_F-orbits by identifying $[\lambda]$ and $[-\lambda]$.

We denote by $\mathcal{R}_{\mathrm{sym}}, \mathcal{R}_{\mathrm{asym}}$ and $\mathcal{R}_{\mathrm{asym}/\pm}$ be certain choices of sets of representatives in Φ of the above equivalent classes respectively.

Proposition 2.1. *The set $\Gamma_F\backslash\Phi$ of Γ_F-orbits of the root system Φ is bijective to the collection of non-trivial double cosets in $\Gamma_E\backslash\Gamma_F/\Gamma_E$, by*

$$\Gamma_F\backslash\Phi \to (\Gamma_E\backslash\Gamma_F/\Gamma_E) - \{\Gamma_E\},\ {}^{\Gamma_F}\lambda = {}^{\Gamma_F}\left[\begin{smallmatrix}1\\g\end{smallmatrix}\right] \mapsto \Gamma_E g \Gamma_E.$$

Proof. The set of roots Φ can be identified with the set of off-diagonal elements of $\Gamma_F/\Gamma_E \times \Gamma_F/\Gamma_E$, with Γ_F-action by ${}^g(g_1\Gamma_E, g_2\Gamma_E) = (gg_1\Gamma_E, gg_2\Gamma_E)$. By elementary group theory, we know that the orbits are bijective to the non-trivial double cosets in $\Gamma_E\backslash\Gamma_F/\Gamma_E$. □

We denote $(\Gamma_E\backslash\Gamma_F/\Gamma_E)'$ the collection of non-trivial double cosets, and $[g]$ the double coset $\Gamma_E g\Gamma_E$. We call $g \in \Gamma_F$ *symmetric* if $[g] = [g^{-1}]$ and *asymmetric* otherwise. Clearly such symmetry descends to an analogous property on $(\Gamma_E\backslash\Gamma_F/\Gamma_E)'$. By Proposition 2.1 the symmetry of $(\Gamma_E\backslash\Gamma_F/\Gamma_E)'$ is equivalent to the symmetry of $\Gamma_F\backslash\Phi$. Let

(i) $(\Gamma_E\backslash\Gamma_F/\Gamma_E)_{\mathrm{sym}}$ be the set of symmetric non-trivial double cosets,

(ii) $(\Gamma_E\backslash\Gamma_F/\Gamma_E)_{\mathrm{asym}}$ be the set of asymmetric non-trivial double cosets, and

(iii) $(\Gamma_E\backslash\Gamma_F/\Gamma_E)_{\mathrm{asym}/\pm}$ be the set of equivalent classes of $(\Gamma_E\backslash\Gamma_F/\Gamma_E)_{\mathrm{asym}}$ by identifying $[g]$ with $[g^{-1}]$.

We denote by $\mathcal{D}_{\mathrm{sym}}, \mathcal{D}_{\mathrm{asym}}$ and $\mathcal{D}_{\mathrm{asym}/\pm}$ be certain choices of sets of representatives in Γ_F/Γ_E of the above equivalent classes respectively.

We hence observe, by the identification in Proposition 2.1, that we can choose a collection $\mathcal{R} = \mathcal{R}_{\mathrm{sym}} \bigsqcup \mathcal{R}_{\mathrm{asym}}$ such that for all $\lambda \in \mathcal{R}$, the field E_λ is an extension of E. More precisely, if λ corresponds to $[g]$, then $E_\lambda = {}^gEE$.

2.2 Galois groups

Let E/F be a field extension of degree n, with ramification index e and residue degree f. The multiplicative group $F^\times$ decomposes into product of subgroups $\langle\varpi_F\rangle \times \mu_F \times U_F^1$. They are namely the group generated by a prime element, the group of roots of unity, and the 1-unit group. We may identify μ_F with $\mathbf{k}_F^\times$ in the canonical way. We have similar decomposition for $E^\times$. In most of the article, we assume that E/F is tame, i.e. $p \nmid e$. By [20] II.5 we can always assume our choices of ϖ_E and ϖ_F satisfying

$$\varpi_E^e = \zeta_{E/F}\varpi_F \text{ for some } \zeta_{E/F} \in \mu_E. \tag{2.2.1}$$

Let L be the Galois closure of E/F. Hence L/E is unramified and L/F is also a tame extension. With the choice of ϖ_F and ϖ_E as in (2.2.1), we define the following F-operators on L

(i) $\phi : \zeta \mapsto \zeta^q$ for all $\zeta \in \mu_L$, $\varpi_E \mapsto \zeta_\phi \varpi_E$ and

(ii) $\sigma : \zeta \mapsto \zeta$ for all $\zeta \in \mu_L$, $\varpi_E \mapsto \zeta_e \varpi_E$.

Here ζ_ϕ is in μ_E satisfying $(\zeta_\phi \varpi_E)^e = \zeta_{E/F}^q \varpi_F$ and ζ_e is a choice of a primitive eth root of unity in $\bar{F}^\times$. More generally we write ${}^{\phi^i}\varpi_E = \zeta_{\phi^i}\varpi_E$ such that $\zeta_{\phi^i} = \zeta_\phi^{1+q+\dots+q^{i-1}}$ is an eth root of $\zeta_{E/F}^{q^i-1}$. Notice that we have an action of ϕ on σ by $\phi : \sigma \mapsto \phi \circ \sigma \circ \phi^{-1} = \sigma^q$. Therefore we can write our Galois group as

$$\Gamma_{L/F} = \langle \sigma \rangle \rtimes \langle \phi \rangle \quad \text{and} \quad \Gamma_{L/E} = \langle \phi^f \rangle \subseteq \langle \phi \rangle. \tag{2.2.2}$$

Proposition 2.2. *(i) We can choose* $\{\sigma^k \phi^i | k = 0, \dots, e-1,\ i = 0, \dots, f-1\}$ *as coset representatives for the quotient* $\Gamma_{E/F} = \Gamma_F/\Gamma_E$.

(ii) Let $q^f \backslash \langle \sigma \rangle$ *be the set of orbits of* $\langle \sigma \rangle$ *under the action of* ϕ^f*, i.e.* $\sigma \mapsto \sigma^{q^f}$*, then the double coset* $\Gamma_E \backslash \Gamma_F / \Gamma_E$ *is bijective to the set* $(q^f \backslash \langle \sigma \rangle) \times \langle \phi \rangle$.

Proof. In general, if we have an abelian group B acting on a group A as automorphisms, and C a subgroup of B, then the canonical maps

$$(A \rtimes B)/(1 \times C) \to A \times (B/C),\ (a, b)(1 \times C) \mapsto (a, bC)$$

and

$$(1 \times C)\backslash(A \rtimes B)/(1 \times C) \to (C\backslash A) \times (B/C),\ (1 \times C)(a, b)(1 \times C) \mapsto (C\backslash a, bC)$$

are bijective. We take $A \rtimes B = \Gamma_{L/F}$ and $C = \Gamma_{L/E}$ as in (2.2.2). □

With such identification we can write down the symmetric double cosets explicitly.

Proposition 2.3. *The double coset* $[g] = [\sigma^k \phi^i]$ *is symmetric if and only if*

(i) $i = 0$ *or, when* f *is even,* $i = f/2$*, and*

(ii) e *divides* $(q^{ft} + 1)k$ *in case* $i = 0$*, and divides* $(q^{f(2t+1)/2} + 1)k$ *in case* $i = f/2$*, for some* $t = 0, \dots, |L/E| - 1$.

Proof. Since ϕ acts as $\sigma \mapsto \sigma^q$, we can show that the inverse of $\sigma^k\phi^i$ in $\Gamma_{L/F}$ is $\sigma^{-k\bar{q}^i}\phi^{-i}$ where $\bar{q}$ is the multiplicative inverse of q in $(\mathbb{Z}/e)^\times$. Also recall that $\Gamma_{L/E} = \langle\phi^f\rangle$, so the double cosets $\Gamma_E\sigma^k\phi^i\Gamma_E$ and $\Gamma_E(\sigma^k\phi^i)^{-1}\Gamma_E$ are equal if and only if $i \equiv -i \mod f$ and $q^{ft}k \equiv -k\bar{q}^i \mod e$ for some t. This implies the assertion by simple calculation. □

By slightly abusing notation, we call those symmetric $[\sigma^k]$ ramified and those symmetric $[\sigma^k\phi^{f/2}]$ unramified.

Our main results rely on knowing the parity of the number of double cosets $\Gamma_E\backslash\Gamma_F/\Gamma_E$ when E/F is totally ramified. If $|E/F| = e$ and $\#\mathbf{k}_F = q$, then by Proposition 2.2(i) it is equivalent to check the number of q-orbits of $\mathbb{Z}/e$. If we partition the set $\mathbb{Z}/e$ according to multiples, i.e. write

$$\mathbb{Z}/e = \bigsqcup_{d|e}(e/d)\,(\mathbb{Z}/d)^\times\,,$$

then because $\gcd(e,q) = 1$ each subset consists of union of q-orbits. For each divisor d of e, let $\mathrm{ord}(q,d)$ be the multiplicative order of q in $(\mathbb{Z}/d)^\times$, and $\phi(d)$ be the order of $(\mathbb{Z}/d)^\times$.

Lemma 2.4. *(i) The number of double cosets $\Gamma_E\backslash\Gamma_F/\Gamma_E$ equals*

$$\sum_{d|e}\phi(d)/\mathrm{ord}(q,d).$$

(ii) If $d = 1$ or 2, then $\phi(d)/\mathrm{ord}(q,d) = 1$. If $d \geq 3$ and q is a square, then $\phi(d)/\mathrm{ord}(q,d)$ is even.

Proof. Assertion (i) is clear. For $d = 1$ or 2, (ii) is also clear. For $d \geq 3$, let $\langle q\rangle$ be the cyclic subgroup in $(\mathbb{Z}/d)^\times$ generated by q. Suppose $q = r^2$. If $\langle q\rangle$ is a proper subgroup of $\langle r\rangle$, then $\phi(d)/\mathrm{ord}(q,d)$ is a multiple of the index $|\langle r\rangle/\langle q\rangle| = 2$. If $\langle q\rangle = \langle r\rangle$, then $q^k \equiv r \mod d$ for some k. The order $\mathrm{ord}(r,d)$ is then odd, and so is $\mathrm{ord}(q,d)$. But $\phi(d)$ is always even. This proves assertion (ii). □

Proposition 2.5. *The parity of*

$$\#(\Gamma_E\backslash\Gamma_F/\Gamma_E)_{\mathrm{sym-unram}} = \#\{[\sigma^k\phi^{f/2}] \in (\Gamma_E\backslash\Gamma_F/\Gamma_E)_{\mathrm{sym}}\}$$

is equal to that of $e(f-1)$.

Proof. If f is odd, then the set $\{[\sigma^k\phi^{f/2}]\}$ is empty, so the statement is true. Now we assume that f is even. Since $\phi^{f/2}$ normalizes Γ_E, we have indeed a bijection

$$\{[\sigma^k\phi^{f/2}] \in \Gamma_E\backslash\Gamma_F/\Gamma_E\} \to \Gamma_E\backslash\Gamma_K/\Gamma_E,\ [\sigma^k\phi^{f/2}] \mapsto [\sigma^k].$$

Since those asymmetric $[\sigma^k\phi^{f/2}]$ pair up, it suffices to show that the parity of $\Gamma_E\backslash\Gamma_K/\Gamma_E$ is the same as the parity of e. To this end we apply Lemma 2.4. Here we have q^f, which is a square, in place of q in Lemma 2.4. If e is odd, then the parity is

$$\phi(1)/\mathrm{ord}(q,1) + \sum_{d|e,\, d\geq 3} \phi(d)/\mathrm{ord}(q,d) = 1 + \sum \text{ even } = \text{ odd}.$$

If e is even, then the parity is

$$\phi(1)/\mathrm{ord}(q,1) + \phi(2)/\mathrm{ord}(q,2) + \sum_{d|e,\, d\geq 3} \phi(d)/\mathrm{ord}(q,d) = 1 + 1 + \sum \text{ even } = \text{ even}.$$

□

2.3 L-groups

For a non-Archimedean local field F, we define the Weil group of F as follows. Let F^{ur} be the maximal unramified extension of F in $\bar{F}$. The Galois group $\Gamma_F/\Gamma_{F^{\mathrm{ur}}}$ is denoted by I_F and is called the tame inertia group of F. We have the exact sequence [30]

$$1 \to I_F \to \Gamma_F \to \hat{\mathbb{Z}} := \varprojlim_{n} \mathbb{Z}/n \to 1.$$

The Weil group W_F of F is defined by the preimage of the subgroup $\mathbb{Z}$ of $\hat{\mathbb{Z}}$ in Γ_F. One of the major results of local class field theory is the isomorphism of (topological) groups

$$W_F^{\mathrm{ab}} := W_F/[W_F, W_F] \cong F^\times.$$

One remark is that when we are dealing with finite extensions, there is no difference between Galois groups and Weil groups, i.e. if E/F is a finite extension, then we have

$$W_F/W_E \cong \Gamma_F/\Gamma_E \text{ and } W_E\backslash W_F/W_E \cong \Gamma_E\backslash\Gamma_F/\Gamma_E.$$

For $G = \mathrm{GL}_n$ we define

$$\hat{G} = \mathrm{GL}_n(\mathbb{C}) \quad \text{and} \quad {}^L G = \mathrm{GL}_n(\mathbb{C}) \times W_F,$$

namely the dual-group and the L-group of G. IF T is the induced torus $\mathrm{Res}_{E/F}\mathbb{G}_m$, we write

$$\hat{T} = \mathrm{Hom}_E(T, \mathbb{G}_m) \otimes \mathbb{C}^\times \cong (\mathbb{C}^\times)^{[E/F]} \quad \text{and} \quad {}^L T = \hat{T} \rtimes W_F$$

as its dual-group and L-group. Here the action of W_F on $\hat{T}$ is the one factoring through Γ_F, which is given by (2.1.2).

Chapter 3

Essentially tame local Langlands correspondence

As a consequence of the local Langlands correspondence for GL_n proved in [10], [11], [23], we know that the supercuspidal spectrum $\mathcal{A}_n^0(F)$ of $G(F)$ is bijective to the collection $\mathcal{G}_n^0(F)$ of complex smooth n-dimensional irreducible representations of the Weil group W_F of F, i.e. there exists a canonical bijection

$$\mathcal{G}_n^0(F) \xrightarrow{\sim} \mathcal{A}_n^0(F).$$

In this chapter, we recall the main results in [6], [7], [9], where we describe the above correspondence when restricted to the essentially tame case. We first introduce this restricted bijection briefly in section 3.1. To describe such bijection explicitly, we need to introduce the third collection which consists of the equivalence classes of admissible characters. We recall these characters in section 3.2. The main machinery to obtain a supercuspidal representation from an admissible character is the automorphic induction constructed in [12], [13], [14]. This operation corresponds to the usual induction of representations on the 'Galois side' according to the functoriality principle. We study this topic in section 3.3. Finally in section 3.4 we can reduce the description of the correspondence to expressing certain characters called rectifiers, which are the main objects we would study in this article.

The last section 3.5 serves as an appendix, where we recall briefly the Langlands correspondence of discrete series in the Archimedean case. If $F = \mathbb{R}$ and $G = \mathrm{GL}_n$, then $G(F)$ possesses (relative) discrete series representations only if $n = 1$ or 2. The case $n = 1$ is just local class field theory. We will consider the case when $n = 2$.

3.1 The correspondence

For each Langlands parameter $\sigma \in \mathcal{G}_n^0(F)$, let $f(\sigma)$ be the number of unramified characters χ of W_F such that $\chi \otimes \sigma \cong \sigma$. We call σ *essentially tame* if p does not divide $n/f(\sigma)$. Let $\mathcal{G}_n^{\mathrm{et}}(F)$ be the set of isomorphism classes of essentially tame irreducible representations of degree n. Similarly, for each irreducible supercuspidal $\pi \in \mathcal{A}_n^0(F)$, let $f(\pi)$ be the number of unramified characters χ of $F^\times$ that $\chi \otimes \pi \cong \pi$. Here χ is regarded as a representation of $G(F)$ by composing with the determinant map. We call π *essentially tame* if p does not divide $n/f(\pi)$. Let $\mathcal{A}_n^{\mathrm{et}}(F)$ be the set of isomorphism classes of essentially tame irreducible supercuspidals.

Theorem 3.1 (Essentially tame local Langlands correspondence for GL_n)**.** *For each positive integer n there exists a unique bijection*

$$\mathcal{L}_n = {}_F\mathcal{L}_n^{\mathrm{et}} : \mathcal{G}_n^{\mathrm{et}}(F) \to \mathcal{A}_n^{\mathrm{et}}(F),\ \sigma \mapsto \pi \tag{3.1.1}$$

which satisfies the following properties.

(i) $\mathcal{L}_1 : \mathcal{G}_1(F) \to \mathcal{A}_1(F)$ is the local class field theory;

(ii) $\mathcal{L}_n$ is compatible with the automorphisms of the base field F;

(iii) $\mathcal{L}_n$ is compatible with contragredient, i.e. $\mathcal{L}_n(\sigma^\vee) = \pi^\vee$;

(iv) $\mathcal{L}_n(\chi \otimes \sigma) = \mathcal{L}_1(\chi) \otimes \pi$ for $\chi \in \mathcal{G}_1(F)$;

(v) $\det(\sigma) = \omega_\pi$ *the central character of* π*;*

(vi) $\mathcal{L}_n$ *is compatible with automorphic induction [12] and base-change [1] for cyclic extensions.*

Proof. See Proposition 3.2 and section 3.5 of [6]. □

We will be more specific in section 3.3 on the compatibility of automorphic induction in property (vi) above.

3.2 Admissible characters

Let E/F be a finite tamely ramified extension, and let ξ be a (quasi-)character of $E^\times$. We call ξ *admissible* over F if it satisfies the following conditions. For every field K between E/F,

(i) if $\xi = \eta \circ N_{E/K}$ for some character η of $K^\times$, then $E = K$;

(ii) if $\xi|_{U_E^1} = \eta \circ N_{E/K}$ for some character η of U_K^1, then E/K is unramified.

Clearly if ξ is admissible over F, then by definition it is admissible over K for every field K between E/F.

Let $P(E/F)$ be the set of admissible characters of $E^\times$ over F. Two admissible characters $\xi \in P(E/F)$ and $\xi' \in P(E'/F)$ are called F-equivalent if there is $g \in \Gamma_F$ that ${}^gE = E'$ and ${}^g\xi = \xi'$. We denote the F-equivalence class of ξ by (E, ξ). Let $P_n(F)$ be the set of F-equivalence classes of admissible characters, i.e.

$$P_n(F) = F\text{-equivalence}\backslash \left(\bigsqcup_E P(E/F) \right)$$

where E goes through tame extensions over F of degree n.

Proposition 3.2. *The map* $\Sigma_n : P_n(F) \to \mathcal{G}_n^{\mathrm{et}}(F)$, $(E, \xi) \mapsto \sigma_\xi := \mathrm{Ind}_{W_E}^{W_F} \xi$ *is a bijection, with* $f(\sigma_\xi) = f(E/F)$*.*

Proof. The proof is in the Appendix of [6]. □

We usually write $\mathrm{Ind}_{E/F}\xi$ as the induced representation $\mathrm{Ind}_{W_E}^{W_F}\xi$.

3.3 Automorphic induction

Let K/F be a cyclic extension of degree d and κ be a character of $F^\times$ of order d and with kernel $N_{K/F}(K^\times)$. In other words, κ is a character of local class field theory of the extension K/F. Write $G = \mathrm{GL}_n$ and $H = \mathrm{Res}_{K/F}\mathrm{GL}_m$ where $m = n/d$. Suppose that $\pi \in \mathcal{A}_n^0(F)$ and satisfies $\kappa\pi = \pi$, then by [12] there is $\rho \in \mathcal{A}_m^0(K)$, i.e. a supercuspidal of $H(F) = \mathrm{GL}_m(K)$, such that π is automorphically induced by ρ. We write $A_{K/F}(\rho) = \pi$.

We can characterize such automorphic induction using character relations as follows. Suppose Ψ is an non-zero intertwining operator between $\kappa\pi$ and π, in the sense that

$$\Psi \circ \kappa\pi(x) = \pi(x) \circ \Psi \text{ for all } g \in G(F).$$

This condition determines Ψ up to a scalar. We then define the (κ, Ψ)-trace of π to be the distribution

$$\Theta_\pi^{\kappa,\Psi}(f) = \mathrm{tr}(\Psi \circ \pi(f)) \text{ for all } f \in \mathcal{C}_c^\infty(G(F)).$$

By 3.9 Corollary of [12] we can view $\Theta_\pi^{\kappa,\Psi}$ as a function on the subset $G(F)_{\mathrm{rss}}$ of regular semi-simple elements of $G(F)$. Similarly, we can view the distribution Θ_ρ of the character of p as a function on $H(F)_{\mathrm{rss}}$. Let $G(F)_{\mathrm{ell}}$ be the subset of elliptic elements in $G(F)_{\mathrm{rss}}$. The character relation of automorphic induction is

$$\Theta_\pi^{\kappa,\Psi}(h) = c(\rho,\kappa,\Psi)\Delta^2(h)\Delta^1(h)^{-1} \sum_{g\in\Gamma_{K/F}} \Theta_\rho^g(h) \text{ for all } h \in H(F) \cap G(F)_{\mathrm{ell}}. \qquad (3.3.1)$$

Here Δ^1 and Δ^2 are the transfer factors defined in 3.2, 3.3 of [12]. We will recall their constructions in section 4.8. The constant $c(\rho,\kappa,\Psi)$ is a normalization depending only on ρ, κ and Ψ, and is called the automorphic induction constant.

From (3.3.1) we know that all ρ^g, where g runs through $\Gamma_{K/F}$, are automorphically induced to π. From [12] we know that given π the corresponding ρ is K/F-regular, i.e. all ρ^g are mutually inequivalent. Conversely if ρ is K/F-regular, then π is uniquely determined by (3.3.1). The relation between ρ and π should be independent of κ.

We explain the meaning of that the correspondence $\mathcal{L}_n$ is compatible with automorphic induction. Let $\mathcal{A}^0_m(K)_{K/F\text{-reg}}$ be the set of isomorphism classes of K/F-regular supercuspidals ρ of $\mathrm{GL}_m(K)$, and $\mathcal{G}^0_m(K)_{K/F\text{-reg}}$ be the set of isomorphism classes of irreducible representations τ of W_K of degree m which are K/F-regular, i.e. if we denote by τ^g the representation

$$\tau^g(h) = \tau(ghg^{-1}),\, h \in W_K\, g \in W_F,$$

then all τ^g are mutually inequivalent. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{G}^0_m(K)_{K/F\text{-reg}} & \xrightarrow{\mathcal{L}_m} & \mathcal{A}^0_m(K)_{K/F\text{-reg}} \\ \Big\downarrow \mathrm{Ind}_{K/F} & & \Big\downarrow \mathrm{A}_{K/F} \\ \mathcal{G}^0_n(F) & \xrightarrow{\mathcal{L}_n} & \mathcal{A}^0_n(F) \end{array}.$$

There is an dual diagram of the above, with induction on the Galois side replaced by restriction. The corresponding operation on the p-adic side is called the base-change. For a comprehensive study on this topic, we refer the readers to [1] and skip further discussion to avoid introducing more notations. We would only indicate at the place where we make use of base change.

We briefly describe how an admissible character determines an essentially tame supercuspidal using automorphic induction. Suppose ξ is an admissible character $E^\times$ over F. We split the extension E/F into intermediate sub-extensions

$$F = K_{-1} \hookrightarrow K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_{l-1} \hookrightarrow K_l \hookrightarrow E \tag{3.3.2}$$

such that

(I) E/K_l is totally ramified of odd degree [6],

(II) each K_i/K_{i-1}, $i = 1, \ldots, l$, is totally ramified quadratic [7], and

(III) K_0/F is unramified [9].

The extension in (II) and (III) are cyclic. Therefore each $A_{K_i/K_{i-1}}$, $i = 0, \ldots, l$, is defined by automorphic induction. The extension E/K_l in case (I) is not necessarily cyclic, but it is so if we base change K_l by an unramified extension. By a trick similar to 3.5 of [6], we can defined a map

$$A_{E/K_l} : P(E/F) \hookrightarrow P(E/K_l) \hookrightarrow \mathcal{A}_1(E)_{E/K_l\text{-reg}} \to \mathcal{A}^0_{|E/K_l|}(K_l)$$

(with the first two maps being natural inclusions) such that the diagram

$$\begin{array}{ccc} P(E/F) & \xrightarrow{\mathcal{L}_1^{-1}} & \mathcal{G}_1(E)_{E/K_l\text{-reg}} \\ \downarrow{\scriptstyle A_{E/K_l}} & & \downarrow{\scriptstyle \mathrm{Ind}_{E/K_l}} \\ \mathcal{A}^0_{|E/K_l|}(K_l) & \xrightarrow{\mathcal{L}^{-1}_{|E/K_l|}} & \mathcal{G}^0_{|E/K_l|}(K_l) \end{array}$$

commute. Using condition (i) in the definition of the admissibility of ξ, we can compose the maps above and define

$$A_{E/F} = A_{K_0/F} \circ A_{K_1/K_0} \circ \cdots \circ A_{K_l/K_{l-1}} \circ A_{E/K_l} : P(E/F) \to \mathcal{A}^0_n(F).$$

The 'union' of the above maps

$$\bigsqcup_{|E/F|=n \text{ tame}} A_{E/F} : \bigsqcup_{|E/F|=n \text{ tame}} P(E/F) \to \mathcal{A}^0_n(F)$$

descends to a map

$$A_n = A_n(F) : P_n(F) \to \mathcal{A}^0_n(F).$$

Proposition 3.3. *The map A_n is injective with image in $\mathcal{A}^{\mathrm{et}}_n(F)$.*

Proof. Indeed we constructed A_n such that it equals to the composition

$$P_n(F) \xrightarrow{\Sigma_n} \mathcal{G}^{\mathrm{et}}_n(F) \xrightarrow{\mathcal{L}_n} \mathcal{A}^{\mathrm{et}}_n(F).$$

We refer the details to section 3 of [6]. □

3.4 Rectifiers

The bijection A_n depends on the character relation of the automorphic induction and is not quite explicit. In section 2 of [6] we constructed directly an essentially tame supercuspidal π_ξ from the admissible character ξ using solely representation theoretic method. The supercuspidal is a compact induction of the form

$$\pi_\xi = \mathrm{cInd}_{\mathbf{J}(\xi)}^{G(F)} \Lambda_\xi \tag{3.4.1}$$

for certain representation Λ_ξ on a compact-mod-center subgroup $\mathbf{J}(\xi)$ of $G(F)$. This representation $(\mathbf{J}(\xi), \Lambda_\xi)$ depends on the admissible character ξ, and is an extended maximal type in the sense of [3].

Proposition 3.4. *The map* $\Pi_n : P_n(F) \to \mathcal{A}_n^{\mathrm{et}}(F)$, $(E, \xi) \mapsto \pi_\xi$ *is a bijection, with* $f(\pi_\xi) = f(E/F)$.

Proof. The proof is summarized in chapter 2 of [6]. □

The whole theory comes from the fact that the maps A_n and Π_n are unequal. In other words, the composition of the bijections in Proposition 3.2, Theorem 3.1 and the inverse of the one in Proposition 1.1.3

$$\mu : P_n(F) \xrightarrow{\Sigma_n} \mathcal{G}_n^{\mathrm{et}}(F) \xrightarrow{\mathcal{L}_n} \mathcal{A}_n^{\mathrm{et}}(F) \xrightarrow{\Pi_n^{-1}} P_n(F)$$

does not give the identity map on $P_n(F)$. Bushnell and Henniart proved in [9] that for any admissible character ξ of $E^\times$, there is a unique tamely ramified character ${}_F\mu_\xi$ of $E^\times$, depending on the wild part of ξ, i.e. the restriction $\xi|_{U_E^1}$, such that ${}_F\mu_\xi \cdot \xi$ is also admissible and

$$\mu(E, \xi) = (E, {}_F\mu_\xi \cdot \xi).$$

We call the character ${}_F\mu_\xi$ the rectifier of ξ. In the series [6], [7], [9], we express ${}_F\mu_\xi$ explicitly, and hence also the correspondence $\mathcal{L}$.

The rectifier ${}_F\mu_\xi$ is a product

$$ {}_F\mu_\xi = ({}_{K_l}\mu_\xi)({}_{K_l/K_{l-1}}\mu_\xi)\cdots({}_{K_1/K_0}\mu_\xi)({}_{K_0/F}\mu_\xi). \tag{3.4.2} $$

Here the intermediate subfields are those in the sequence (3.3.2). The first factor ${}_{K_l}\mu_\xi$ is the rectifier by considering ξ as admissible over K_l. This rectifier is constructed in [6]. The remaining factors are of the form ${}_{K_i/K_{i-1}}\mu_\xi$. These characters, called the ν-rectifiers, are constructed in [7] for $i = 1, \dots, l$ and in [9] for $i = 0$. We briefly describe the construction of the rectifiers and the ν-rectifiers by the following inductive process. Suppose that K/F is a cyclic extensions of E/F of degree d and $m = n/d$ and assume that we have constructed ${}_K\mu_\xi$. Suppose that the essentially tame supercuspidal π

(i) is automorphically induced from $\rho = {}_K\pi_\xi$ and

(ii) is compactly induced from an extended maximal type $\Lambda = \Lambda_{\nu\xi}$ depending on an admissible character $\nu\xi$, where ν is some tame character of $E^\times$. We know that such ν exists by Proposition 3.4 of [6].

By choosing suitable normalized Ψ we can equate the characters of automorphic induction (3.3.1) from ρ and the (twisted) Mackey induction formula from Λ, i.e.

$$ \Theta_\pi^\kappa(h) = \sum_{x \in G(F)/\mathbf{J}(\xi)} \kappa(\det x^{-1}) \operatorname{trace}\Lambda(x^{-1}hx) \text{ for all } h \in H(F) \cap G(F)_{\mathrm{ell}}. \tag{3.4.3} $$

Here we regard $\operatorname{trace}\Lambda$ as a function of $G(F)$ vanishing outside $\mathbf{J}(\xi) = \mathbf{J}_{\nu\xi}$. By decomposing both formulae into finer sums and studying each terms in the sums, we solve for the character ν which is hence the ν-rectifier ${}_{K/F}\mu_\xi$. We remark that we will specify the correct Ψ using Whittaker model in chapter 8.

To express the values of the rectifiers and the ν-rectifier, we need to first understand the construction of the compact induction (3.4.1). We will come to this point in chapter 7.

3.5 The Archimedean case

We consider when $G(F) = \mathrm{GL}_2(\mathbb{R})$ when $T(F) = \mathbb{C}^\times$. Each character of $\mathbb{C}^\times$ is of the form

$$\chi_{2s,n} : \mathbb{C}^\times \to \mathbb{C}^\times,\ re^{i\theta} \mapsto r^{2s}e^{in\theta},\ s \in \mathbb{C},\ n \in \mathbb{Z}.$$

We call $\chi_{2s,n}$ admissible over $\mathbb{R}$ (or regular) if $\chi_{2s,n}$ does not factor through

$$N_{\mathbb{C}/\mathbb{R}} : z \mapsto z\bar{z} = |z|^2.$$

This is equivalent to require that $n \neq 0$. It is easy to check that $\chi_{2s,n}$ and $\chi_{2s,-n}$ are conjugate under the non-trivial action $^- \in \Gamma_{\mathbb{C}/\mathbb{R}}$. We define the equivalence class of $\chi_{2s,n}$ by $(\mathbb{C}/\mathbb{R}, \chi_{2s,n})$ and denote the set of equivalent classes by

$$P_2(\mathbb{R}) = \{(\mathbb{C}/\mathbb{R}, \chi_{2s,n}) | s \in \mathbb{C}, n > 0\}.$$

By [30] any irreducible 2-dimensional representation of $W_{\mathbb{R}}$ is induced from a character of $W_{\mathbb{C}}$. We denote

$$\varphi_{2s,n} = \mathrm{Ind}_{\mathbb{C}/\mathbb{R}}\chi_{2s,n} \tag{3.5.1}$$

We know that $\varphi_{2s,n} \cong \varphi_{2s,-n}$. Therefore the set of equivalence classes of 2-dimensional representation of $W_{\mathbb{R}}$ is

$$\mathcal{G}_2^0(\mathbb{R}) = \{\varphi_{2s,n} | s \in \mathbb{C}, n > 0\}$$

and the map

$$P_2(\mathbb{R}) \to \mathcal{G}_2^0(\mathbb{R}),\ \chi \mapsto \mathrm{Ind}_{\mathbb{C}/\mathbb{R}}\chi$$

is clearly bijective.

Each character of $\mathbb{R}^\times$ is of the form

$$\mu_{s,m} : \mathbb{R}^\times \to \mathbb{C}^\times,\ \pm r \mapsto \pm^m r^s,\ s \in \mathbb{C},\ m \in \mathbb{Z}/2 = \{0, 1\}.$$

Using the notation in Theorem 5.11 of [17], we consider the representation

$$\sigma(\mu_{s+\frac{n}{2},[n+1]}, \mu_{s-\frac{n}{2},0}),\ s \in \mathbb{C},\ n \geq 1.$$

where $[n] \in \mathbb{Z}/2$ is the parity of n. These exhaust the collection $\mathcal{A}_2^0(\mathbb{R})$ of all (relative) discrete series representations by Harish-Chandra classification. The Langlands correspondence is

$$\mathcal{L} : \mathcal{G}_2^0(\mathbb{R}) \to \mathcal{A}_2^0(\mathbb{R}),\ \varphi_{2s,n} \mapsto \sigma(\mu_{s+\frac{n}{2},[n+1]}, \mu_{s-\frac{n}{2},0}).$$

Since

$$\sigma(\mu_{s-\frac{n}{2},[-n+1]}, \mu_{s+\frac{n}{2},0}) \cong \sigma(\mu_{s+\frac{n}{2},[n+1]}, \mu_{s-\frac{n}{2},0})$$

by Theorem 5.11 of [17], $\mathcal{L}$ is injective.

Let's recall the construction of the discrete series representations, and refer to chapter 3 of [21] for details. For each $\varphi = \varphi_{2s,n} \in \mathcal{G}_2^0(\mathbb{R})$ coming from $\chi = \chi_{2s,n}$ by (3.5.1), we introduce a 'ρ-shift' on χ as follows. Let

$$\rho : \mathbb{C}^\times \to \mathbb{C}^\times,\ z \mapsto (z\bar{z}^{-1})^{1/2}.$$

Hence indeed $\rho = \chi_{0,1}$. This is the based χ-data used in [29] for the admissible embedding ${}^LT \hookrightarrow {}^LG$, a notion which will be introduced in section 4.5. We then define

$$\chi_0 = \chi\rho^{-1} = \chi_{2s,n-1}.$$

For every such χ_0 we assign

$$\pi(\chi_0) = \sigma(\mu_{s+\frac{n}{2},[n+1]}, \mu_{s-\frac{n}{2},0}) = \mathcal{L}(\mathrm{Ind}_{\mathbb{C}/\mathbb{R}}\chi_0\rho) \tag{3.5.2}$$

whose character formula is given by

$$-\frac{\chi_0(z)\rho(z) - \chi_0(\bar{z})\rho(\bar{z})}{\Delta(z)\rho(z)} = -\frac{\chi(z) - \chi(\bar{z})}{\Delta(z)\rho(z)}. \tag{3.5.3}$$

Here $z \in \mathbb{C}^\times - \mathbb{R}$ embedded in $\mathrm{GL}(2,\mathbb{R})$ as a compact-mod-center Cartan subgroup, and $\Delta(z)$ is the Weyl determinant. The character formula is expressed as on the left side of (3.5.3) in order to match the character formula of discrete series for general reductive groups in [21] under the Harish-Chandra's classification. Therefore from (3.5.2) we see that, in order to obtain the correct correspondence, we have rectify the correspondence by the character ρ.

Chapter 4

Endoscopy

Let K/F be a cyclic extension of degree d and $m = n/d$. The aim of this chapter is to compute the Langlands-Shelstad transfer factor, in the sense of [22] and [18], in the case when $G = \mathrm{GL}_n$ and $H = \mathrm{Res}_{K/F}\mathrm{GL}_m$ regarded as a twisted endoscopic group of G. We express this transfer factor Δ_0 as a product of several other transfer factors, namely

$$\Delta_0 = \Delta_{\mathrm{I}}\Delta_{\mathrm{II}}\Delta_{\mathrm{III}_1}\Delta_{\mathrm{III}_2}\Delta_{\mathrm{IV}}.$$

These factors are functions evaluated at the regular semi-simple elements in an elliptic torus T regarded as contained in both G and H. We would also establish some properties of these transfer factors.

The topics of this chapter are divided as follows.

(i) We define the transfer factor Δ_0 and its significance in terms of the transfer principle in section 4.1.

(ii) We construct Δ_{I} in terms of several auxiliary data in section 4.2. Then in section 4.3 we show that if we choose specific data then we can trivialize Δ_{I}, i.e. $\Delta_{\mathrm{I}} \equiv 1$.

(iii) In section 4.4 and 4.5, we interpret the induction of representations of Weil groups in terms of admissible embeddings of L-groups constructed by χ-data. We state this result as Theorem 4.10 in section 4.6, where we can also express the transfer factor Δ_{III_2} explicitly. In section 4.7 we state a restriction property of Δ_{III_2} which is related to condition (v) of the local Langlands correspondence in Theorem 3.1 (see Remark 9.6 for instance).

(iv) We relate the above transfer factors to the ones Δ^1 and Δ^2 defined in [12] in section 4.8.

4.1 The transfer principle

Let d be a divisor of n and let K/F be a cyclic extension of degree d. Write $m = n/d$ and $H = \mathrm{Res}_{K/F}\mathrm{GL}_m$. Then we can regard H as a twisted endoscopic group of G by the theory in [18] as follows. Take a torus T over F of rank n (but not necessarily split over F). We can embed T into H and G by certain isomorphisms $\iota_H : T_H \to T$ and $\iota_G : T_G \to T$ such that the corresponding F-isomorphism $\iota := \iota_G\iota_H^{-1} : T_H \to T_G$ is admissible with respect to certain fixed splittings [1] spl_H and spl_G of H and G respectively (see section 3 of [18]). On the dual side, we take

$$\hat{G} = \mathrm{GL}_n(\mathbb{C}), \quad \hat{H} = \mathrm{GL}_m(\mathbb{C})^d \quad \text{and} \quad \hat{T} = (\mathbb{C}^\times)^n.$$

Here W_F acts on $\hat{H}$ by cyclicly permutes the d factors, so the action factors through that of the cyclic group W_F/W_K. For convenience, we can choose splittings $\mathrm{spl}_{\hat{H}}$ and $\mathrm{spl}_{\hat{G}}$ both consist of diagonal subgroups as their respective tori and embed $\hat{T}$ into both $\hat{G}$ and $\hat{H}$ diagonally.

Together with G there is a triple of datum (G, θ, κ) such that, in our case here, θ is the trivial automorphism of G and κ is a character of order d and whose kernel equals $N_{E/F}(E^\times)$. By H being a twisted endoscopic group of (G, θ, κ) we mean that there exists a quadruple of data $(H, \mathcal{H}, s, \hat{\xi})$ which satisfy the conditions in section 2.1 of [18]. In our

[1] The word ‘splitting’ is also called ‘pinning’ in some literature, coming from the French terminology ‘épinglage’.

case we can take

$$\mathcal{H} = {}^L H = \mathrm{GL}_m(\mathbb{C})^d \rtimes \Gamma_{K/F},\ s = \begin{pmatrix} & & 1_m \\ 1_m & & \\ & \ddots & \\ & & 1_m \end{pmatrix}$$

and

$$\hat{\xi} : {}^L H \to {}^L G,\ \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_d \end{pmatrix} \rtimes_{\hat{H}} w \mapsto \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_d \end{pmatrix} \bar{N}_{HG}(w) \rtimes_{\hat{G}} w$$

where 1_m is the identity matrix of size $m \times m$, the entries $h_i \in \mathrm{GL}_m(\mathbb{C})$ and $\bar{N}_{HG}(w)$ is the appropriate permutation matrix which measures the difference of the actions of $w \in W_F$ on $\hat{H}$ and on $\hat{G}$.

Suppose $\gamma \in T(F)$ is strongly G-regular, i.e. $\iota_G(\gamma)$ is strongly regular in $\iota_G(T)$. With ι being admissible, it can be shown that $\iota_H(\gamma)$ is also strongly regular. We can compute the transfer factor

$$\Delta_0(\iota_H(\gamma), \iota_G(\gamma))$$

using the recipes in [22] or [18]. It is a product of five factors, $\Delta_{\mathrm{I}}, \Delta_{\mathrm{II}}, \Delta_{\mathrm{III}_1}, \Delta_{\mathrm{III}_2}$ and Δ_{IV}, all evaluated at the point $(\iota_H(\gamma), \iota_G(\gamma))$. For simplicity we write

(i) $\Delta_*(\gamma) = \Delta_*(\iota_H(\gamma), \iota_G(\gamma))$ for every subscribe $* = 0, \mathrm{I}, \dots, \mathrm{IV}$ as above,

(ii) $\Delta_{i,j,\dots}$ for the product $\Delta_i \Delta_j \cdots$, and

(iii) $\Delta_{\mathrm{III}} = \Delta_{\mathrm{III}_1, \mathrm{III}_2}$.

We can compute some of the factors with little effort.

(i) Since we regard H as a subgroup of G and the embeddings $T \hookrightarrow H \hookrightarrow G$ are fixed, we have

$$\Delta_{\mathrm{III}_1}(\gamma) = 1.$$

(ii) The factor Δ_{IV} is the quotient of Weyl determinants. It equals to the factor Δ^1 in the automorphic induction formula (3.3.1). This factor turns out to be of very little interest to us.

We have to choose some auxiliary data to compute Δ_{I}, Δ_{II} and Δ_{III_2}. These will occupy the subsequent sections of this chapter.

Before we end this section we briefly describe the transfer principle. Take $\gamma \in G(F)_{\mathrm{rss}}$ and write $G(F)_\gamma$ the centralizer of γ in $G(F)$. For every $f \in \mathcal{C}_c^\infty(G(F))$ we define the κ-orbital integral

$$O_\gamma^\kappa(f) = \int_{G(F)_\gamma \backslash G(F)} \omega(g) f(g^{-1}\gamma g) \frac{dg}{dt}$$

where dg and dt are Haar measures on $G(F)$ and $G(F)_\gamma$ respectively. Similarly, for $\gamma_H \in H(F)_{\mathrm{rss}}$ and $f^H \in \mathcal{C}_c^\infty(H(F))$, we define the orbital integral

$$O_{\gamma_H}(f^H) = \int_{H(F)_{\gamma_H} \backslash H(F)} f(h^{-1}\gamma_H h) \frac{dh}{dt_H}$$

where dh and dt_H are Haar measures on $H(F)$ and $H(F)_{\gamma_H}$ respectively. Suppose $\iota(\gamma_H) = \gamma$. We call f^H and f are matching functions if they satisfy

$$O_{\gamma_H}(f^H) = \Delta_0(\gamma) O_\gamma^\kappa(f). \tag{4.1.1}$$

The Fundamental Lemma [31] asserts that for every f such f^H exists.

Given $\rho \in \mathcal{A}_m^0(K)$, we call an admissible representation π of $G(F)$ a spectral transfer of ρ if their characters satisfy

$$\Theta_\rho(f^H) = \text{constant} \cdot \Theta_\pi^\kappa(f) \tag{4.1.2}$$

We will show in Proposition 4.19 that the Langlands-Shelstad transfer factor Δ_0 equals to the Henniart-Herb transfer factor

$$\Delta_G^H(\gamma) = \Delta^1(\gamma)\Delta^2(\gamma), \text{ for all } \gamma \in H(F) \cap G(F)_{\mathrm{ell}}$$

up to a constant. More precisely, we will show that $\Delta_{\mathrm{IV}} = \Delta^1$ and the factors $\Delta_{\mathrm{I,II,III}}$ and Δ^2 differ by a constant. Hence $\Delta_{\mathrm{I,II,III}}$ also satisfies the properties of Δ^2. Using the technique involving the Weyl integration formula (see the proof of Theorem 3.11 of [12]), we can show that the matching of orbital integrals (4.1.1) and the spectral transfer

(4.1.2) gives rise to an analogous formula of (3.3.1).

$$\Theta_\pi^\kappa(h) = \text{constant} \cdot \Delta_{\mathrm{I,II,III}}(h)\Delta_{\mathrm{IV}}(h)^{-1} \sum_{g\in\Gamma_{K/F}} \Theta_\rho^g(h), \text{ for all } h \in H(F) \cap G(F)_{\mathrm{ell}} \quad (4.1.3)$$

Therefore by [12] the spectral transfer π of ρ exists, and is exactly the automorphic induction of ρ. In chapter 8 we will show that the automorphic induction formula (3.3.1) and the spectral transfer formula (4.1.3) are equal when both are suitably normalized.

4.2 The splitting invariant Δ_{I}

We now sketch the construction of Δ_{I} in [22]. It consists of several ingredients.

(i) We choose an F-splitting $\mathbf{spl}_G = (\mathbf{B}, \mathbf{T}, \{\mathbf{X}_\alpha\})$ of G and a Borel subgroup B of G (not necessarily defined over F) such that $h^{-1}(B,T)h = (\mathbf{B}, \mathbf{T})$ for some $h \in G$. We write

$$\omega : W_F \to \Omega(G, \mathbf{T}) \rtimes W_F,\ w \mapsto \omega(w) = \bar{\omega}(w) \rtimes w$$

such that the action $\omega(w)$ on $t \in \mathbf{T}$ is the transferred action of ${}^{w_T}t$ by $\mathrm{Int}(h)$.

(ii) We recall in section (2.1) of [22] the Springer section $n_{\mathrm{S}} : \Omega(G, \mathbf{T}) \to N_G(\mathbf{T})$ and define

$$n : W_F \to N_G(\mathbf{T}) \rtimes W_F,\ w \mapsto n(w) = \bar{n}(w) \rtimes w := n_{\mathrm{S}}(\bar{\omega}(w)) \rtimes w.$$

This map may not be a morphism of groups.

(iii) We take a collection $\{a_\lambda \in \bar{F}^\times | \lambda \in \Phi(G,T)\}$, called a set of a-data, which satisfies

$$a_{g\lambda} = {}^g(a_\lambda) \text{ for all } g \in \Gamma_F \text{ and } a_{-\lambda} = -a_\lambda. \quad (4.2.1)$$

We may regard the indexes λ as in $\Phi(G, \mathbf{T})$ by using the transport $\mathrm{Int}(h)$. Then write

$$x(g) = \prod_{\lambda\in\Phi(\mathbf{B},\mathbf{T}),\, g^{-1}\lambda\notin\Phi(\mathbf{B},\mathbf{T})} a_\lambda^{\lambda^\vee} \in \mathbf{T}.$$

Here is a remark about the a-data in (4.2.1). By Proposition 2.2 every λ can be written as $\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$ with $g, h \in \Gamma_F/\Gamma_E$. We sometimes denote a_λ by $a_{g,h}$. Hence by (4.2.1) the whole collection $\{a_\lambda | \lambda \in \Phi\}$ of a-data depends only on the subcollection

$$\{a_{1,g} | g \in \mathcal{D}_{\text{sym}} \sqcup \mathcal{D}_{\text{asym}/\pm}\}.$$

Proposition 4.1. *The map*

$$g \to hx(g)\bar{n}(g)^g h^{-1},\ g \in \Gamma_F$$

is a 1-cocycle of Γ_F *whose image is in* T*.*

Proof. See section (2.3) of [22]. □

We denote by $\lambda(\{a_\lambda\}, T)$ the class defined by this cocycle in $H^1(\Gamma_F, T)$. Following the recent preprint [19], we call this class the splitting invariant. It can be shown that the splitting invariant is independent of choice of the Borel subgroup B (again see section (2.3) of [22]).

4.3 Trivializing the splitting invariant

Let E/F be a tamely ramified extension. The aim in this section is to choose an embedding of the torus $T(F) = E^\times$ into $\mathrm{GL}_n(F)$ and certain a-data $\{a_\lambda\}$ to trivialize the splitting invariant class $\lambda(\{a_\lambda\}, T)$. We can even make things better: the 1-cocycle representing this class is trivial by selecting suitable a-data. The idea originates from the construction in [15].

We choose the ordered F-basis of E by

$$\begin{aligned} \mathfrak{b} &= \{1, \zeta, \dots, \zeta^{f-1}, \varpi_E, \varpi_E\zeta, \dots, \varpi_E\zeta^{f-1}, \dots, \varpi_E^{e-1}\zeta^{f-1}\} \\ &= \{\varpi_E^i \zeta^j | 0 \le i < e,\ 0 \le j < f\}. \end{aligned} \tag{4.3.1}$$

We identify G with $\mathrm{Aut}_F(E)$ the group of F-automorphisms of E, such that we can take an F-splitting $\mathbf{spl}_G$ of G containing $\mathbf{T}$ as the diagonal subgroup with respect to the above basis. We write

$$\vec{v} = (1, \zeta, \ldots, \zeta^{f-1}, \varpi_E, \varpi_E\zeta, \ldots, \varpi_E\zeta^{f-1}, \ldots, \varpi_E^{e-1}\zeta^{f-1}) \in E^n,$$

and embed $\iota : T \to G$ by

$$x \cdot \vec{v} = \vec{v} \cdot \iota(x), \tag{4.3.2}$$

where the left side is coordinate-wise multiplication and the right side is a multiplication of a row vector with a matrix. We can assume that T lies in H and that ι is admissible.

We now define a total order $<_\phi$ on the 'Galois set' $\Gamma_{E/F} = \{\phi^i\sigma^k | 0 \leq i < f, 0 \leq k < e\}$ by the multiplicative ϕ-orbits of $\langle\sigma\rangle$. Take a set $q^f\backslash(\mathbb{Z}/e)$ of representatives in $\{0, 1, \ldots, e\}$ of the q^f-orbits of $\mathbb{Z}/e$ (see Proposition 2.2(ii)) and choose an arbitrary order on $\{\sigma^k | k \in q^f\backslash(\mathbb{Z}/e)\}$. For simplicity we may choose the orbit of 1 to be the minimum. We then consider a multiplicative ϕ-orbit of σ^k for a fixed $k \in q^f\backslash(\mathbb{Z}/e)$. This is the subset $\{\phi^i\sigma^k | 0 \leq i < q^{fs_k}\}$ where s_k is the cardinality of the q^f-orbit $k \in \mathbb{Z}/e$. We order the ϕ-orbit of σ^k by

$$\phi^i\sigma^k <_\phi \phi^j\sigma^k \text{ if and only if } 0 \leq i < j < q^{fs_k}.$$

Finally if $g, h \in \Gamma_F/\Gamma_E$ lie in different ϕ-orbits, i.e. $g \in \phi\backslash\sigma^k$, $h \in \phi\backslash\sigma^l$, and $k \neq l$, then we order

$$g <_\phi h \text{ if and only if } k <_\phi l.$$

For example, if we choose the order $0 <_\phi 1 <_\phi \cdots <_\phi k <_\phi \cdots$ on $q^f\backslash(\mathbb{Z}/e)$, then the order on Γ_F/Γ_E looks like

$$\begin{aligned}
&1 <_\phi \phi <_\phi \cdots <_\phi \phi^{f-1} \\
&<_\phi \sigma <_\phi \phi\sigma <_\phi \cdots <_\phi \phi^{f-1}\sigma <_\phi \sigma^{1+_\phi 1} <_\phi \phi\sigma^{1+_\phi 1} <_\phi \cdots <_\phi \sigma^{1_l} <_\phi \phi^{f-1}\sigma^{1_l} \\
&<_\phi \cdots \\
&<_\phi \sigma^k <_\phi \phi\sigma^k <_\phi \cdots <_\phi \phi^{f-1}\sigma <_\phi \sigma^{k+_\phi 1} <_\phi \phi\sigma^{k+_\phi 1} <_\phi \cdots <_\phi \sigma^{k_l} <_\phi \phi^{f-1}\sigma^{k_l} \\
&<_\phi \cdots .
\end{aligned}$$

Here for each $k \in \mathbb{Z}/e$ we denote

(i) $k_{+_\phi 1}$ the 'successor' of α under the order induced by $<_\phi$ from Γ_F/Γ_E to the subset $\langle\sigma\rangle \cong \mathbb{Z}/e$, and

(ii) k_l the maximum within the q^f-orbit of k, i.e. $k_l \equiv q^{f(s_k-1)}k \mod e$.

If $\Gamma_{E/F} = \{g_1 = 1, \ldots, g_n\}$ following the above order $<_\phi$, then by putting

$$\mathbf{g} = (\vec{v}^t, {}^{g_2}\vec{v}^t, \ldots, {}^{g_n}\vec{v}^t)^t \in \mathrm{GL}_n(E),$$

we have $\mathbf{g}\iota(x)\mathbf{g}^{-1} = \mathrm{diag}(x, {}^{g_2}x, \ldots, {}^{g_n}x)$ for all $x \in E^\times$. The (α, j)-block of $\mathbf{g}$ is

$$\mathbf{g}_{\alpha j} = \begin{pmatrix} {}^{\sigma^\alpha}(\varpi_E^j, \varpi_E^j\zeta, \ldots, \varpi_E^j\zeta^{f-1}) \\ {}^{\phi\sigma^\alpha}(\varpi_E^j, \varpi_E^j\zeta, \ldots, \varpi_E^j\zeta^{f-1}) \\ \vdots \\ {}^{\phi^{f-1}\sigma^\alpha}(\varpi_E^j, \varpi_E^j\zeta, \ldots, \varpi_E^j\zeta^{f-1}) \end{pmatrix}$$

which equals $g_{\alpha j}u$ where $g_{\alpha j} = \mathrm{diag}({}^{\sigma^\alpha}\varpi_E^j, \ldots, {}^{\phi^{f-1}\sigma^\alpha}\varpi_E^j)$ and

$$u = \begin{pmatrix} 1 & {}^{\sigma^\alpha}\zeta & \ldots & {}^{\sigma^\alpha}\zeta^{f-1} \\ 1 & {}^{\phi\sigma^\alpha}\zeta & \ldots & {}^{\phi\sigma^\alpha}\zeta^{f-1} \\ \vdots & & & \\ 1 & {}^{\phi^{f-1}\sigma^\alpha}\zeta & \ldots & {}^{\phi^{f-1}\sigma^\alpha}\zeta^{f-1} \end{pmatrix}.$$

Therefore we have $\det \mathbf{g} = \det g(\det u)^e$, where

$$\det g = \prod_{k=0}^{f-1} \prod_{0\le\beta<_\phi\alpha<e} ({}^{\phi^k\sigma^\alpha}\varpi_E - {}^{\phi^k\sigma^\beta}\varpi_E)$$

and

$$\det u = \prod_{0\le i<k\le f-1} ({}^{\phi^k}\zeta - {}^{\phi^i}\zeta).$$

Therefore if we take

$$b_{\phi^k\sigma^\alpha} = \prod_{0\le\beta<\alpha} ({}^{\phi^k\sigma^\alpha}\varpi_E - {}^{\phi^k\sigma^\beta}\varpi_E)^{-1} \prod_{0\le i<k} ({}^{\phi^k}\zeta - {}^{\phi^i}\zeta)^{-1},\ \phi^k\sigma^\alpha \in \Gamma_{E/F} \tag{4.3.3}$$

and write

$$h^{-1} = (b_1\vec{v}^t, b_{g_2}{}^{g_2}\vec{v}^t, \ldots, b_{g_n}{}^{g_n}\vec{v}^t)^t,$$

then $\det h = 1$ and $h^{-1}\iota(x)h = \mathbf{g}\iota(x)\mathbf{g}^{-1}$ for all $x \in E^\times$.

Proposition 4.2. *The cocycle (rather than the class it defines)*

$$g \mapsto hx(g)\bar{n}(g)^g h^{-1} : \Gamma_F \to T$$

is trivial by choosing suitable a-data.

The proof occupies the remaining of this section. First notice that If L/F is the Galois closure of E/F and $g \in \Gamma_L$, then g acts trivially on $h \in G(E)$, and $x(g)$ and $\bar{n}(g)$ are identity matrix. Hence it is enough to show that the cocycle is trivial on the generators σ and ϕ of $\Gamma_{L/F}$. In other words, we will prove that

$$x(\sigma)\bar{n}(\sigma)^{\sigma}h^{-1} = h^{-1} \quad \text{and} \quad x(\phi)\bar{n}(\phi)^{\phi}h^{-1} = h^{-1}. \tag{4.3.4}$$

Notice that for each equation in (4.3.4) we can concentrate on the rows consisting of a σ-orbit or a ϕ-orbit of the cosets in $\Gamma_{E/F}$ by left multiplication respectively. We would apply the following Lemma.

Lemma 4.3. *Let τ be an arbitrary element in $\Gamma_{L/F}$. If $\{g, \tau g, \ldots, \tau^{k-1}g\}$ is an τ-orbit, then with respect to such basis,*

$$\bar{n}(\tau) = \begin{pmatrix} & & & 1 \\ -1 & & & \\ & \ddots & & \\ & & -1 & \end{pmatrix} \text{ and } x(\tau) = \operatorname{diag}\left(\prod_{i=1}^{k-1} a_{g,\tau^i g}, a^{-1}_{g,\tau g}, a^{-1}_{g,\tau^2 g}, \ldots, a^{-1}_{g,\tau^{k-1}g}\right).$$

Proof. Recall that

$$x(\tau) = \prod_{\lambda>0,\, \tau^{-1}\lambda<0} a_\lambda^{\lambda^\vee}$$

and when $\lambda = \left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right] = \left[\begin{smallmatrix} gW_E \\ hW_E \end{smallmatrix}\right]$, we have $a_\lambda = a_{g,h}$. Choose the basis of (B,T) over $\bar{F}^{\langle\tau\rangle}$ given by the τ-orbit $\{g, \tau g, \ldots, \tau^{k-1}g\}$, then those positive $\lambda = \left[\begin{smallmatrix} \tau^i g \\ \tau^j g \end{smallmatrix}\right]$ with $\tau^{-1}\lambda < 0$ are those λ with $i = 0$ and $j > 0$. Therefore

$$x(\tau) = \prod_{j=1}^{k-1} a_{g,\tau^j g}^{\left[\begin{smallmatrix} g \\ \tau^j g \end{smallmatrix}\right]^\vee}$$

as above. The Springer element $\bar{n}(\tau)$ is given by the cyclic permutation

$$(g, \tau g, \ldots, \tau^{k-1}g) = (g, \tau^{k-1}g)(g, \tau^{k-2}g)\cdots(g, \tau g),$$

whose section is given by

$$\begin{pmatrix} & 1 & & & 1 \\ & & \ddots & & \\ -1 & & & 1 & \end{pmatrix} \begin{pmatrix} & 1 & & 1 \\ & & \ddots & \\ -1 & & 1 & \\ & & & 1 \end{pmatrix} \cdots \begin{pmatrix} -1 & 1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -1 & & 1 \\ & \ddots & \\ & & -1 \end{pmatrix}.$$

□

We start to solve for $\{a_\lambda\}$ in the first equation of (4.3.4), i.e. when $g = \sigma$. A σ-orbit is of the form $\{\sigma^k \phi^j | 0 \le k < e\}$ with a fixed $j = 0, \ldots, f-1$. Under the order $<_\phi$, the minimum element in this orbit is ϕ^j. Focusing on the rows of (4.3.4) corresponding to this orbit, then the equation reads as

$$\begin{aligned} &\operatorname{diag}\left(\prod_{j=1}^{e-1} a_{\phi^j,\sigma^k\phi^j}, a^{-1}_{\phi^j,\sigma\phi^j}, \ldots, a^{-1}_{\phi^j,\sigma^{e-1}\phi^j}\right) \begin{pmatrix} -1 & & 1 \\ & \ddots & \\ & & -1 \end{pmatrix} \cdot \\ &{}^{\sigma}(b_{\phi^j}{}^{\phi^j}\vec{v}^t, b_{\sigma\phi^j}{}^{\sigma\phi^j}\vec{v}^t, \ldots, b_{\sigma^{e-1}\phi^j}{}^{\sigma^{e-1}\phi^j}\vec{v}^t)^t \\ =&(b_{\phi^j}{}^{\phi^j}\vec{v}^t, b_{\sigma\phi^j}{}^{\sigma\phi^j}\vec{v}^t, \ldots, b_{\sigma^{e-1}\phi^j}{}^{\sigma^{e-1}\phi^j}\vec{v}^t)^t. \end{aligned} \tag{4.3.5}$$

Here we cheat a little bit: the rows below the first one may not be ordered as above, but it does not matter to the final result. The $(j+1)$th row for $1 \le j < e$ reads

$$-a^{-1}_{\phi^j,\sigma^k\phi^j}\, {}^{\sigma}b_{\sigma^{k-1}\phi^j} = b_{\sigma^k\phi^j}$$

and we solve from (4.3.3) that

$$a_{\phi^j,\sigma^k\phi^j} = -{}^{\sigma}b_{\sigma^{k-1}\phi^j} b^{-1}_{\sigma^k\phi^j} = {}^{\phi^j}\varpi_E - {}^{\sigma^k\phi^j}\varpi_E. \tag{4.3.6}$$

The first row reads

$$\left(\prod_{j=1}^{e-1} a_{\phi^j,\sigma^k\phi^j}\right) {}^{\sigma}b_{\sigma^{e-1}\phi^j} = b_{\phi^j},$$

and by putting (4.3.6), it changes into

$${}^{\sigma}\left(\prod_{j=0}^{e-1} b_{\sigma^k\phi^j}\right) = (-1)^{e-1} \prod_{j=0}^{e-1} b_{\sigma^k\phi^j},$$

which can be easily checked to be true.

We then solve for $\{a_\lambda\}$ in the second equation of (4.3.4), i.e. when $g = \phi$. Take the ϕ-orbit $\{\phi^j\sigma^\alpha | j = 0, \dots, f-1,\ \alpha \in q^f\backslash k\}$ of σ^k, ordered by $<_\phi$ defined above. We have equations similar to (4.3.5) corresponding to this orbit. We then check the equations of the rows, which are distinguished by the following cases.

(i) For fixed α, if $j = 1, \dots, f-1$, then

$$a_{\sigma^k, \phi^j\sigma^\alpha} = -{}^{\phi}b_{\phi^{j-1}\sigma^\alpha} b^{-1}_{\phi^j\sigma^\alpha} = \zeta - {}^{\phi^j}\zeta.$$

(ii) If $\alpha \neq k$ and $j = 0$, then

$$a_{\sigma^k, \sigma^\alpha} = -{}^{\phi}b_{\phi^{f-1}\sigma^{\alpha_{-\phi}1}} b^{-1}_{\sigma^\alpha}.$$

Here $\alpha_{-\phi}1$ satisfies $\alpha = \alpha_{-\phi}1 q^f$ the 'predecessor' of α with respect to $<_\phi$ in the ϕ-orbit of $\{\sigma^k\}$. Notice that ϕ permutes the orbits $<_\phi$-strictly smaller than the orbit of σ^k, and move the orbit of σ^k 'one step forward', so we have

$$\begin{aligned} {}^{\phi}b_{\phi^{f-1}\sigma^{\alpha_{-\phi}1}} &= {}^{\phi}\left(\prod_{0 \le \beta < \alpha_{-\phi}1} ({}^{\phi^{f-1}\sigma^{\alpha_{-\phi}1}}\varpi_E - {}^{\phi^{f-1}\sigma^\beta}\varpi_E)^{-1} \prod_{0 \le i < f-1} ({}^{\phi^{f-1}}\zeta - {}^{\phi^i}\zeta)^{-1} \right) \\ &= \prod_{0 \le \beta < \alpha} ({}^{\sigma^\alpha}\varpi_E - {}^{\sigma^\beta}\varpi_E)^{-1} ({}^{\sigma^\alpha}\varpi_E - {}^{\sigma^k}\varpi_E) \prod_{0 \le i < f-1} (\zeta - {}^{\phi^{i+1}}\zeta)^{-1}. \end{aligned}$$

Therefore

$$a_{\sigma^k, \sigma^\alpha} = ({}^{\sigma^k}\varpi_E - {}^{\sigma^\alpha}\varpi_E) \prod_{0 \le i < f-1} (\zeta - {}^{\phi^{i+1}}\zeta)^{-1}.$$

Notice that it equals

$$({}^{\sigma^k}\varpi_E - {}^{\sigma^\alpha}\varpi_E) \left(\prod_{0 \le i < f-1} (-{}^{\phi}b_{\phi^i\sigma^\alpha} b^{-1}_{\phi^{i+1}\sigma^\alpha}) \right)^{-1}. \tag{4.3.7}$$

(iii) If $\alpha = k$ and $j = 0$, the equation reads

$$\prod_{i=0}^{s_k f - 1} a_{\sigma^k, \phi^i\sigma^k} = {}^{\phi}b^{-1}_{\phi^{f-1}\sigma^{k_l}} b_{\sigma^k} \tag{4.3.8}$$

Here k_l is the $<_\phi$-largest in the orbit of σ^k. We have

$$
\begin{aligned}
{}^{\phi}b_{\phi^{f-1}\sigma^{k_l}} &= {}^{\phi}\left(\prod_{0\le_\phi\beta<_\phi k_l}\left({}^{\phi^{f-1}\sigma^{k_l}}\varpi_E - {}^{\phi^{f-1}\sigma^{\beta}}\varpi_E\right)^{-1}\prod_{0\le i<f-1}\left({}^{\phi^{f-1}}\zeta - {}^{\phi^i}\zeta\right)^{-1}\right)\\
&= \prod_{0\le_\phi j<_\phi k}\left({}^{\sigma^k}\varpi_E - {}^{\sigma^j}\varpi_E\right)^{-1}\prod_{k\le_\phi\beta<\alpha_l}\left({}^{\sigma^k}\varpi_E - {}^{\sigma^{\beta+_\phi 1}}\varpi_E\right)^{-1}\prod_{1\le i<f}\left(\zeta - {}^{\phi^i}\zeta\right)^{-1}
\end{aligned}
$$

and

$$b_{\sigma^k} = \prod_{0\le j<k}({}^{\sigma^k}\varpi_E - {}^{\sigma^j}\varpi_E)^{-1},$$

so right side of (4.3.8) is

$$\prod_{k\le\beta<k_l}({}^{\sigma^k}\varpi_E - {}^{\sigma^{\beta+_\phi 1}}\varpi_E)\prod_{1\le i<f}(\zeta - {}^{\phi^j}\zeta)$$

which equals the left side. Indeed each ${}^{\sigma^k}\varpi_E - {}^{\sigma^{\beta+_\phi 1}}\varpi_E$ are coming from case (4.3.7) and each $\zeta - {}^{\phi^j}\zeta$ are coming from $a_{\sigma^k,\phi^j\sigma^{k_l}}$ for $j = 1,\dots,f-1$.

We have solved both equations in (4.3.4) for particular a-data of the form $a_{g,\sigma^k g}$ and $a_{g,\phi^i g}$, with $g \in \Gamma_F/\Gamma_E$. The situation of other a-data are more complicated, but we do not need to deal with them. We conclude that we have proved Proposition 4.2, i.e. the splitting invariant $\lambda(\{a_\lambda\},T)$ is trivial. Recall that $\Delta_{\mathrm{I}}(\gamma)$ is defined to be the Tate-Nakayama product

$$\langle\lambda(\{a_\lambda\},T), s_T\rangle$$

for certain s_T depending on the datum s of the endoscopic group H. Therefore we assume from now on that Δ_{I} is always trivial.

When E/F is totally ramified, we have $\Gamma_{E/F} = \{1,\sigma,\dots,\sigma^{e-1}\}$. In this case we choose a-data

$$a_{\sigma^i,\sigma^j} = {}^{\sigma^i}\varpi_E - {}^{\sigma^j}\varpi_E. \tag{4.3.9}$$

4.4 Induction as admissible embedding

Let T be a maximal torus of G defined over F. By taking a conjugate of T in G, which is still denoted by T for brevity, we assume that T is contained in a Borel subgroup B defined over F. Choose an W_F-invariant splitting $(\mathcal{T}, \mathcal{B}, \{\hat{X}_\alpha\})$ of $\hat{G}$ and an isomorphism $\iota : \hat{T} \to \mathcal{T}$ which maps the basis determined by $\hat{B}$ to the basis determined by $\mathcal{B}$. For notation convenience we usually omit ι and write $t = \iota(t) \in \mathcal{T}$ for $t \in \hat{T}$, but bear in mind that $\hat{T}$ and $\mathcal{T}$ may have inequivalent W_F-actions.

An *admissible embedding* from ${}^LT$ to ${}^LG$ is a morphism of groups $\chi : {}^LT \to {}^LG$ of the form

$$\chi(t \rtimes w) = t\bar{\chi}(w) \rtimes w,\ t \in \hat{T},\ w \in W_F$$

for some map $\bar{\chi} : W_F \to \hat{G}$. By expanding $\chi(s \rtimes v)\chi(t \rtimes w) = \chi((s \rtimes v)(t \rtimes w))$, we can show that

$$(\mathrm{Int}\bar{\chi}(v))({}^{v_{\hat{G}}}t) = {}^{v_{\hat{T}}}t \text{ and } \bar{\chi}(vw) = \bar{\chi}(v)^{v_{\hat{G}}}\bar{\chi}(w)\ t \in \hat{T},\ v, w \in W_F. \tag{4.4.1}$$

Conversely if $\bar{\chi}$ satisfies (4.4.1), then χ is an admissible embedding. We can show that $\bar{\chi}$ has image in $N_{\hat{G}}(\hat{T})$. More precisely, if $N_{\hat{G}}(\hat{T}) \rtimes W_F$ acts on $\hat{T}$ by ${}^{x \rtimes w}t = \mathrm{Int}(x)({}^{w_{\hat{G}}}t)$, then the morphism $W_F \to \mathrm{Aut}(\hat{T})$, $w \mapsto w_{\hat{T}}$ factors through

$$W_F \to N_{\hat{G}}(\hat{T}) \rtimes W_F,\ w \mapsto \bar{\chi}(w) \rtimes w.$$

Let $\mathcal{H}$ be a subgroup of $\hat{G}$. Two admissible embeddings χ_1, χ_2 are called $\mathrm{Int}(\mathcal{H})$-equivalent if there is $x \in \mathcal{H}$ such that

$$\chi_1(t \rtimes w) = (x \rtimes 1)\chi_2(t \rtimes w)(x \rtimes 1)^{-1} \text{ for all } t \rtimes w \in {}^LT.$$

Using (4.4.1) we can show that this condition is equivalent to require an $x \in \mathcal{H}$ giving

$$\bar{\chi}_1(w) = x\bar{\chi}_2(w)^{w_{\hat{G}}}x^{-1} \text{ for all } w \in W_F.$$

Remark 4.4. We choose splittings on G and $\hat{G}$ so that we have a duality on the bases of T and $\mathcal{T}$ for explicit computations. For example, we choose a basis of $\mathcal{T}$ for the construction of the Steinberg section (see section 2.1 of [22]). Our main results would be independent of these choices. For instance, the $\hat{G}$-conjugacy class of an admissible embedding is independent of the choices of the Borel subgroup B containing T and the splitting $(\mathcal{T}, \mathcal{B}, \{\hat{X}_\alpha\})$ of $\hat{G}$ (see (2.6.1) and (2.6.2) of [22]). $\square$

Langlands and Shelstad constructed certain admissible embeddings using χ-data (see section (2.5) of [22]). We will give the construction in section 4.5. Let's assume such construction for a moment, and denote $\mathrm{AE}({}^LT, {}^LG, (\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B}))$, or just $\mathrm{AE}({}^LT, {}^LG)$ for simplicity, the set of admissible embeddings ${}^LT \to {}^LG$ (associated to the choice of the isomorphism $(\hat{T}, \hat{B}) \to (\mathcal{T}, \mathcal{B})$). The description of this collection is not difficult.

Proposition 4.5. *The set* $\mathrm{AE}({}^LT, {}^LG)$ *is a* $Z^1(W_F, \hat{T})$*-torsor, and the set of its* $\mathrm{Int}(\mathcal{T})$*-equivalence classes* $\mathrm{Int}(\mathcal{T})\backslash\mathrm{AE}({}^LT, {}^LG)$ *is an* $H^1(W_F, \hat{T})$*-torsor.*

Proof. We fix an embedding $\chi_0 \in \mathrm{AE}({}^LT, {}^LG)$ and take $\bar{\chi}_0 : W_F \to \hat{G}$ as in the definition of admissible embedding. Then for each $\chi \in \mathrm{AE}({}^LT, {}^LG)$, the difference $\bar{\chi}\bar{\chi}_0{}^{-1}$ is a 1-cocycle of W_F valued in $\hat{T}$, i.e. $\bar{\chi}\bar{\chi}_0{}^{-1} \in Z^1(W_F, \hat{T})$. Indeed for a fixed $w \in W_F$ both $\bar{\chi}(w)$ and $\bar{\chi}_0(w)$ project to the same element in $\Omega(\hat{G}, \mathcal{T}) = N_{\hat{G}}(\mathcal{T})/\mathcal{T}$. Using (4.4.1) we have that

$$\begin{aligned} \bar{\chi}\bar{\chi}_0{}^{-1}(vw) &= \bar{\chi}(v)^{v_{\hat{G}}}\bar{\chi}(w)^{v_{\hat{G}}}\bar{\chi}_0(w)^{-1}\bar{\chi}_0(v)^{-1} \\ &= \bar{\chi}(v)\bar{\chi}_0(v)^{-1 v_{\hat{T}}}\bar{\chi}(w)^{v_{\hat{T}}}\bar{\chi}_0(w)^{-1} \end{aligned}$$

for all $v, w \in W_F$. We can readily verify that the map

$$\mathrm{AE}({}^LT, {}^LG) \to Z^1(W_F, \hat{T}),\ \chi \mapsto \bar{\chi}\bar{\chi}_0{}^{-1}$$

is bijective. From the equality $t\bar{\chi}(v)^{v_{\hat{G}}}t^{-1}\bar{\chi}_0{}^{-1}(v) = t\bar{\chi}(v)\bar{\chi}_0{}^{-1}(v)^{v_{\hat{T}}}t^{-1}$ for all $t \in \mathcal{T}$, we know that two embeddings are $\mathrm{Int}(\mathcal{T})$-equivalent if and only if the corresponding 1-cocycles differ by a coboundary in $Z^1(W_F, \hat{T})$. $\square$

Remark 4.6. For $G = \mathrm{GL}_n$ we can construct an explicit embedding ${}^LT \to {}^LG$. Choose $\mathcal{T}$ to be the diagonal subgroup of $\hat{G}$. We embed $\hat{T}$ into $\hat{G}$ with image $\mathcal{T}$ and define

$$W_F \to N_{\hat{G}}(\hat{T}),\ w \mapsto N(w)$$

the permutation matrix, i.e. those with entries being either 0 or 1, whose assignment is according to the W_F-action on $\hat{T} \cong \mathbb{C}^{[E/F]}$, i.e. $\mathrm{Int}(N(v))t = {}^{v_{\hat{T}}}t$ for all $t \in \hat{T}$. Clearly the map ${}^LT \to {}^LG$, $t \rtimes w \mapsto tN(w) \times w$ defines an admissible embedding. $\square$

Now we take the elliptic torus $T = \mathrm{Res}_{E/F}\mathbb{G}_m$. By Shapiro's Lemma (see the Exercise in VII §5 of [28]), we have a special case of Langlands correspondence for torus

$$\mathrm{Hom}(E^\times, \mathbb{C}^\times) = H^1(W_F, \hat{T}). \tag{4.4.2}$$

The precise correspondence is given as follows. Suppose ξ is a character of $E^\times$. We regard ξ as a character of W_E by class field theory [30]. Take a collection of coset representatives $\{g_1, \ldots, g_n\}$ of W_F/W_E. Define for each g_i a map $u_{g_i} : W_F \to W_E$ given by

$$wg_i = g(w, g_i)u_{g_i}(w) \text{ for } g(g_i, w) \in \{g_1, \ldots, g_n\}. \tag{4.4.3}$$

Then define

$$\tilde{\xi} : W_F \to \hat{T} \cong \mathbb{C}^n,\ w \mapsto \left(\xi(u_{g_1}(w)), \ldots, \xi(u_{g_n}(w))\right).$$

It can be checked that $\tilde{\xi}$ is a 1-cocycle in $Z^1(W_F, \hat{T})$, and different choices of coset representatives give cocycles different from $\tilde{\xi}$ by a 1-coboundary. Hence the 1-cohomology class of $\tilde{\xi}$ is defined. By abusing of language, we call $\tilde{\xi}$ a *Langlands parameter* of ξ. Moreover, combining Proposition 4.5 and (4.4.2) we have

$$\mathrm{Hom}(E^\times, \mathbb{C}^\times) = \mathrm{Int}(\mathcal{T})\backslash \mathrm{AE}({}^LT, {}^LG). \tag{4.4.4}$$

Explicitly, if we have a character μ of $E^\times$, then define

$$\chi : {}^LT \to {}^LG,\ t \rtimes w \mapsto t \begin{pmatrix} \mu(u_{g_1}(w)) & & \\ & \ddots & \\ & & \mu(u_{g_n}(w)) \end{pmatrix} N(w) \times w.$$

Here $N(w)$ is the permutation matrix as introduced in Remark 4.6. Notice that this bijection is non-canonical. Write $\mathrm{proj} : {}^LG \to \hat{G}$, $g \times w \mapsto g$, which is a morphism of groups because $G = \mathrm{GL}_n$ splits over F. Combining the bijections (4.4.2) and (4.4.4), we have the following result.

Proposition 4.7. *Suppose that ξ and μ come from $\tilde{\xi}$ and χ by the bijections (4.4.2) and (4.4.4). The composition*

$$H^1(W_F, \hat{T}) \times \mathrm{Int}(\mathcal{T})\backslash \mathrm{AE}({}^LT, {}^LG) \to \mathrm{Int}(\hat{G})\backslash \mathrm{Hom}_{W_F}(W_F, {}^LG)$$

such that $(\tilde{\xi}, \chi) \mapsto \chi \circ \tilde{\xi}$ gives an isomorphism $\mathrm{proj} \circ \chi \circ \tilde{\xi} \cong \mathrm{Ind}_{E/F}(\xi\mu)$ as representations of W_F.

Proof. Choose a suitable basis on the representation space of $\mathrm{Ind}_{E/F}(\xi\mu)$. For example, if we realize our induced representation by the subspace of functions

$$\{f : W_F \to \mathbb{C} | f(xg) = \xi\mu(x)f(g) \text{ for all } x \in W_E, g \in W_F\},$$

then we choose those f_i determined by $f_i(g_j) = \delta_{ij}$ (Kronecker delta) as basis vectors. The matrix of $\mathrm{Ind}_{E/F}(\xi\mu)$ is therefore

$$\begin{pmatrix} \mu\xi(u_{g_1}(w)) & & \\ & \ddots & \\ & & \mu\xi(u_{g_n}(w)) \end{pmatrix} N(w)$$

the same matrix as the image of $\chi \circ \tilde{\xi}$. □

Remark 4.8. We can recover ξ from $\mathrm{Ind}_{E/F}\xi$ as follows. We choose the first k coset representatives $g_1 = 1, g_2 \ldots, g_k$ to be those in the normalizer $N_{W_F}(W_E) = \mathrm{Aut}_F(E)$, and (by choosing suitable basis) consider the matrix coefficients of $\mathrm{Res}_{E/F}\mathrm{Ind}_{E/F}\xi$. The first k diagonal entries are always non-zero and give the characters ξ^{g_i}. □

4.5 Langlands-Shelstad χ-data

In this section we recall the construction of admissible embeddings ${}^LT \to {}^LG$ given in chapter 2 of [22]. The construction applies for general connected reductive algebraic group G defined and quasi-split over F. We take a maximal torus T in G also defined over F. Take a maximal torus $\mathcal{T}$ of $\hat{G}$ and choose a splitting $(\mathcal{T}, \mathcal{B}, \{\hat{X}_\alpha\})$ for $\hat{G}$. We emphasis again that different choices yield $\mathrm{Int}(\hat{G})$-equivalent embeddings. For computational convenience we choose $\mathcal{T}$ to be the diagonal group and $\mathcal{B}$ to be the group of upper triangular matrices. The tori $\hat{T}$ and $\mathcal{T}$ are isomorphic as groups but with different W_F-actions.

Recall that the existence of an admissible embedding $\chi : {}^LT \to {}^LG$ with restriction $\hat{T} \to \mathcal{T}$ is equivalent to the existence of a 1-cocycle $\bar{\chi} \in Z^1(W_F, N_{\hat{G}}(\mathcal{T}))$ as in (4.4.1). We construct one directly as follows. Recall the Springer section in section **??**

$$n : W_F \to N_{\hat{G}}(\mathcal{T}) \rtimes W_F,\ w \mapsto n(w) = \bar{n}(w) \rtimes w.$$

It is not necessarily a morphism of groups, yet $\bar{n}$ satisfies the first equation in (4.4.1) in place of $\bar{\chi}$. We write

$$t_{\mathrm{b}}(v, w) = n(v)n(w)n(vw)^{-1} = \bar{n}(v)^{v_{\hat{G}}}\bar{n}(w)\bar{n}(vw)^{-1}, \tag{4.5.1}$$

a 2-cocycle of W_F, whose values are in $\{\pm 1\}^n \subseteq \mathcal{T}$ by Lemma 2.1.A of [22]. Hence the problem of seeking such $\bar{\chi}$ is equivalent to looking for a map $r_{\mathrm{b}} : W_F \to \hat{T}$ that splits t_{b}^{-1}, i.e.

$$r_{\mathrm{b}}(v)^{v_{\hat{T}}} r_{\mathrm{b}}(w) r_{\mathrm{b}}(vw)^{-1} = t_{\mathrm{b}}(v, w)^{-1}. \tag{4.5.2}$$

The idea in [22] to construct such splitting r_{b} is to choose a set of characters

$$\{\chi_\lambda : E_\lambda^\times \to \mathbb{C}^\times | \lambda \in \Phi\},$$

called χ-*data*, such that the following conditions hold.

(i) For each $\lambda \in \Phi$, we have $\chi_{-\lambda} = \chi_\lambda^{-1}$ and $\chi_{w\lambda} = \chi_\lambda^{w^{-1}}$ for all $w \in W_F$.

(ii) If λ is symmetric, then $\chi|_{E_{\pm\lambda}^\times}$ equals the quadratic character $\delta_{E_\lambda/E_{\pm\lambda}}$ attached to the extension $E_\lambda/E_{\pm\lambda}$.

We can choose $\mathcal{R} = \mathcal{R}_{\mathrm{sym}} \sqcup \mathcal{R}_{\mathrm{asym}\pm}$ to be a subset of Φ consisting of representatives of $W_F \backslash \Phi_{sym}$ and $W_F \backslash \Phi_{asym/\pm}$. Then by condition (i) the set of χ-data depends completely on the subset

$$\{\chi_g | g \in \mathcal{R} = \mathcal{R}_{\text{sym}} \sqcup \mathcal{R}_{\text{asym}\pm}\}. \tag{4.5.3}$$

For a chosen χ-data $\{\chi_\lambda\}_{\lambda\in\mathcal{R}}$, following section (2.5) of [22] we define for each $\lambda \in \mathcal{R}$ a map

$$r_\lambda : W_F \to \mathcal{T}, \quad w \mapsto \prod_{g_i \in W_F/W_{\pm\lambda}} \chi_\lambda(v_1(u_{g_i}(w)))^{g_i\lambda}, \tag{4.5.4}$$

where u_{g_i} is the map (4.4.3) for $W_F/W_{\pm\lambda}$ and v_1 is defined similarly for $W_{\pm\lambda}/W_\lambda$. We then define

$$r_{\text{g}} = \prod_{\lambda\in\mathcal{R}} r_\lambda. \tag{4.5.5}$$

Such construction yields (Lemma 2.5.A of [22]) a 2-cocycle

$$t_{\text{g}}(v, w) = r_{\text{g}}(v)^{v_{\hat{T}}} r_{\text{g}}(w) r_{\text{g}}(vw)^{-1} \in Z^2(W_F, \{\pm 1\}^n). \tag{4.5.6}$$

In constructing the 2-cocycles (4.5.1) and (4.5.6) we implicitly used two different notions of gauges (defined right before Lemma 2.1.B of [22]) on the root system Φ. To relate them we introduce a map (see section (2.4) of [22]) $s = s_{\text{b/g}} : W_F \to \{\pm 1\}^n$ such that

$$s(v)^{v_{\hat{T}}} s(w) s(vw)^{-1} = t_{\text{b}}(v, w) t_{\text{g}}(v, w)^{-1}. \tag{4.5.7}$$

Write $r_{\text{b}} = s_{\text{b/g}} r_{\text{g}}$ and $\bar{\chi} = r_{\text{b}} \bar{n}$.

Proposition 4.9. *The map χ defines an admissible embedding ${}^LT \to {}^LG$.*

Proof. It suffices to show that $\bar{\chi}$ satisfies the two conditions in (4.4.1). The first condition is just from the definition of $n(w)$, while the second condition is a straightforward calculation using (4.5.1), (4.5.2), (4.5.6), and (4.5.7). □

4.6 Explicit Δ_{III_2}

Let $\chi : {}^LT \to {}^LG$ be the admissible embedding defined by some chosen χ-data $\{\chi_g\}$ (notation as in (4.5.3)). Let ξ be a character of $E^\times$ and $\tilde{\xi} \in Z^1(W_F, \hat{T})$ be a Langlands parameter of ξ. (The choice of $\tilde{\xi}$ is known to be irrelevant.) In Proposition 4.7 we described an induced representation $\mathrm{Ind}_{E/F}\xi$ as certain embedding of the image of $\tilde{\xi}$ into $\mathrm{GL}_n(\mathbb{C})$. Here we have the reverse.

Theorem 4.10. *Given χ-data $\{\chi_g\}$, define*

$$\mu = \mu_{\{\chi_g\}} = \prod_{[g]\in(W_E\backslash W_F/W_E)'} \mathrm{Res}^{E_g^\times}_{E^\times}\chi_g.$$

Let χ be the admissible embedding defined by $\{\chi_g\}$. Then for all character ξ of $E^\times$, the composition

$$W_F \xrightarrow{\tilde{\xi}} {}^LT \xrightarrow{\chi} {}^LG \xrightarrow{\text{proj}} \mathrm{GL}_n(\mathbb{C})$$

is isomorphic to $\mathrm{Ind}_{E/F}(\xi\mu)$ as representations of W_F.

Remark 4.11. Notice that the product in Proposition 4.10 is uniquely determined by $\{\chi_\lambda\}$, i.e. independent of the representative g of $[g] \in (W_E\backslash W_F/W_E)'$ which is itself a coset representative of W_F/W_E. Indeed if ${}^x\left[\begin{smallmatrix}1\\g\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\h\end{smallmatrix}\right]$ for some $x \in W_E$, then ${}^xE_g = E_h$ and so $\mathrm{Res}^{E_h^\times}_{E^\times}\chi_h = \mathrm{Res}^{{}^xE_g^\times}_{E^\times}\chi_g^{x^{-1}} = \mathrm{Res}^{E_g^\times}_{E^\times}\chi_g$ by (i) of the definition of χ-data. □

Remark 4.12. Suppose we have fixed a character ξ of $E^\times$. Take a subset $\{g_1 = 1, g_2 \dots, g_k\}$ of coset representatives of W_F/W_E in the normalizer $N_{W_F}(W_E) = \mathrm{Aut}_F(E)$ and write $\mu_1 = \mu_{\{\chi_g\}}$ as in Proposition 4.10. Then all other characters μ_j such that $\mathrm{Ind}_{E/F}(\xi\mu_j) \cong \mathrm{Ind}_{E/F}(\xi\mu_1)$ are of the form $\mu_j = \xi^{g_j-1}\mu_1$, $j = 1, \dots, k$. This character μ_j also has a factorization in Proposition 4.10 with the same χ-data of μ, except when $g = g_j$ the character χ_g is changed according to the following.

(i) If g is symmetric, then χ_g is replaced by $\xi^{g-1}\chi_g$.

(ii) If g is asymmetric, then χ_g is replaced by $\xi^g\chi_g$ and so $\chi_{g^{-1}}$ by $\xi^{-1}\chi_{g^{-1}}$.

□

Proof. (of Proposition 4.10) We first abbreviate $H = W_F$ and $K = W_E$. For each $\lambda = \left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]$ we denote $K_g = K \cap gKg^{-1}$, which equals W_λ. If $[g] \in (K\backslash H/K)_{\mathrm{sym}}$, then because $KgK = Kg^{-1}K$ we can replace g by an element in Kg such that $g^2 \in K$. Subsequently we have $g \in W_{\pm\lambda}$ and $g^2 \in K_g = W_\lambda$. We denote $K_{\pm g}$ the group generated by $K \cap gKg^{-1}$ and g, which equals $W_{\pm\lambda}$.

By (i) of the definition of χ-data, we rewrite the product in Proposition 4.10 as

$$\prod_{[g]\in(K\backslash H/K)_{asym/\pm}} (\mathrm{Res}^{E_g^\times}_{E^\times}\chi_g)(\mathrm{Res}^{E_{g^{-1}}^\times}_{E^\times}\chi_g^g)^{-1} \prod_{[g]\in(K\backslash H/K)_{sym}} (\mathrm{Res}^{E_g^\times}_{E^\times}\chi_g). \tag{4.6.1}$$

Recall that our dual group $\mathcal{T}$ is the diagonal subgroup. In order to check that χ gives rise to a character μ as (4.6.1) it is enough to consider the first entry of r_{g} (see (4.5.5) and the discussion in Remark 4.8). From (4.5.4) we have

$$r_{\mathrm{g}}(w) = \left(\prod_{[g]\in(K\backslash H/K)_{asym/\pm}} \prod_{g_i\in K_g\backslash H} \chi_g(u_{g_i}(w))^{\left[\substack{g_i\\ g_ig}\right]}\right) \left(\prod_{[g]\in(K\backslash H/K)_{sym}} \prod_{g_i\in K_{\pm g}\backslash H} \chi_g(v_1u_{g_i}(w))^{\left[\substack{g_i\\ g_ig}\right]}\right).$$

By restricting $w \in W_E$, we get the first entry of $r_{\mathrm{g}}(w)$, namely

$$r_{\mathrm{g}}(w)_1 = \left(\prod_{g\in(K\backslash H/K)_{asym/\pm}} \left(\prod_{g_i\in K/K_g} \chi_g(u_{g_i}(w))\right)\left(\prod_{\substack{g_i\in H/K_g\\ g_ig\in K}} \chi_g(u_{g_i}(w))^{-1}\right)\right) \left(\prod_{g\in(K\backslash H/K)_{sym}} \left(\prod_{g_i\in K/K_{\pm g}} \chi_g(v_1u_{g_i}(w))\right)\left(\prod_{\substack{g_i\in H/K_{\pm g}\\ g_ig\in K}} \chi_g(v_1u_{g_i}(w))^{-1}\right)\right). \tag{4.6.2}$$

We now analysis the products in (4.6.2) and match them to those in (4.6.1). First, for $g \in (K\backslash H/K)_{asym/\pm}$, the first product of (4.6.2)

$$\prod_{g_i\in K/K_g} u_{g_i}(w),\ w \in K$$

is the transfer map $T^K_{K_g} : K^{\mathrm{ab}} \to (K_g)^{\mathrm{ab}}$. By class field theory (see [30]), it corresponds to the inclusion $E^\times \hookrightarrow E_g^\times$. Therefore

$$\prod_{g_i\in K/K_g} \chi_g(u_{g_i}(w)) = \mathrm{Res}^{E_g^\times}_{E^\times}\chi_g(w),$$

which is the first factor in (4.6.1). Next we consider the inverse of the second product of (4.6.2)

$$\prod_{\substack{g_i \in H/K_g \\ g_i g \in K}} u_{g_i}(w),\ w \in K$$

For $g_i \in H/K_g$ such that $g_i g \in K$, we can write $g_i = x_i g$ for some x_i running through a set in K of representatives of $K/K_{g^{-1}}$. If u_{x_i} is the map (4.4.3) for $K/K_{g^{-1}}$ then we have

$$w x_i g = x_{j(w,x_i)} g u_{x_i}(w), \tag{4.6.3}$$

where $u_{x_i(w)} \in K_{g^{-1}}$. On the other hand, by regarding $x_i g \in H/K_g$ we have

$$w(x_i g) = g_{j(w,x_i g)} u_{x_i g}(w) \tag{4.6.4}$$

where $u_{x_i g}(w) \in K_g$ and $g_{j(w,x_i g)}$ is of the form $x_j g$ for some j. By comparing (4.6.3) and (4.6.4) we have $u_{x_i g}(w) = g u_{x_i}(w) g^{-1}$. Therefore

$$\prod_i u_{g_i}(w) = g\left(\prod_i u_{x_i}(w)\right) g^{-1} = g T^K_{K_{g^{-1}}}(w) g^{-1}$$

and hence

$$\prod_{\substack{g_i \in H/K_g \\ g_i g \in K}} \chi_g(u_{g_i}(w)) = \chi_g^g(T^K_{K \cap gKg^{-1}}(w)) = (\mathrm{Res}^{E^\times_{g^{-1}}}_{E^\times} \chi_g^g)(w)$$

which is the inverse of the second factor in (4.6.1). Finally, for $g \in (K\backslash H/K)_{sym}$, we choose coset representatives $g_1, \ldots, g_k,\ g_1 g, \ldots, g_k g$ for H/K_g such that $g_1, \ldots, g_k$ are those of $H/K_{\pm g}$. Moreover we can assume that

$$g_1, \ldots, g_h, g_{h+1} g, \ldots, g_{2h} g \in K.$$

Hence the third product in (4.6.2) is

$$\prod_{g_i \in K/K_{\pm g}} \chi_g(v_1(u_{g_i}(w))) = \prod_{i=1}^{h} \chi_g(v_1(u_{g_i}(w))). \tag{4.6.5}$$

Here u_{g_i} is the map (4.4.3) for $H/K_{\pm g}$ and so $v_1 u_{g_i}$ is the one for H/K_g. For the fourth product in (4.6.2), because $\chi_g^g = \chi_g^{-1}$ (by (ii) of the definition of χ-data) and

$g^{-1}(v_1(u_{g_i}(w)))g = v_1(u_{g_ig}(w))$, we have indeed

$$\prod_{\substack{g_i \in H/K_{\pm g} \\ g_ig \in K}} \chi_g(v_1 u_{g_i}(w))^{-1} = \prod_{i=h+1}^{2h} \chi_g(v_1(u_{g_ig}(w))). \tag{4.6.6}$$

Therefore the product of (4.6.5) and (4.6.6) is $\chi_g(T^K_{K_g}(w)) = (\mathrm{Res}^{E_g^\times}_{E^\times}\chi_g)(w)$ which is the last factor of (4.6.1). □

We have similar result for H as follows. We regard $H = \mathrm{GL}_m$ as an reductive group over K. Let $\tilde{\xi}_K \in Z^1(W_K, \hat{T})$ be a Langlands parameter of the character ξ of $E^\times$. Take the sub-collection

$$\{\chi_g | [g] \in (W_E \backslash W_K / W_E)'\}$$

of χ-data and define the admissible embedding

$$(\chi_H)_K : {}^LT_K = \hat{T} \rtimes W_K \to {}^LH_K = \hat{H} \times W_K.$$

Then similar to Proposition 4.10,

$$W_K \xrightarrow{\tilde{\xi}_K} {}^LT_K \xrightarrow{(\chi_H)_K} {}^LH_K \to \mathrm{GL}_m(\mathbb{C}) \tag{4.6.7}$$

is isomorphic to $\mathrm{Ind}_{E/K}(\xi\mu_H)$ as representations of W_K, where

$$\mu_H = \prod_{[g] \in (W_E \backslash W_K / W_E)'} \mathrm{Res}^{E_g^\times}_{E^\times}\chi_g.$$

We also write μ_G to be the character μ in Proposition 4.10.

Corollary 4.13. *For all $\gamma \in E^\times$ regular, we have*

$$\Delta_{\mathrm{III}_2}(\gamma) = \mu_G(\gamma)^{-1}\mu_H(\gamma) = \prod_{[g] \in W_E \backslash W_F / W_E - W_E \backslash W_K / W_E} \left(\mathrm{Res}^{E_g^\times}_{E^\times}\chi_g\right)(\gamma).$$

Proof. We regard both G and H as reductive groups over F. Let $\chi_G : {}^LT \to {}^LG$ be the admissible embedding defined by $\{\chi_\lambda | \lambda \in \Phi(G,T)\}$ and $\chi_H : {}^LT_F \to {}^LH_F$ be the one defined by $\{\chi_\lambda | \lambda \in \Phi(H,T)/F\}$. Recall by definition in (3.5) of [22] that $\Delta_{\mathrm{III}_2}(\gamma) = \langle \mathbf{a}, \gamma \rangle$ where $\mathbf{a}$ is the class in $H^1(W_F, \hat{T})$ defined by the cocycle a satisfying

$$\hat{\xi} \circ \chi_H = a\chi_G.$$

We make use of the bijection (4.4.4) defined by torsor. Explicitly we consider the commutative diagram

$$\begin{array}{ccccccc}
W_K & \xrightarrow{\tilde{\xi}_K} & {}^LT_K & \xrightarrow{(\chi_H)_K} & {}^LH_K & \longrightarrow & \mathrm{GL}_m(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow & & \\
W_F & \xrightarrow{\tilde{\xi}} & {}^LT_F & \xrightarrow{\chi_H} & {}^LH_F & \xrightarrow{\hat{\xi}} & \hat{G} \times W_F \to \mathrm{GL}_n(\mathbb{C})
\end{array}$$

such that

(i) the upper row is (4.6.7);

(ii) the vertical maps are natural inclusions;

(iii) the morphism $\hat{\xi}$ is defined as an endoscopic data for H.

Following the diagram, we see that the lower row is isomorphic to $\mathrm{Ind}_{E/F}(\xi\mu_H)$ as representations of W_F. Comparing this to the representation $\mathrm{Ind}_{E/F}(\xi\mu_G)$ given in Proposition 4.10, we then proved the assertion immediately. □

We can change the notation of Δ_{III_2} in Corollary 4.13 in terms of roots

$$\Delta_{\mathrm{III}_2}(\gamma) = \left(\prod_{\lambda \in \mathcal{R}(G)_{\mathrm{asym}/\pm} - \Phi(H)} \chi_\lambda(\lambda(\gamma)) \prod_{\lambda \in \mathcal{R}(G)_{\mathrm{sym}} - \Phi(H)} \chi_\lambda|_{E^\times}(\gamma) \right)^{-1}, \gamma \in E^\times.$$

In [22], the transfer factor Δ_{II} is defined as

$$\Delta_{\mathrm{II}}(\gamma) = \prod_{\lambda \in \mathcal{R}(G)_{\mathrm{asym}/\pm} - \Phi(H)} \chi_\lambda(\lambda(\gamma)) \prod_{\lambda \in \mathcal{R}(G)_{\mathrm{sym}} - \Phi(H)} \chi_\lambda \left(\frac{\lambda(\gamma) - 1}{a_\lambda} \right), \gamma \in E^\times \text{ regular}.$$

Therefore

$$\Delta_{\mathrm{III}_2}(\gamma)\Delta_{\mathrm{II}}(\gamma) = \prod_{\lambda \in \mathcal{R}(G)_{\mathrm{sym}} - \Phi(H)} \chi_\lambda \left(\frac{\lambda(\gamma) - 1}{a_\lambda} \right) = \prod_{g \in \mathcal{D}(F)_{\mathrm{sym}} - \mathcal{D}(K)} \chi_g \left(\frac{\gamma - {}^g\gamma}{a_{1,g}} \right). \quad (4.6.8)$$

For $g \in W_F$ symmetric we can assume that $g^2 \in W_E$ and ${}^ga_{1,g} = -a_{1,g}$, we have $(\gamma - {}^g\gamma)/a_{1,g} \in E^\times_{\pm g}$. Therefore each $\chi_g((\gamma - {}^g\gamma)/a_g)$ and also $\Delta_{\mathrm{I,II,III}}(\gamma) = \Delta_{\mathrm{III}_2}\Delta_{\mathrm{II}}(\gamma)$ is a sign.

Proposition 4.14. *Suppose E/F is totally ramified. We choose the embedding $\iota : E^\times \hookrightarrow G(F)$ defined by (4.3.2), a-data defined by (4.3.9) and arbitrary χ-data. Then $\Delta_{\mathrm{I,II,III}}(\varpi_E) = 1$.*

Proof. This follows directly from (4.6.8) when $\gamma = \varpi_E$. □

Recall we have computed that $\Delta_{\mathrm{III}_1}(\gamma)$ and $\Delta_{\mathrm{I}}(\gamma)$ are all 1. Therefore we have $\Delta_{\mathrm{I,II,III}}(\gamma) = \Delta_{\mathrm{II,III}_2}(\gamma)$ which is also a sign.

Proposition 4.15. *$\Delta_{\mathrm{I,II,III}}(\gamma)$ is independent of the choices of admissible embedding $T_H \to T_G$, a-data and χ-data.*

Proof. We only sketch the reasons and refer to chapter 3 of [22] for details. We recall the definition of various transfer factors and check that

(i) only Δ_{III_1} and Δ_{I} depend on the admissible embedding $T_H \to T_G$

(ii) only Δ_{I} and Δ_{II} depend on a-data, and

(iii) only Δ_{II} and Δ_{III_2} depend on χ-data.

The effects of the choices cancel when we multiply the various factors together. □

The product $\Delta_{\mathrm{I,II,III}}$ still depends on the chosen F-splitting $\mathbf{spl}_G$ as shown in the factor Δ_I. We refer to Lemma 3.2.A of [22] for the detail of such dependence.

4.7 A restriction property of Δ_{III_2}

The product $\mu = \mu_{\{\chi_g\}}$ in Proposition 4.10, as $\{\chi_g\}$ runs through all χ-data, does not produce arbitrary character of $E^\times$. Its restriction on $F^\times$ has a specific form by Proposition 4.18. First recall the following known results. Given a group H we write 1_H the trivial representation of H. If K is a subgroup of H of finite index, denote $T_K^H : H^{\mathrm{ab}} \to K^{\mathrm{ab}}$ the transfer morphism. For any $g \in H$, we write ${}^gK = gKg^{-1}$.

Proposition 4.16. *Let σ and π be finite dimensional representations of K and H respectively. We have the following formulae.*

(i) (Mackey's Formula)

$$\mathrm{Res}_K^H \mathrm{Ind}_K^H \sigma \cong \bigoplus_{[g]\in K\backslash H/K} \mathrm{Ind}_{K\cap {}^gK}^K \mathrm{Res}_{K\cap {}^gK}^{{}^gK}({}^g\sigma).$$

(ii) $\det \mathrm{Ind}_K^H \sigma \cong (\det \mathrm{Ind}_K^H 1_K)^{\dim \sigma} \otimes (\det \sigma \circ T_K^H)$.

(iii) $(\det \mathrm{Res}_K^H \pi) \circ T_K^H = (\det \pi)^{|H/K|}$.

Proof. Formulae (i) and (ii) are well-known, for example (i) is proved in [27] 7.3, and (ii) can be found in the Exercise in [28] VII §8. Formula (iii) is direct from (ii) if we take $\sigma = \mathrm{Res}_K^H \pi$. □

In particular, if χ is a character of K, then by (ii) we have

$$\chi \circ T_K^H \cong \left(\det \mathrm{Ind}_K^H \chi\right) \left(\det \mathrm{Ind}_K^H 1_K\right). \tag{4.7.1}$$

Lemma 4.17. *We have the formula*

$$\det \mathrm{Ind}_K^H 1_K = \prod_{[g]\in (K\backslash H/K)'} \det \mathrm{Ind}_{K_g}^H 1_{K_g}.$$

Proof. Applying Mackey's formula on $\sigma = 1_K$ we obtain

$$\mathrm{Res}_K^H \mathrm{Ind}_K^H 1_K \cong \bigoplus_{[g]\in K\backslash H/K} \mathrm{Ind}_{K_g}^K 1_{K_g}.$$

We take determinant and then transfer morphism T_K^H on both sides. By (ii) and (iii) of Proposition 4.16 we obtain

$$\left(\det \mathrm{Ind}_K^H 1_K\right)^{|H/K|} = \prod_{[g]\in K\backslash H/K} \left(\det \mathrm{Ind}_{K_g}^H 1_{K_g}\right) \left(\det \mathrm{Ind}_K^H 1_K\right)^{|K/K_g|}.$$

Because the sum of $|K/K_g|$ for $[g]$ runs through $K\backslash H/K$ is $|H/K|$, the factor $\left(\det \mathrm{Ind}_K^H 1_K\right)$ on both sides vanish. What remains gives the desired formula. □

Notice that $\det \operatorname{Ind}_{K_g}^H 1_{K_g}$ is independent of the choice of representative g of the double coset $[g]$ if we interpret the character as the sign of the canonical H-action on H/K_g, For all representatives of $[g]$, the corresponding actions are equivalent each other. We write $\delta_{H/K}$ for the character $\det \operatorname{Ind}_K^H 1_K$. If $H = W_F$ and $K = W_E$ for some field extension E/F, then we write $\delta_{E/F}$ as δ_{W_F/W_E}. We can easily check the formula

$$\delta_{E/F} = \delta_{E/K}|_{F^\times} \cdot \delta_{K/F}^{|E/K|}.$$

Proposition 4.18. *For all χ-data $\{\chi_g\}$, if μ is the character of $E^\times$ defined by $\{\chi_g\}$ as in Proposition 4.10, then $\mu|_{F^\times} = \delta_{E/F}$.*

Proof. We first abbreviate $H = W_F$, $K = W_E$. For each $\lambda = \left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]$ we denote $K_g = K \cap {}^gK = W_{+\lambda}$ and $K_{\pm g} = W_{\pm\lambda}$. The isomorphism in Proposition 4.18 can be rewritten as

$$\prod_{[g]\in(K\backslash H/K)'} \chi_g \circ T_{K_g}^H = \delta_{H/K}.$$

By Lemma 4.17 we have to show that

$$\prod_{[g]\in(K\backslash H/K)'} \chi_g \circ T_{K_g}^H = \prod_{[g]\in(K\backslash H/K)'} \delta_{H/K_g} \tag{4.7.2}$$

By comparing (4.7.2) termwise, we claim that

(i) If $[g] \in (K\backslash H/K)_{asym/\pm}$, then

$$\left(\chi_g \circ T_{K_g}^H\right)\left(\chi_{g^{-1}} \circ T_{K_{g^{-1}}}^H\right) = \delta_{H/K_g}\delta_{H/K_{g^{-1}}} = 1.$$

(ii) If $[g] \in (K\backslash H/K)_{sym}$, then $\chi_g \circ T_{K_g}^H = \delta_{H/K_g}$.

If $[g] \in (K\backslash H/K)_{asym/\pm}$, then we have $K_{g^{-1}} = {}^gK_g$, which is the stabilizer of the root $\left[\begin{smallmatrix}1\\g^{-1}\end{smallmatrix}\right]$. Because $\chi_{g^{-1}} = ({}^g\chi_g)^{-1}$ by (i) of the definition of χ-data, we have

$$\left(\chi_g \circ T_{K_g}^H\right)\left(\chi_{g^{-1}} \circ T_{K_{g^{-1}}}^H\right) = \left(\chi_g \circ T_{K_g}^H\right)\left({}^g\chi_g^{-1} \circ T_{{}^gK_g}^H\right) \equiv 1.$$

On the other hand, since the H-action on ${}^H\left[\begin{smallmatrix} & g \\ 1 & \end{smallmatrix}\right] = {}^H\left[\begin{smallmatrix} & 1 \\ g^{-1} & \end{smallmatrix}\right]$ is equivalent to that on ${}^H\left[\begin{smallmatrix} & 1 \\ g & \end{smallmatrix}\right]$, we have $\mathrm{Ind}_{K_g}^H 1_{K_g} \cong \mathrm{Ind}_{K_{g^{-1}}}^H 1_{K_{g^{-1}}}$. Therefore $\delta_{H/K_g}\delta_{H/K_{g^{-1}}} = 1$. We have proved the first claim.

If $[g] \in (K\backslash H/K)_{sym}$, then we have an isomorphism $\mathrm{Ind}_{K_g}^{K_{\pm g}} 1_{K_g} \cong 1_{K_{\pm g}} \oplus \delta_{K_{\pm g}/K_g}$ as representations of $K_{\pm g}$. Here $\delta_{K_{\pm g}/K_g}$ is the quadratic character of $K_{\pm g}/K_g$. We denote this character by δ. Hence $\mathrm{Ind}_{K_g}^H 1_{K_g} \cong \mathrm{Ind}_{K_{\pm g}}^H 1_{K_{\pm g}} \oplus \mathrm{Ind}_{K_{\pm g}}^H \delta$ and

$$\delta_{H/K_g} \cong \delta_{H/K_{\pm g}} \cdot \det \mathrm{Ind}_{K_{\pm g}}^H \delta \tag{4.7.3}$$

by taking determinant. Now (ii) of the definition of χ-data, namely $\chi_g \circ T_{K_g}^{K_{\pm g}} = \delta$, gives $\chi_g \circ T_{K_g}^H = \delta \circ T_{K_{\pm g}}^H$. By (4.7.1), this is just the right side of (4.7.3). We have proved the second claim and therefore Proposition 4.18. □

Strictly speaking, Δ_{III_2} is not defined on $F^\times$. However if we use its definition $\langle \mathbf{a}, \cdot \rangle$ in the proof of Corollary 4.13 and identify this character with $\mu_G^{-1}\mu_H$, then by abusing notation we have

$$\Delta_{\mathrm{III}_2}|_{F^\times} = \delta_{K/F}^{|E/K|}$$

as a consequence of Proposition 4.18.

4.8 Relations of transfer factors

Recall that to establish the automorphic induction for $G = \mathrm{GL}_n$ and $H = \mathrm{Res}_{K/F}\mathrm{GL}_m$ we have to define transfer factors in the sense of [12]. It is of the form

$$\Delta_G^H(\gamma) = \Delta^1(\gamma)\Delta^2(\gamma) \text{ for } \gamma \text{ regular.}$$

By the definition of Δ^1 in (3.2) of [12] and that of Δ_{IV} in (3.6) of [22], they are both equal to the quotient of the Weyl-determinants

$$\left(|D_G(\gamma)|_F |D_H(\gamma)|_E^{-1}\right)^{1/2}.$$

To define Δ^2, we have to fix a transfer system $(\sigma, \chi_{K/F}, e)$ such that

(i) σ is a generator of $\Gamma_{K/F}$;

(ii) $\chi_{K/F}$ is a character of $F^\times$ with kernel $N_{K/F}(K^\times)$;

(iii) $e \in K^\times$ such that ${}^\sigma e = (-1)^{m(d-1)}e$.

There are certain choices of e from 3.2 Lemma of [12].

(i) if $m(d-1)$ is even or if $\mathrm{char}(F) = 2$, then we can choose $e = 1$, or

(ii) if E/F is unramified, then we can choose $e \in U_E$.

However, these choices are by no means canonical. We define Δ^2 to be

$$\Delta^2(\gamma) = \chi_{K/F}\left(e\left(\prod_{0\le i<j\le d-1}\prod_{k,\ell=1}^{m}\left(({}^{\sigma^i}\gamma)_k - ({}^{\sigma^j}\gamma)_\ell\right)\right)\right) \text{ for } \gamma \text{ regular,}$$

where $({}^{\sigma^i}\gamma)_k$, for $k = 1, \dots, m$, are the m distinct eigenvalues of ${}^{\sigma^i}\gamma$.

Proposition 4.19. *The product $\Delta_{\mathrm{II,III}}$ equals Δ^2 up to a choice of the transfer system $(\sigma, \chi_{K/F}, e)$.*

Proof. We fix σ and $\chi_{K/F}$ in the transfer system. The product within $\chi_{K/F}$ of $\Delta^2(\gamma)$ is just

$$\prod_{\lambda\in\Phi_+(G,T)-\Phi(H,T)}(\lambda(\gamma)-1). \tag{4.8.1}$$

We then check the contribution of each Γ_F-orbits of roots in the product (4.8.1). If $[\lambda]$ is an asymmetric orbit, then up to a sign its contribution is

$$\prod_{\mu\in[\lambda]}(\mu(\gamma)-1).$$

Since the product runs through a Γ_F-orbit, we have

$$\chi_{K/F}\left(\prod_{\mu\in[\lambda]}(\mu(\gamma)-1)\right) = 1.$$

If $[\lambda]$ is symmetric, then only those positive roots in this orbit contribute to the product. Write this subset by $[\lambda]_+$. Therefore by choosing a constant $e_{[\lambda]}$ such that

$$e_{[\lambda]} \prod_{\mu \in [\lambda]_+} (\mu(\gamma) - 1) \in F^\times,$$

the contribution equals

$$\chi_{K/F} \left(e_{[\lambda]} \prod_{\mu \in [\lambda]_+} (\mu(\gamma) - 1) \right) = \pm 1.$$

We can choose $e_{[\lambda]}$ such that this sign equals

$$\chi_\lambda \left(\frac{\lambda(\gamma) - 1}{a_\lambda} \right).$$

Hence by choosing e in the transfer system as the product of $e_{[\lambda]}$ as $[\lambda]$ runs through all symmetric orbits, we have both $\Delta^2(\gamma)$ and $\Delta_{\mathrm{II,III}}(\gamma)$ equal

$$\prod_{[\lambda] \text{ sym}} \chi_\lambda \left(\frac{\lambda(\gamma) - 1}{a_\lambda} \right).$$

□

Chapter 5

An interlude

In this small chapter we briefly look ahead to the end of the article and state the main results in more detail, based on the background we have provided in chapter 2-4.

Recall that in order to describe the essentially tame local Langlands correspondence, we have to make use of the automorphic induction of cyclic extensions (with the help of base change), which is known to be part of the transfer principle. On one hand, the formula of the automorphic induction is given in (3.3.1) as

$$\Theta_\pi^{\kappa,\Psi}(\gamma) = c_\theta \Delta^2(\gamma)\Delta^1(\gamma)^{-1} \sum_{g\in\Gamma_{K/F}} \Theta_\rho^g(\gamma), \text{ for all } \gamma \in H(F) \cap G(F)_{\mathrm{ell}} \tag{5.0.1}$$

Here we replace the notation of the constant $c(\rho, \kappa, \Psi)$ in (3.3.1) by a more precise notation c_θ used in [9]. The notation θ is a choice of simple character of ξ, which is an inflation of the wild part $\xi|_{U_E^1}$ of ξ to a character of a compact subgroup of $G(F)$. This simple character comes from the first step of constructing the supercuspidal π from ξ, which is briefly described in section 7.2.

On the other hand, we can deduce the spectral transfer formula as in (4.1.3)

$$\Theta_\pi^{\kappa}(\gamma) = \epsilon_{\mathrm{L}}(V_{G/H}) \Delta_{\mathrm{I,II,III}}(\gamma)\Delta_{\mathrm{IV}}(\gamma)^{-1} \sum_{g\in\Gamma_{K/F}} \Theta_\rho^g(\gamma), \text{ for all } \gamma \in H(F) \cap G(F)_{\mathrm{ell}} \tag{5.0.2}$$

Here the normalization $\epsilon_{\mathrm{L}}(V_{G/H})$ is the Langlands constant, which depends on the Whittaker datum coming from a chosen F-Borel subgroup. We may choose this Borel from our F-splitting $\mathbf{spl}_G$ as in section 4.3, and call the Whittaker datum the standard one. From section 5.3 of [18] we know that this normalized transfer factor is independent of the chosen splitting $\mathbf{spl}_G$ giving rise to the Whittaker datum.

The first result is to compare these two formulae (5.0.1) and (5.0.2), i.e. we would find out the difference of the two normalizations in terms of a constant. As we have mentioned this constant, denoted by $\kappa(x)$ in the Introduction, is deduced by studying different Whittaker data and depends on the element $x \in G(F)$ which conjugates these data. In section 8.2 we will first figure out the correct Whittaker datum which gives the normalization in (5.0.1), then we will compute the constant $\kappa(x)$ in the subsequent sections of chapter 8.

The theory is that, the product $\kappa(x)c_\theta\Delta^2$ depends only on the standard Whittaker datum, which is coming from the group G and is independent of the representation π. However the factors $\kappa(x)$ and c_θ certainly depend on π. Similar happens on the other side. We have chosen the splitting $\mathbf{spl}_G$, the admissible embedding of the maximal torus and the a-data as in 4.2 and 4.3 in favor of our calculations, for example we can trivialize Δ_{I}. Recall that our second result is to choose suitable χ-data whose corresponding admissible embedding of L-groups yields the Langlands parameter of the specified representation π. Since the transfer factors Δ_{II} and Δ_{III_2} are built by χ-data, they also depend on the representation π.

Most of the idea of our second result is in the Introduction. Suppose that ξ is the admissible character which gives rise to π. With those chosen χ-data $\{\chi_{\lambda,\xi}\}$ in Theorem 9.1 we have the following consequence on the rectifier. Let K be an intermediate subfield between E/F. If we regard an F-admissible character ξ as being admissible over K, then we have

$$ {}_K\mu_\xi = \prod_{\substack{\lambda \in W_F \backslash \Phi \\ \lambda|_K \equiv 1}} \chi_{\lambda,\xi}|_{E^\times}. $$

Notice that the index set $\{\lambda \in W_F\backslash\Phi,\ \lambda|_K \equiv 1\}$ above can be identified with

$$W_K\backslash\Phi_K(\mathrm{GL}_{|E/K|}, \mathrm{Res}_{E/K}\mathbb{G}_m).$$

If we regard $\mathrm{GL}_{|E/K|}$ as an F-group, i.e. take $H = \mathrm{Res}_{K/F}\mathrm{GL}_{|E/K|}$, then H is a twisted endoscopic group of G as we have seen. We can define the transfer factor Δ_{III_2} for (G, H) and obtain the relation

$${}_F\mu_\xi(\gamma)_K\mu_\xi(\gamma)^{-1} = \prod_{\substack{\lambda\in W_F\backslash\Phi \\ \lambda|_K\neq 1}} \chi_{\lambda,\xi}(\gamma) = \Delta_{\mathrm{III}_2}(\gamma)$$

for every $\gamma \in T(F) = E^\times$ regular in $G(F)$.

We can express the explicit values of the normalization constant and the χ-data in terms of t-factors, whose constructions are summarized as follows and whose details are referred to chapter 6 and 7. Denote $\Psi_{E/F} = E^\times/F^\times U_E^1$. For each admissible character ξ there is a finite symplectic $\mathbf{k}_F\Psi_{E/F}$-module $V = V_\xi$ emerges from the bijection Π_n in Proposition 3.4. Equipped with V is an alternating bilinear form h_θ naturally defined by the chosen simple character θ of ξ. We can embed each V into a fixed finite $\mathbf{k}_F\Psi_{E/F}$-module U which is independent of the admissible character ξ. Practically we can think of U as the largest possible V as ξ runs through all admissible characters in $P(E/F)$. The module U admits a complete decomposition (in Theorem 6.5)

$$U = \bigoplus_{\lambda\in W_F\backslash\Phi} U_\lambda$$

called the *residual root space decomposition*, which is analogous to the root space decomposition of a Lie algebra in the absolute case. By restriction the submodule V admits a decomposition

$$V = \bigoplus_{\lambda\in W_F\backslash\Phi} V_\lambda. \tag{5.0.3}$$

We can show that V_λ is either trivial or isomorphic to U_λ. We will prove in Theorem 7.4 that such decomposition on V is orthogonal with respect to h_θ, with symplectic isotypic components of the form

$$\mathbf{V}_\lambda = \begin{cases} V_\lambda \oplus V_{-\lambda}, & \text{if } {}^{W_F}\lambda \neq {}^{W_F}(-\lambda), \text{ i.e. } \lambda \text{ is asymmetric} \\ V_\lambda, & \text{if } {}^{W_F}\lambda = {}^{W_F}(-\lambda), \text{ i.e. } \lambda \text{ is symmetric} \end{cases}.$$

Let W_λ be the complementary module of V_λ in the sense that

$$V_\lambda \oplus W_\lambda = U_\lambda.$$

We remark that we already had a decomposition of V in terms of the jump data of ξ in [9], see (7.2.5) for instance. The decomposition (5.0.3) is a finer one, and is complete in the sense that the isotypic components are all known.

Let μ and ϖ be the images in $\Psi_{E/F}$ of the subgroup of the roots of unity μ_E and the subgroup generated by a chosen prime element ϖ_E respectively. What we are interested are the $\mathbf{k}_F\mu$-module and the $\mathbf{k}_F\varpi$-module structures of these V_λ.

(i) By regarding each symplectic component $\mathbf{V}_\lambda$ as $\mathbf{k}_F\mu$-module and $\mathbf{k}_F\varpi$-module respectively, we have the symplectic signs

$$t^0_\mu(\mathbf{V}_\lambda),\ t^1_\mu(\mathbf{V}_\lambda),\ t^0_\varpi(\mathbf{V}_\lambda), \text{ and } t^1_\varpi(\mathbf{V}_\lambda).$$

Here $t^0_\mu(\mathbf{V}_\lambda)$ and $t^0_\varpi(\mathbf{V}_\lambda)$ are signs ± 1, while $t^1_\mu(\mathbf{V}_\lambda) : \mu \to \{\pm 1\}$ and $t^1_\varpi(\mathbf{V}_\lambda) : \varpi \to \{\pm 1\}$ are quadratic characters. These are all defined by the algorithm in section 3 of [9] and are computed in Proposition 6.10 for each $\lambda \in W_F\backslash\Phi$.

(ii) For each W_λ we attach a Gauss sum $t(W_\lambda)$ with respect to certain quadratic form on W_λ. Usually $t(W_\lambda)$ is a sign ± 1, except in one case it is at most a 4th root of unity. These factors are defined in section 4 of [7] and are computed in section 6.3.

We then assign a collection of tamely ramified characters $\{\chi_{\lambda,\xi}\}_{\lambda \in W_F\backslash\Phi}$ in terms of the t-factors above. The precise values are given in Theorem 9.1. We also need certain t-factors to compute the constant $\kappa(x)$. During the way of computations in chapter 8 and 9 we will make strong use of the jump data of ξ, and run into some technical but funny sign checking.

Chapter 6

Finite symplectic modules

Let $\mathfrak{A}$ be the hereditary order of $\mathrm{End}_F(E)$ corresponding to the $\mathfrak{o}_F$-lattice chain $\{\mathfrak{p}_E^k | k \in \mathbb{Z}\}$ in E and $\mathfrak{P}_{\mathfrak{A}}$ be its Jacobson radical, as introduced in chapter 1 of [3]. Denote $\Psi_{E/F} = E^\times / F^\times U_E^1$. The main result, Theorem 6.5 of section 6.1, is to deduce a $\mathbf{k}_F \Psi_{E/F}$-module structure of the quotient space $U := \mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$ naturally derived from the conjugate action of $E^\times$ on $\mathrm{End}_F(E)$. We would describe the complete decomposition of U into $\mathbf{k}_F \Psi_{E/F}$-submodules, called the residual root space decomposition.

In section 6.2 and 6.3 we attach certain invariants called t-factors on the submodules of U. These invariants arise from certain symplectic structure imposed on V. We will bring the t-factors into play frequently in subsequent chapters.

6.1 The standard module

Again E/F is tamely ramified. We consider the following F-vector spaces with $E^\times$-actions. For all $t \in E^\times$,

$$\begin{aligned} & E \otimes E \text{ with } {}^t(x \otimes y) \mapsto tx \otimes yt^{-1} \text{ for all } x, y \in E, \\ & \mathrm{End}_F(E) \text{ with } ({}^tA)(v) = tA(t^{-1}v) \text{ for all } A \in \mathrm{End}_F(E), \text{ and} \\ & \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} {}^gEE \text{ with } {}^t({}^gx_{[g]}y_{[g]})_{[g]} = ({}^gtt^{-1g}x_{[g]}y_{[g]})_{[g]} \text{ for all } x_{[g]}, y_{[g]} \in E. \end{aligned} \tag{6.1.1}$$

The following fact is well known.

Proposition 6.1. *The F-linear maps*

(i) $E \otimes E \to \mathrm{End}_F(E)$, $x \otimes y \mapsto (v \mapsto \mathrm{tr}_{E/F}(yv)x)$, *and*

(ii) $E \otimes E \to \bigoplus_{[g] \in \Gamma_E \backslash \Gamma_F / \Gamma_E} {}^g EE$, $x \otimes y \mapsto ({}^g xy)_{[g]}$

are isomorphisms of $E^\times$-modules.

Proof. Indeed (i) is isomorphic by the non-degeneracy of the trace form $\mathrm{tr}_{E/F}$, while (ii) is isomorphic by considering all possible F-algebra embeddings $E \otimes E \to \bar{F}$. Notice that the isomorphism (ii) is moreover an F-algebra one. The $E^\times$-invariance of both morphisms are clear. □

We identify $\mathrm{End}_F(E)$ with $\mathfrak{g}(F) = \mathfrak{gl}_n(F)$ by choosing an F-basis of E and its subalgebra E with a Cartan subalgebra $\mathfrak{g}(F)_0$ of $\mathfrak{gl}_n(F)$. Recall that the roots of the elliptic maximal F-torus $T = \mathrm{Res}_{E/F}\mathbb{G}_m$ in the F-reductive group $G = \mathrm{GL}_n$ are of the form $\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$ with $g, h \in \Gamma_F$ such that

$$\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](t) = {}^g t ({}^h t)^{-1} \text{ for all } t \in E^\times = T(F).$$

Proposition 6.2. *The F-Lie algebra $\mathfrak{g}(F) = \mathfrak{gl}_n(F)$ decomposes into $\mathrm{Ad}(E^\times)$-invariant subspaces*

$$\mathfrak{g}(F) = \mathfrak{g}(F)_0 \oplus \bigoplus_{[\lambda] \in \Gamma_F \backslash \Phi} \mathfrak{g}(F)_{[\lambda]}.$$

This decomposition is compatible with the one of $\mathrm{End}_F(E)$ induced by Proposition 6.1 such that

$$\mathfrak{g}(F)_0 \cong E \text{ and } \mathfrak{g}(F)_{\left[\left[\begin{smallmatrix} 1 \\ g \end{smallmatrix}\right]\right]} \cong {}^g EE.$$

Proof. The first assertion can be derived by a simple Galois descent argument from the absolute case. More precisely, for every orbit $[\lambda] := {}^{\Gamma_F}\lambda \in \Gamma_F \backslash \Phi$ there is a subspace $\mathfrak{g}(F)_{[\lambda]}$ in $\mathfrak{g}(F)$ such that

$$\mathfrak{g}(F)_{[\lambda]} \otimes_F \bar{F} = \bigoplus_{\mu \in [\lambda]} \mathfrak{g}(\bar{F})_\mu$$

the direct sum of the root space $\mathfrak{g}(\bar{F})_\mu$ for $\mu \in \Phi$. The second assertion is clear by Proposition 2.2(ii). $\square$

We call the decomposition of $\mathrm{End}_F(E) \cong \mathfrak{g}(F)$ in Proposition 6.2 the rational root space decomposition. We are going to show that such decomposition descends to the ones of certain $\mathfrak{o}_F$-sublattices of $\mathfrak{g}(F)$. Let $\mathfrak{A}$ be the hereditary order of $\mathrm{End}_F(E)$ corresponding to the $\mathfrak{o}_F$-lattice chain $\{\mathfrak{p}_E^k | k \in \mathbb{Z}\}$ in E, as introduced in (1.1) of [3]. Let K be the maximal unramified extension of F in E. Consider the following $\mathfrak{o}_F$-lattices contained in the F-vector spaces in (6.1.1)

$$\begin{aligned} &\mathfrak{o}_{E\otimes E} := (\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E) + \sum_{\substack{1\leq k\leq \ell\leq e-1\\ k+\ell\geq e}} \mathfrak{o}_K \varpi_F^{-1}(\varpi_E^k \otimes \varpi_E^\ell) \subseteq E \otimes E, \\ &\mathfrak{A} \subseteq \mathrm{End}_F(E), \text{ and} \\ &\bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathfrak{o}_{{}^gEE} \subseteq \bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} {}^gEE. \end{aligned} \tag{6.1.2}$$

They are all $E^\times$-conjugate invariant. If we identify $\mathrm{End}_F(E) \cong \mathfrak{gl}_n(F)$ and choosing suitable basis, then the lattice $\mathfrak{A}$ can be expressed in matrices partitioned into $e \times e$ blocks of size $f \times f$, with entries in $\mathfrak{o}_F$, and is blocked upper triangular mod $\mathfrak{p}_F$.

Proposition 6.3. *The isomorphisms in Proposition 6.1 induce isomorphisms of $\mathfrak{o}_F$-lattices as well as of $E^\times$-modules in (6.1.2).*

Proof. The $\mathfrak{o}_F$-morphism $\mathfrak{o}_{E\otimes E} \to \mathfrak{A}$ restricted from (i) of Proposition 6.1 is clearly injective and $E^\times$-invariant. To show the surjectivity, we choose an $\mathfrak{o}_F$-basis $\{w_1, \dots, w_n\}$ of $\mathfrak{o}_E$ such that

$$\begin{aligned} &v_E(w_1) = \cdots = v_E(w_f) = 0,\ v_E(w_{f+1}) = \cdots = v_E(w_{2f}) = 1, \dots, \\ &v_E(w_{(e-1)f+1}) = \cdots = v_E(w_{ef}) = e-1, \end{aligned}$$

We choose another $\mathfrak{o}_F$-basis $\{w_i^*\}$ of $\mathfrak{o}_E$ dual to $\{w_i\}$ in the sense that

$$\mathrm{tr}_{E/F}(w_i^* w_j) = 0 \text{ for } i \neq j \text{ and } \mathrm{tr}_{E/F}(w_i^* w_i) = \begin{cases} 1 & \text{for} \quad i = 1, \dots, f \\ \varpi_F & \text{for} \quad i = f+1, \dots, n \end{cases}.$$

Then we can readily show that, under the isomorphism $E \otimes E \to \mathrm{End}_F(E) \to \mathfrak{gl}_n(F)$, the element $\sum_{i,j} a_{ij}(w_i^* \otimes w_j)$ in $E \otimes E$ is mapped to the matrix (A_{ij}) where $A_{ij} = a_{ij}\mathrm{tr}(w_i^* w_i)$. We check the F-valuation of these entries. Suppose that w_i^* and w_j corresponds to the kth and ℓth block respectively with respect to the chosen bases.

1. If $k = 1$, then we have $v_E(w_i^*) = v_E(w_i) = 0$ and so $a_{ij}\mathrm{tr}(w_i^* w_i) = a_{ij} \in \mathfrak{o}_F$.
2. If $\ell \geq k \gneq 1$, then $a_{ij} \in \mathfrak{o}_F \varpi_F^{-1}$, $v_E(w_i^*) = e + 1 - k$ and $v_E(w_j) = \ell - 1$. Hence $a_{ij}\mathrm{tr}(w_i^* w_i) \in \mathfrak{o}_F$.
3. When $k > \ell$, we have $a_{ij} \in \mathfrak{o}_F$ and $a_{ij}\mathrm{tr}(w_i^* w_i) = a_{ij}\varpi_F \in \mathfrak{p}_F$.

We have just shown that $\mathfrak{o}_{E\otimes E} \to \mathfrak{A}$ is surjective and therefore isomorphic.

To deal with another isomorphism, we use the standard technique in chapter 4 of [26]. Let $M \subseteq N$ be two free $\mathfrak{o}_F$-modules of the same rank. Suppose the quotient N/M is isomorphic to $\oplus_j \mathfrak{o}_F/\mathfrak{p}_F^{n_j}$ as $\mathfrak{o}_F$-modules. Define the order ideal of $\mathfrak{o}_F$ to be $\mathrm{ord}_{\mathfrak{o}_F}(N/M) = \prod_j \mathfrak{p}_F^{n_j}$. We can compute the order alternatively as follows. If $M = \oplus_j \mathfrak{o}_F x_j$ and $N = \oplus_j \mathfrak{o}_F y_j$ such that $x_i = \sum_j a_{ij} y_j$ for some $a_{ij} \in \mathfrak{o}_F$, then $\mathrm{ord}_{\mathfrak{o}_F}(N/M) = \mathfrak{o}_F \det(a_{ij})$. Suppose we have a trace form of $W = M \otimes_{\mathfrak{o}_F} F$, i.e. a non-degenerate symmetric F-bilinear form $\mathrm{tr} : W \times W \to F$, we define the discriminant ideal of M with respect to the form tr to be $d(M) = \det \mathrm{tr}(x_i x_j)\mathfrak{o}_F$.

Lemma 6.4. *(i) We have the relation $d(M) = \mathfrak{o}_F(N/M)^2 d(N)$.*

(ii) If P and Q are free $\mathfrak{o}_F$-modules, then

(a) $d(P \oplus Q) = d(P)d(Q)$, and

(b) $d(P \otimes_{\mathfrak{o}_F} Q) = d(P)^{\mathrm{rank} Q} d(Q)^{\mathrm{rank} P}$.

Proof. (i) comes from the identity $\det \mathrm{tr}(x_i x_j) = \det(a_{ij})^2 \det \mathrm{tr}(y_i y_j)$, (iia) is easy, and (iib) is an elementary calculation of the determinant of tensor product of matrices. □

In the case when E is a tame extension of F and $M = \mathfrak{o}_E$, we denote the discriminant ideal $d(\mathfrak{o}_E)$ of $\mathfrak{o}_F$ by $d(E/F)$, which is known to be $\mathfrak{p}_F^{f(e-1)}$.

Now we show that the second $\mathfrak{o}_F$-morphism $\mathfrak{o}_{E\otimes E} \to \bigoplus_{[g]} \mathfrak{o}_{{}^gEE}$ is isomorphic. It is clearly injective and $E^\times$-invariant. To show that the map is surjective, we compare the sizes of $\bigoplus_{[g]} \mathfrak{o}_{{}^gEE}$ and the image of $\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$. We apply Lemma 6.4(ii) on

$$M = \mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E,\ N = \bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathfrak{o}_{{}^gEE} \text{ and } P = Q = \mathfrak{o}_E.$$

Let m be the F-valuation of the ideal

$$d(E/F)^{2n} \prod_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} d({}^gEE/F)^{-1}.$$

In the tame case, we can compute

$$m = 2nf(e-1) - (e-1) \sum_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} f({}^gEE/F).$$

The sum

$$\sum_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} f({}^gEE/F)$$

equals fn, as we know that

$$\sum_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} |{}^gEE/F| = \dim_F(E \otimes_F E) = n^2$$

and each ${}^gEE/F$ has the same ramification degree e. Therefore by (i) of Lemma 6.4 the order of

$$\left(\bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathfrak{o}_{{}^gEE} \right) / (\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E)$$

is $q^{m/2} = q^{f^2e(e-1)/2}$. Using the expression of (6.1.2), we can check that the order of $\mathfrak{o}_{E\otimes E}/(\mathfrak{o}_E \otimes_{\mathfrak{o}_F} \mathfrak{o}_E)$ is also $q^{f^2e(e-1)/2}$. Therefore the morphism $\mathfrak{o}_{E\otimes E} \to \bigoplus_{[g]} \mathfrak{o}_{{}^gEE}$ is surjective and hence isomorphic. $\square$

Let $\mathfrak{P}_{\mathfrak{A}}$ be the Jacobson radical of $\mathfrak{A}$. By similar arguments we can show that the $\mathfrak{o}_F$-sublattices of the lattices in (6.1.2)

$$\mathfrak{P}_{E\otimes E} := \mathfrak{p}_E \otimes_{\mathfrak{o}_F} \mathfrak{p}_E \subseteq \mathfrak{o}_{E\otimes E},$$

$$\mathfrak{P}_{\mathfrak{A}} \subseteq \mathfrak{A}, \text{ and}$$

$$\bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathfrak{p}_{{}^gEE} \subseteq \bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathfrak{o}_{{}^gEE}$$

are all isomorphic. We therefore have an $\mathbf{k}_F$-isomorphism

$$\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}} \cong \bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathfrak{o}_{{}^gEE}/\mathfrak{p}_{{}^gEE} = \bigoplus_{[g]\in\Gamma_E\backslash\Gamma_F/\Gamma_E} \mathbf{k}_{{}^gEE}, \tag{6.1.3}$$

which is moreover $E^\times$-equivalent. The $E^\times$-conjugate action on both sides factor through the finite group

$$\Psi_{E/F} := E^\times/F^\times U_E^1.$$

We call the decomposition of $\mathbf{k}_F\Psi_{E/F}$-module in (6.1.3) the residual root space decomposition.

Let's describe the action of $\Psi_{E/F}$ on $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$ more precisely. Since $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$ is isomorphic to M^e where $M = \mathfrak{gl}_f(\mathbf{k}_F)$, we regard M^e as being embedded into diagonal blocks in $\mathfrak{gl}_n(\mathbf{k}_F)$. We first consider M as a $\mathbf{k}_F\mu$-module. By embedding $\mathbf{k}_E \hookrightarrow \mathfrak{gl}_f(\mathbf{k}_F)$ (the choice of such embedding is irrelevant), we have (from section 7.3 of [9])

$$M \cong \bigoplus_{i\in\mathbb{Z}/f} M_i$$

where $M_i \cong \mathbf{k}_E$ as $\mathbf{k}_F$-vector spaces and $\zeta \in \mu_E$ acts by $v \mapsto \zeta^{q^i-1}v$ for all $v \in M_i$. Each of the characters $\zeta \mapsto \zeta^{q^i-1}$, $i \in \mathbb{Z}/f$, is trivial on μ_F, hence is a character of

$$\mu := \mu_E/\mu_F.$$

The $\Psi_{E/F}$-module we are interested in is

$$U = \mathrm{Ind}_\mu^{\Psi_{E/F}} M.$$

Clearly $U \cong \mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$ as $\mathbf{k}_F$-vector spaces.

Theorem 6.5. *(i) The $\Psi_{E/F}$-module U decomposes into submodules*

$$U \cong \bigoplus_{\substack{i\in\mathbb{Z}/f \\ k\in q^f\backslash(\mathbb{Z}/e)}} U_{ki}$$

such that for each component U_{ki} the $\Psi_{E/F}$-action is given by $\lambda_{ki} : \zeta \mapsto \zeta^{q^i-1}$ for all $\zeta \in \mu_E$ and $\varpi_E \mapsto \zeta_e^k \zeta_{\phi^i}$.

(ii) The decomposition of U is equivalent to the residual root space decomposition for $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$.

Proof. (i) We have the decomposition $U = \bigoplus_{i\in\mathbb{Z}/f} U_i$ for $U_i = \mathrm{Ind}_{\mu}^{\Psi_{E/F}} M_i$. Each M_i is isomorphic to $\mathbf{k}_E$, hence U_i is an e-dimensional $\mathbf{k}_E$-vector space. By (2.2.1), we can choose a $\mathbf{k}_E$-basis for U_i such that ϖ_E acts as conjugation of the matrix

$$\begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ \zeta_{E/F} & & & \end{pmatrix},$$

i.e. ϖ_E cyclicly permutes the components $\mathbf{k}_E$, with the first component being permuted to the last followed by an action of $\zeta_{E/F} \in \mu_E$, which is the multiplication of $\zeta_{E/F}^{q^i-1}$. The eigenvalues of such matrix is $\zeta_e^k\zeta_{\phi^i}$ for some fixed eth root ζ_{ϕ^i} of $\zeta_{E/F}^{q^i-1}$ and some $k = 0, \ldots, e-1$. Hence those $\zeta_e^k\zeta_{\phi^i}$ in the same $\Gamma_{\mathbf{k}_E}$-orbit, i.e. those $k \in \mathbb{Z}/e$ in the same q^f-orbit, form a simple $\Psi_{E/F}$-module.

(ii) If we identify $g = \sigma^k\phi^i$ as in Proposition 2.2(i), then the action of $E^\times$ on $\mathbf{k}_{^gEE}$ as a component of $\mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$ is induced from the conjugate action on ^gEE and hence is given by ${}^{\sigma^k\phi^i}\zeta\zeta^{-1} = \zeta^{q^i-1}$ for all $\zeta \in \mu_E$ and ${}^{\sigma^k\phi^i}\varpi_E\varpi_E^{-1} = \zeta_e^k\zeta_{\phi^i}$. Hence $\mathbf{k}_{^gEE} = \mathbf{k}_E[\zeta_e^k]$, which is just U_{ki}.

□

The $\mathbf{k}_F\Psi_{E/F}$-module U is called the standard module. Using the notation in Lemma 2.2, we write U_{ki} as $U_{[\sigma^k\phi^i]}$. This notation is well-defined, i.e. the finite module $U_{[g]}$ is independent of the coset representative of $[g]$, because if $[g] = [h]$ then ${}^gEE$ and ${}^hEE$ have the same residue field. For any subset $\mathcal{D} \subseteq (W_E \backslash W_F / W_E)'$ we write

$$U_{\mathcal{D}} = \bigoplus_{[g]\in\mathcal{D}} U_{[g]}.$$

In particular, for every intermediate field extensions $F \subseteq K \subseteq L \subseteq E$, if

$$\mathcal{D} = (W_E \backslash W_K / W_E) - (W_E \backslash W_L / W_E),$$

we denote $U_{\mathcal{D}}$ by $U_{L/K}$. If V is a submodule of U, we write $V_{\mathcal{D}} = U_{\mathcal{D}} \cap V$.

6.2 Symplectic modules

In this section we introduce a symplectic structure on certain submodules of the standard module $U = \mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$. We first give a brief summary on finite symplectic modules, whose details are referred to [9]. We consider a finite cyclic group Γ whose order is not divisible by a prime p. We call a finite $\mathbb{F}_p\Gamma$-module symplectic if there is a non-degenerate Γ-invariant alternating form $h : V \times V \to \mathbb{F}_p$, i.e.

$$h(\gamma v_1, \gamma v_2) = h(v_1, v_2) \text{ for all } \gamma \in \Gamma,\ v_i \in V.$$

Let $V_\lambda = \mathbb{F}_p[\lambda(\Gamma)]$ be the simple $\mathbb{F}_p\Gamma$-module defined by $\lambda \in \mathrm{Hom}(\Gamma, \bar{\mathbb{F}}_p^\times)$. Its $\mathbb{F}_p$-linear dual is just $V_{\lambda^{-1}}$.

Proposition 6.6. *(i) Any indecomposable symplectic $\mathbb{F}_p\Gamma$-module is one of the following two kinds.*

(a) A hyperbolic *module is of the form $V_{\pm\lambda} \cong V_\lambda \oplus V_{\lambda^{-1}}$ such that either $\lambda^2 = 1$ or $V_\lambda \ncong V_{\lambda^{-1}}$.*

(b) An anisotropic *module is of the form V_λ with $\lambda^2 \neq 1$ and $V_\lambda \cong V_{\lambda^{-1}}$.*

(ii) If V_λ is anisotropic, then $|\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p|$ is even and $\lambda(\Gamma) \subseteq \ker(N_{\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]_\pm})$ the kernel of the norm map of the quadratic extension $\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]_\pm$.

(iii) *The* Γ*-isometry class of a symplectic* $\mathbb{F}_p\Gamma$*-module* (V, h) *is determined by the underlying* $\mathbb{F}_p\Gamma$*-module* V*.*

Proof. All the proofs can be found in chapter 3 of [9]. □

We also call a symplectic $\mathbb{F}_p\Gamma$-module hyperbolic (resp. anisotropic) if it is a direct sum of hyperbolic (resp. anisotropic) indecomposable submodules. A special case is that if V is anisotropic, then $V \oplus V$ is hyperbolic. By slightly abusing our terminology, we treat this as a special case of hyperbolic module, namely an *even anisotropic* module. Therefore by Proposition 6.6 (iii), given a finite symplectic module whose Γ-action is known, we do not have to know the alternating form exactly.

For each symplectic $\mathbb{F}_p\Gamma$-module V we attach a sign $t^0_\Gamma(V) \in \{\pm 1\}$ and a character $t^1_\Gamma(V) : \Gamma \to \{\pm 1\}$, called the *t-factors* of V. We choose a generator γ of Γ and set $t_\Gamma(V) = t^0_\Gamma(V) t^1_\Gamma(V)(\gamma)$. We give the algorithm in [9] on computing the t-factors.

(i) If Γ acts on V trivially, then

$$t^0_\Gamma(V) = 1 \text{ and } t^1_\Gamma(V) \equiv 1.$$

(ii) Let V be an indecomposable symplectic $\mathbb{F}_p\Gamma$-module, then

(a) If $V = V_\lambda \oplus V_{\lambda^{-1}}$ is hyperbolic, then

$$t^0_\Gamma(V) = 1 \text{ and } t^1_\Gamma(V) = \mathrm{sgn}_{\lambda(\Gamma)}(V_\lambda),$$

where $\mathrm{sgn}_{\lambda(\Gamma)}(V_\lambda) : \Gamma \to \{\pm 1\}$ such that $\gamma \mapsto \mathrm{sgn}_{\lambda(\gamma)}(V_\lambda)$ is the sign of the multiplicative action of $\lambda(\gamma)$ on V_λ.

(b) If $V = V_\lambda$ is anisotropic, then

$$t^0_\Gamma(V) = -1 \text{ and } t^1_\Gamma(V)(\gamma) = \left(\frac{\gamma}{\mathcal{K}}\right) \text{ for any } \gamma \in \Gamma,$$

where $\mathcal{K} = \ker(N_{\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]_\pm})$ for the quadratic extension $\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p[\lambda(\Gamma)]_\pm$ and $(-)$ is the Jacobi symbol, i.e. for every finite cyclic group H,

$$\left(\frac{x}{H}\right) = \begin{cases} 1, & \text{if } x \in H^2 \\ -1, & \text{otherwise} \end{cases}.$$

(iii) If V decomposes into an orthogonal sum $V_1 \oplus \cdots \oplus V_m$ of indecomposable symplectic $\mathbb{F}_p\Gamma$-modules, then

$$t^i_\Gamma(V) = t^i_\Gamma(V_1) \cdots t^i_\Gamma(V_m) \text{ for } i = 0, 1.$$

If $p = 2$, then the order of Γ is odd. In this case $t^1_\Gamma(V)$ is always trivial because all sign characters and Jacobi symbols are trivial.

Remark 6.7. If $V_\lambda = \mathbb{F}_p[\lambda(\Gamma)]$ is anisotropic, then $V \oplus V$ is hyperbolic, or precisely even anisotropic. The t-factors are the same whether we consider $V \oplus V$ as hyperbolic or anisotropic. It is clear that $t^0_\Gamma(V \oplus V) = 1$ in both cases, while $t^1_\Gamma(V \oplus V) = \mathrm{sgn}_{\lambda(\Gamma)} V$ in the hyperbolic case and $t^1_\Gamma(V \oplus V) = t^1_\Gamma(V)^2 = 1$ in the anisotropic case. Indeed $\mathrm{sgn}_{\lambda(\Gamma)} V \equiv 1$. It is clear for $p = 2$. If p is odd, then by Proposition 6.6(ii) we have that $s = |\mathbb{F}_p[\lambda(\Gamma)]/\mathbb{F}_p|$ is even and $\lambda(\Gamma) \subseteq \mu_{p^{s/2}+1}$. Therefore $|\mathbb{F}_p[\lambda(\Gamma)]^\times/\mu_{p^{s/2}+1}| = p^{s/2} - 1$ is even and $\mathrm{sgn}_{\lambda(\Gamma)} V = (\mathrm{sgn}_{\lambda(\Gamma)} \mu_{p^{s/2}+1})^{p^{s/2}-1} = 1$. □

Suppose now Γ is a cyclic subgroup of $\Psi_{E/F}$. We study the symplectic Γ-submodules of the standard $\mathbf{k}_E\Psi_{E/F}$-module U. The cyclic subgroups

$$\mu := \mu_E/\mu_F \text{ and } \varpi := \langle \varpi_E \rangle / \langle \varpi_F \rangle$$

of $\Psi_{E/F}$ are of our particular interest. For $\Gamma = \mu$ or ϖ, we regard U as an $\mathbb{F}_p\Gamma$-module by restriction, and compute the t-factors of the symplectic $\mathbb{F}_p\Gamma$-submodule of U. Recall that $[g] \in (\Gamma_E \backslash \Gamma_F / \Gamma_E)'$ is *symmetric* if $[g] = [g^{-1}]$ and is *asymmetric* otherwise.
If $[g]$ is asymmetric, then $U_{\pm[g]} = U_{[g]} \oplus U_{[g^{-1}]}$ is a hyperbolic $\mathbb{F}_p\Gamma$-module. Write $[g] = [\sigma^k \phi^i]$ using the description in Proposition 2.2. If $\Gamma = \mu$ then we have

$$t^0_\mu(U_{\pm[\sigma^k\phi^i]}) = 1 \text{ and } t^1_\mu(U_{\pm[\sigma^k\phi^i]}) : \zeta \mapsto \mathrm{sgn}_{\zeta^{q^i-1}}(U_{[\sigma^k\phi^i]}) \text{ for all } \zeta \in \mu_E.$$

In particular, if $i = f/2$, then both $U_{[\sigma^k \phi^{f/2}]}$ and $U_{[(\sigma^k \phi^{f/2})^{-1}]}$ contain the $\mathbb{F}_p\mu$-module $U_{\phi^{f/2}} \cong \mathbf{k}_E$, which is anisotropic indecomposable (see Proposition 19 of [9]). Hence $U_{\pm[\sigma^k \phi^{f/2}]}$ is even anisotropic and

$$t^0_\mu(U_{\pm[\sigma^k \phi^{f/2}]}) = 1 \text{ and } t^1_\mu(U_{\pm[\sigma^k \phi^{f/2}]}) \equiv 1.$$

If $\Gamma = \varpi$, then we have similarly

$$t^0_\varpi(U_{\pm[\sigma^k \phi^i]}) = 1 \text{ and } t^1_\varpi(U_{\pm[\sigma^k \phi^i]})(\varpi_E) = \mathrm{sgn}_{\zeta_e^k \zeta_{\phi^i}}(U_{[\sigma^k \phi^i]}).$$

Now we assume that $[g]$ is symmetric. We first give some properties of these symmetric $[g]$ and the corresponding submodule $U_{[g]}$. For $[g] = [\sigma^k \phi^i]$ symmetric, let $t = t_k$ be the minimal solution of Proposition 2.3.

Lemma 6.8. *Assume $[g] \neq [\sigma^{e/2}]$, then $|U_{[g]}/\mathbf{k}_E|$ equals $2t_k$ in case $i = 0$, and equals $2t_k + 1$ in case $i = f/2$.*

Proof. Since $U_{[g]}$ is the field extension of $\mathbf{k}_E$ generated by ζ_e^k, the degree of $U_{[g]}/\mathbf{k}_E$ equals the minimal solution s of $e|(q^{fs} - 1)k$. Therefore s must be the one indicated above. □

We write $U_{\pm[g]}$ the subfield of $U_{[g]}$ such that $|U_{[g]}/U_{\pm[g]}| = 2$. Notice that $U_{\pm[g]}$ contains $\mathbf{k}_E$ as a sub-extension of even degree if and only if $i = 0$. In the exception case when $[g] = [\sigma^{e/2}]$, i.e. $[g]$ corresponds to the root λ in Φ that satisfies $\lambda^2 = 1$, we have $U_{[\sigma^{e/2}]} = \mathbf{k}_E$ and the minimal solution in Lemma 2.3(6.8) is $t_{e/2} = 0$. As we will see in Proposition 7.5, not all submodules of U admit symplectic structures. This implies in particular to $U_{[\sigma^{e/2}]}$. We only study $U_{[g]}$ with $[g] \neq [1]$ and, in the case e is even, $[g] \neq [\sigma^{e/2}]$.

Now we compute the t-factors for symmetric $[g]$. Suppose $[g] = [\sigma^k]$ that $k \neq 0$ or $e/2$. The group $\Psi_{E/F}$ acts by the character λ_{k0} where $\lambda_{k0}|_\mu = 1$ and $\lambda_{k0}(\varpi_E) = \zeta_e^k$. Since μ_E acts trivially, we have

$$t^0_\mu(U_{[\sigma^k]}) = 1 \text{ and } t^1_\mu(U_{[\sigma^k]}) \equiv 1.$$

To compute $t^i_\varpi(U_{[\sigma^k]})$, $i = 0, 1$, we consider $U_{[\sigma^k]} = \mathbf{k}_E[\zeta_e^k]$ as an $\mathbb{F}_p\varpi$-module. Each simple submodule $\mathbb{F}_p[\zeta_e^k]$ of $U_{[\sigma^k]}$ is anisotropic. We have the following property about its multiplicity.

Lemma 6.9. *The degree $r = r_{[\sigma^k]} = |\mathbf{k}_E[\zeta_e^k]/\mathbb{F}_p[\zeta_e^k]|$ is odd.*

Proof. We see that $\lambda_{k0}(\varpi)$ is contained in $\ker(N_{U_{[\sigma^k]}/U_{\pm[\sigma^k]}})$ the group of $(q^{ft_k}+1)$-roots in $U_{[\sigma^k]}$. Suppose that r is even. If $\mathbb{F}_p[\zeta_e^k]$ has Q elements, then $U_{[\sigma^k]}$ has $Q^r = q^{2ft_k}$ elements. We have

$$\lambda_{k0}(\varpi) \subseteq \mathbb{F}_p[\zeta_e^k]^\times \cap \ker(N_{U_{[\sigma^k]}/U_{\pm[\sigma^k]}}) = \mu_{Q-1} \cap \mu_{Q^{r/2}+1} \subsetneq \{\pm 1\},$$

forcing $k = 0$ or $e/2$. This is a contradiction. □

By Lemma 6.9 we have

$$t^0_\varpi(U_{[\sigma^k]}) = (-1)^r = -1 \text{ and } t^1_\varpi(U_{[\sigma^k]}) : \varpi_E \mapsto \left(\frac{\zeta_e^k}{\mu_{p^s+1}}\right)^r = \left(\frac{\zeta_e^k}{\mu_{p^s+1}}\right),$$

where $|\mathbb{F}_p[\zeta_e^k]/\mathbb{F}_p| = 2s$ and

$$\mu_{p^s+1} = \ker(N_{\mathbb{F}_p[\zeta_e^k]/\mathbb{F}_p[\zeta_e^k]_\pm})$$

for the quadratic extension $\mathbb{F}_p[\zeta_e^k]/\mathbb{F}_p[\zeta_e^k]_\pm$.

Now suppose $[g] = [\sigma^k\phi^{f/2}]$. We first consider $U_{[\sigma^k\phi^{f/2}]}$ as an $\mathbb{F}_p\mu$-module. Since $\lambda_{k,f/2}(\mu_E) = \ker(N_{\mathbf{k}_E/\mathbf{k}_{E\pm}}) = \mu_{q^{f/2}+1}$ and $\mathbb{F}_p[\lambda_{k,f/2}(\mu_E)] = \mathbf{k}_E$ is anisotropic, we have

$$t^0_\mu(U_{[\sigma^k\phi^{f/2}]}) = (-1)^{2t_k+1} = -1$$

and

$$t^1_\mu(U_{[\sigma^k\phi^{f/2}]}) : \zeta \mapsto \left(\frac{\zeta^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right)^{2t_k+1} = \left(\frac{\zeta^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right) \text{ for all } \zeta \in \mu_E.$$

For $U_{[\sigma^k\phi^{f/2}]}$ as an $\mathbb{F}_p\varpi$-module, the action of ϖ_E is the multiplication of $\lambda_{k,f/2}(\varpi_E) = \zeta_e^k\zeta_{\phi^{f/2}}$. We distinguish this value into the following cases.

(i) If $\zeta_e^k\zeta_{\phi^{f/2}} = 1$, then ϖ_E acts trivially and hence

$$t^0_\varpi(U_{[\sigma^k\phi^{f/2}]}) = 1 \text{ and } t^1_\varpi(U_{[\sigma^k\phi^{f/2}]}) \equiv 1.$$

(ii) If $\zeta_e^k\zeta_{\phi^{f/2}} = -1$, then $U_{[\sigma^k\phi^{f/2}]} = \mathbf{k}_E$ and $\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}] = \mathbb{F}_p$. Since $|\mathbf{k}_E/\mathbb{F}_p|$ is even, the module $U_{[\sigma^k\phi^{f/2}]}$ is even anisotropic. We have

$$t^0_\varpi(U_{[\sigma^k\phi^{f/2}]}) = 1 \text{ and } t^1_\varpi(U_{[\sigma^k\phi^{f/2}]})(\varpi_E) = \mathrm{sgn}_{-1}(U_{\pm[\sigma^k\phi^{f/2}]}) = (-1)^{\frac{1}{2}(q^{f/2}-1)}.$$

(iii) If $\zeta_e^k\zeta_{\phi^{f/2}} \neq \pm 1$, then similar to Lemma 6.9 we have that $|U_{[\sigma^k\phi^{f/2}]}/\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}]|$ is odd. Hence

$$t^0_\varpi(U_{[\sigma^k\phi^{f/2}]}) = -1 \text{ and } t^1_\varpi(U_{[\sigma^k\phi^{f/2}]}) : \varpi_E \mapsto \left(\frac{\zeta_e^k\zeta_{\phi^{f/2}}}{\mu_{p^s+1}}\right),$$

where $|\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}]/\mathbb{F}_p| = 2s$ and

$$\mu_{p^s+1} = \ker(N_{\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}]/\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}]_\pm})$$

for the quadratic extension $\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}]/\mathbb{F}_p[\zeta_e^k\zeta_{\phi^{f/2}}]_\pm$.

We summarize the above values of t-factors in the following

Proposition 6.10. *(i) If $\Gamma = \mu$, then the t-factors are as follows.*

(a) If $[g] = [\sigma^k\phi^i]$ is asymmetric, then $t^0_\mu(U_{\pm[g]}) = 1$ and $t^1_\mu(U_{\pm[g]}) : \zeta \mapsto \mathrm{sgn}_{\zeta^{q^i-1}}(U_{[g]})$.

(b) If $[g] = [\sigma^k]$ is symmetric, then $t^0_\mu(U_{[g]}) = 1$ and $t^1_\mu(U_{[g]}) \equiv 1$.

(c) If $[g] = [\sigma^k\phi^{f/2}]$ is symmetric, then $t^0_\mu(U_{[g]}) = -1$ and $t^1_\mu(U_{[g]}) : \zeta \mapsto \left(\frac{\zeta^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right)$.

(ii) If $\Gamma = \varpi$, then the t-factors are as follows.

(a) If $[g] = [\sigma^k\phi^i]$ is asymmetric, then $t^0_\varpi(U_{\pm[g]}) = 1$ and $t^1_\varpi(U_{\pm[g]})(\varpi_E) = \mathrm{sgn}_{\zeta_e^k\zeta_{\phi^i}}(U_{[g]})$.

(b) If $[g] = [\sigma^k]$ is symmetric, then $t^0_\varpi(U_{[g]}) = -1$ and

$$t^1_\varpi(U_{[g]}) : \varpi_E \mapsto \left(\frac{\zeta_e^k}{\ker(N_{\mathbb{F}_p[\zeta_e^k]/\mathbb{F}_p[\zeta_e^k]_\pm})}\right).$$

(c) *If* $[g] = [\sigma^k \phi^{f/2}]$ *is symmetric,*

(I) *if* $\zeta_e^k \zeta_{\phi^{f/2}} = 1$, *then* $t^0_\varpi(U_{[g]}) = 1$ *and* $t^1_\varpi(U_{[g]}) \equiv 1$*;*

(II) *if* $\zeta_e^k \zeta_{\phi^{f/2}} = -1$, *then* $t^0_\varpi(U_{[g]}) = 1$ *and* $t^1_\varpi(U_{[g]})(\varpi_E) = (-1)^{\frac{1}{2}(q^{f/2}-1)}$*;*

(III) *if* $\zeta_e^k \zeta_{\phi^{f/2}} \neq \pm 1$, *then* $t^0_\varpi(U_{[g]}) = -1$ *and*

$$t^1_\varpi(U_{[g]}) : \varpi_E \mapsto \left(\frac{\zeta_e^k \zeta_{\phi^{f/2}}}{\ker(N_{\mathbb{F}_p[\zeta_e^k \zeta_{\phi^{f/2}}]/\mathbb{F}_p[\zeta_e^k \zeta_{\phi^{f/2}}]_{\pm}})} \right).$$

□

6.3 Complementary modules

Let W be a finite dimensional $\mathbf{k}_F$-vector space, $Q : W \to \mathbf{k}_F$ be a quadratic form and ψ_F be a non-trivial character of $\mathbf{k}_F$. We define the Gauss sum

$$\mathfrak{g}(Q, \psi_F) = \sum_{v \in W} \psi_F(Q(v)).$$

The simplest example is when $W = \mathbf{k}_F$ and $Q(x) = x^2$. In this case we denote the Gauss sum by $\mathfrak{g}(\psi_F)$. Suppose Q is non-degenerate, i.e. $\det Q \neq 0$, then we can show that

$$\mathfrak{g}(Q, \psi_F) = \left(\frac{\det Q}{\mathbf{k}_F^\times} \right) \mathfrak{g}(\psi_F)^{\dim_{\mathbf{k}_F} W}. \tag{6.3.1}$$

We also define the normalized Gauss sum $\mathfrak{n}(Q, \psi_F) = (\#W)^{-1/2} \mathfrak{g}(Q, \psi_F)$ and $\mathfrak{n}(\psi_F) = q^{-1/2} \mathfrak{g}(\psi_F)$. The equation (6.3.1) is still true if we replace $\mathfrak{g}$ by $\mathfrak{n}$. We can easily show that the Gauss sum is a convoluted sum

$$\mathfrak{g}(\psi_F) = \sum_{x \in \mathbf{k}_F} \left(\frac{x}{\mathbf{k}_F^\times} \right) \psi_F(x).$$

From this point we can deduce that

$$\mathfrak{n}(Q, \psi_F)^2 = \left(\frac{-1}{q} \right),$$

i.e. the normalize sum $\mathfrak{n}(Q, \psi_F)$ is a 4th root of unity.

We now assume that W is a $\mathbf{k}_F\Psi_{E/F}$-submodule of the standard module U. Let Q be a non-degenerate $\Psi_{E/F}$-invariant bilinear form of W. Let $Q_{[g]}$ be the restriction of Q on $W_{\pm[g]}$ for $[g]$ asymmetric and on $W_{[g]}$ for $[g]$ symmetric. Hence each $Q_{[g]}$ is non-degenerate if $W_{[g]}$ is non-trivial.

Proposition 6.11. *If $[g] \neq [\sigma^{e/2}]$, then*

(i) $\mathfrak{n}(Q_{[g]}, \psi_F) = 1$ if $[g]$ is asymmetric, $= -1$ if $[g]$ is symmetric, and

(ii) the normalized Gauss sum $\mathfrak{n}(Q_{[g]}, \psi_F)$ depends only on the symmetry of $[g]$ and is independent to the quadratic form Q.

In the exceptional case $[g] = [\sigma^{e/2}]$, the sum $\mathfrak{n}(Q_{[g]}, \psi_F)$ is an arbitrary 4th root of unity.

Proof. The assertion (ii) is direct from (i), so we only prove (i) below. We temporarily write $\mathbf{W}_{[g]}$ the component $U_{[g]} \oplus U_{[g^{-1}]}$ for $[g]$ asymmetric or $U_{[g]}$ for $[g]$ symmetric and $[g] \neq [\sigma^{e/2}]$, i.e. the corresponding root λ satisfies $\lambda^2 \neq 1$.

By Proposition 4.4 of [7], in the case when $\Psi_{E/F}$ is cyclic and each $U_{[g]}$ is a non-trivial indecomposable $\mathbf{k}_F\Psi_{E/F}$-module , we have

$$\left(\frac{\det Q_{[g]}}{\mathbf{k}_F^\times}\right) = \begin{cases} \left(\frac{-1}{q}\right)^{\dim_{\mathbf{k}_F} U_{[g]}}, & \text{if } [g] \text{ is asymmetric} \\ -\left(\frac{-1}{q}\right)^{\dim_{\mathbf{k}_F} U_{[g]}/2}, & \text{if } [g] \text{ is symmetric} \end{cases}. \tag{6.3.2}$$

This result readily generalizes to arbitrary $\Psi_{E/F}$ and any isotypic component $U_{[g]}$. This is clear if $[g]$ is asymmetric. If $[g]$ is symmetric, it suffices to show that $U_{[g]}$ contains an indecomposable $\mathbf{k}_F\Psi_{E/F}$-component of odd degree. If $[g] = [\sigma^k]$ for $k \neq 0$ or $e/2$, then any indecomposable $\mathbf{k}_F\Psi_{E/F}$-component is $\mathbf{k}_F[\zeta_e^k]$, and $|U_{[\sigma^k]}/\mathbf{k}_F[\zeta_e^k]|$ is odd by Lemma 6.9. If $[g] = [\sigma^k\phi^{f/2}]$, then any indecomposable $\mathbf{k}_F\Psi_{E/F}$-component contains $\mathbf{k}_E$ and $|U_{[\sigma^k\phi^{f/2}]}/\mathbf{k}_E|$ is odd by Lemma 6.8.

Therefore (6.3.2) implies that the discriminant of $Q_{[g]}$ mod $\mathbf{k}_F^{\times 2}$ is determined by the underlying $\mathbf{k}_F\Psi_{E/F}$-module structure. Therefore using (6.3.1) we have

$$\mathfrak{n}(Q_{[g]},\psi_F) = \left(\frac{\det Q_{[g]}}{\mathbf{k}_F^\times}\right)\mathfrak{n}(\psi_F)^{\dim_{\mathbf{k}_F}\mathbf{W}_{[g]}} = \pm\left(\frac{-1}{q}\right)^{\dim_{\mathbf{k}_F}\mathbf{W}_{[g]}/2}\left(\frac{-1}{q}\right)^{\dim_{\mathbf{k}_F}\mathbf{W}_{[g]}/2} = \pm 1,$$

where the sign is determined by the symmetry of $[g]$ as in (6.3.2). □

Given a symplectic sub-module V of U, let W be the complementary module of V, in the sense that

$$W = \bigoplus_{[g]\in(W_E\backslash W_F/W_E)'} W_{[g]}$$

where $W_{[g]}$ is either trivial or isomorphic to $U_{[g]}$ and

$$W \oplus V = \bigoplus_{[g]\in(W_E\backslash W_F/W_E)'} U_{[g]}.$$

Suppose that Q is a non-degenerate $\Psi_{E/F}$-invariant quadratic form on W so that each $Q_{[g]}$ is also non-degenerate on $W_{[g]}$. We define our extra t-factors to be

$$t(W_{[g]}) = \mathfrak{n}(Q_{[g]},\psi_F).$$

In particular, we have

(i) If $W_{[g]}$ is trivial, then $t(W_{[g]}) = 1$.

(ii) If $[g]$ is asymmetric, then $t(W_{\pm[g]}) = 1$.

(iii) If $[g]$ is symmetric, then $t(W_{[g]}) = -1$ if $[g] \neq [\sigma^{e/2}]$.

It is clear that the if $[g] \neq [\sigma^{e/2}]$, the t-factor $t(W_{[g]})$ is independent of the quadratic form Q above.

Chapter 7

Essentially tame supercuspidal representations

In section 7.1 we recall the admissible characters defined in section 3.2 and extract more structural information from them. With these information we can describe how to extend from an admissible character in several steps to an essentially tame supercuspidal representation of $G(F)$ in section 7.2. In between the steps we can construct a finite symplectic $\mathbf{k}_F\Psi_{E/F}$-module, of which we can apply the decomposition in section 6.1 and attach t-factors on the indecomposable components. Finally in 7.3 we restate the values of the rectifiers defined by Bushnell-Henniart in terms of t-factors.

7.1 Admissible characters revisited

We further assume that our additive character ψ_F of F is of level 0, i.e. ψ_F is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$. For any tamely ramified extension E/F, we write $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$. Given a character ξ of $E^\times$, the E-level $r_E(\xi)$ of ξ is the minimum integer r such that $\xi|_{U_E^{r+1}} \equiv 1$. We say that ξ is *tamely ramified*, or just tame, if $r_E(\xi) = 0$. Suppose ξ is admissible over F (see section 3.2), then it admits a *Howe-factorization* (see Lemma

2.2.4 of [24]) of the form

$$\xi = (\xi_{d+1} \circ N_{E/F})(\xi_d \circ N_{E/E_d}) \cdots (\xi_0 \circ N_{E/E_0})\xi_{-1}. \tag{7.1.1}$$

We need to specify the notations in (7.1.1).

1. We have a decreasing sequence of fields

$$E = E_{-1} \supseteq E_0 \supsetneq E_1 \cdots \supsetneq E_d \supsetneq E_{d+1} = F. \tag{7.1.2}$$

   Each ξ_i is a character of $E_i^\times$, and ξ_d is a character of $F^\times$.

2. Let r_i be the E-level of ξ_i, i.e. the E-level of $\xi_i \circ N_{E/E_i}$, and r_{d+1} be the E-level of ξ. We assume that ξ_{d+1} is trivial if $r_{d+1} = r_d$. We call the E-levels $r_0 < \cdots < r_d$ the jumps of ξ.

3. If $E_0 = E$, then we replace $(\xi_0 \circ N_{E/E_0})\xi_{-1}$ by ξ_0. If $E_0 \subsetneq E$ we have that ξ_{-1} is tame and E/E_0 is unramified.

We define the wild component of ξ to be $\Xi_0 \circ N_{E/E_0}$ where

$$\Xi_0 = (\xi_{d+1} \circ N_{E_0/F}) \cdots (\xi_1 \circ N_{E_0/E_1})\xi_0$$

and the tame component of ξ to be $\xi_{-1} = \xi(\Xi_0 \circ N_{E/E_0})^{-1}$.

Proposition 7.1. *For $i = 0, \ldots, d+1$, we have the following.*

(i) If s_i is the E_i-level of ξ_i, then $s_i e(E/E_i) = r_i$.

(ii) There is a unique $\alpha_i \in \langle \varpi_{E_i} \rangle \times \mu_{E_i}$ such that $v_{E_i}(\alpha_i) = -s_i$ and

$$\xi_i|_{U_{E_i}^{s_i}}(1+x) = \psi_{E_i}(\alpha_i x).$$

For $i = 0, \ldots, d$, we have the following.

(iii) Each character ξ_i is generic over E_{i+1}, in the sense that $E_{i+1}[\alpha_i] = E_i$.

(iv) We have the relation $\gcd(r_i, e(E/E_{i+1})) = e(E/E_i)$.

Proof. (i) comes from an elementary calculation of the image of N_{E/E_i}. (ii) and (iii) can be found in section 2.2 of [24]. For (iv), that $E_{i+1}[\alpha_i] = E_i$ in (iii) implies that

$$\gcd(v_{E_i}(\alpha_i), e(E_i/E_{i+1})) = 1.$$

Then (ii) and (i) imply the desired result. □

We define the jump data of ξ consisting of a sequence of subfields $E \supseteq E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d \supsetneq F$ and an increasing sequence $1 \leq r_0 < \cdots < r_d$ of positive integers. Hence each admissible character gives rise to a jump data as above. We can define a W_F-action on the sequence

$$g : \{E \supseteq E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d \supsetneq F\} \mapsto \{{}^gE \supseteq {}^gE_0 \supsetneq {}^gE_1 \supsetneq \cdots \supsetneq {}^gE_d \supsetneq F\}$$

for all $g \in W_F$. The jumps of ${}^g\xi$ is clearly the same as those of ξ. Hence we can define a jump data of the equivalence class $(E/F, \xi)$ in the obvious sense.

For each ξ_i where $i = 0, \dots, d+1$, there is $\beta_i \in E_i \cap \mathfrak{p}_E^{-r_i}$ so that

$$\phi_i \circ N_{E/E_i}(1+x) = \psi_F(\mathrm{tr}_{E/F}(\beta_i x)) \quad \text{for all } x \in \mathfrak{p}_E^{\lfloor r_i/2 \rfloor + 1}.$$

Such β_i can be chosen mod $\mathfrak{p}_E^{1-(\lfloor r_i/2 \rfloor+1)}$. We write

$$\beta_i = \zeta_i \varpi_E^{-r_i} u_i \text{ for some } \zeta_i \in \mu_E \text{ and } u_i \in U_E^1.$$

The 'first term' $\zeta_i \varpi_E^{-r_i}$ equals α_i defined in Proposition 7.1(ii). We denote

$$\beta = \beta(\xi) = \beta_{d+1} + \cdots + \beta_0. \tag{7.1.3}$$

7.2 Essentially tame supercuspidal representations

We briefly describe how to parameterize essentially tame supercuspidals by admissible characters, i.e. the map $\Pi_n : P_n(F) \to \mathcal{A}_n^{\mathrm{et}}(F)$ defined in Proposition 3.4. The details can be found in [24], [3], and are also summarized in [6]. We first introduce certain subgroups of $G(F)$. Let $E \supseteq E_0 \supsetneq E_1 \cdots \supsetneq E_d \supsetneq F$ be a decreasing sequence of fields. Write $B_i = \mathrm{End}_{E_i}(E)$ and define

$$\begin{aligned} \mathfrak{B}_i &= \{x \in B_i | x\mathfrak{p}_E^k \subseteq \mathfrak{p}_E^k \text{ for all } k \in \mathbb{Z}\} \text{ and} \\ \mathfrak{P}_{\mathfrak{B}_i} &= \{x \in B_i | x\mathfrak{p}_E^k \subseteq \mathfrak{p}_E^{k+1} \text{ for all } k \in \mathbb{Z}\} \end{aligned} \tag{7.2.1}$$

the hereditary orders in B_i corresponding to the $\mathfrak{o}_{E_i}$-lattice chain $\{\mathfrak{p}_E^k | k \in \mathbb{Z}\}$ of the E_i-vector space E and its radical. We then define subgroups of $B_i^\times$

$$U_{\mathfrak{B}_i} = \{x \in B_i^\times | x\mathfrak{p}_E^k = \mathfrak{p}_E^k \text{ for all } k \in \mathbb{Z}\} \text{ and } U_{\mathfrak{B}_i}^j = 1 + \mathfrak{P}_{\mathfrak{B}_i}^j \text{ for } j > 0. \tag{7.2.2}$$

If E/E_0 is unramified, then we can replace $\mathfrak{p}_E$ by $\mathfrak{p}_{E_0}$ in (7.2.1) and (7.2.2). We denote $A = \mathrm{End}_F(E)$ and define $\mathfrak{A}$, $\mathfrak{P}_{\mathfrak{A}}$, $U_{\mathfrak{A}}$, and $U_{\mathfrak{A}}^j$ as in (7.2.1) and (7.2.2) with B_i replaced by A. The multiplication of E identifies E as a subspace of A. Choose an isomorphism of $A^\times \cong G(F)$ so that $E^\times$ embeds into $G(F)$ by an F-regular morphism. Then $\mathfrak{A}^\times$, $U_{\mathfrak{A}}$, $U_{\mathfrak{A}}^j$, $\mathfrak{B}_i^\times$, $U_{\mathfrak{B}_i}$ and $U_{\mathfrak{B}_i}^j$ embeds into $G(F)$ accordingly.

If the fields E_i are defined by the Howe-factorization of $\xi \in P(E/F)$ as in (7.1.1) with jumps $\{r_0, \dots, r_d\}$, we define two numbers j_i and h_i by

$$j_i = \lfloor \frac{r_i + 1}{2} \rfloor \leq h_i = \lfloor \frac{r_i}{2} \rfloor + 1. \tag{7.2.3}$$

Here $\lfloor x \rfloor$ is the largest integer not greater than x. We construct the subgroups

$$\begin{aligned} &H^1(\xi) = U_{\mathfrak{B}_0}^1 U_{\mathfrak{B}_1}^{h_0} \cdots U_{\mathfrak{B}_d}^{h_{d-1}} U_{\mathfrak{A}}^{h_d}, \\ &J^1(\xi) = U_{\mathfrak{B}_0}^1 U_{\mathfrak{B}_1}^{j_0} \cdots U_{\mathfrak{B}_d}^{j_{d-1}} U_{\mathfrak{A}}^{j_d} \subseteq J(\xi) = U_{\mathfrak{B}_0} U_{\mathfrak{B}_1}^{j_0} \cdots U_{\mathfrak{B}_d}^{j_{d-1}} U_{\mathfrak{A}}^{j_d}, \text{ and} \\ &\mathbf{J}(\xi) = E^\times J(\xi) = E_0^\times J(\xi). \end{aligned} \tag{7.2.4}$$

We abbreviate these groups by H^1, J^1, J and $\mathbf{J}$ if the admissible character ξ is fixed. Notice that H^1, J^1, J are compact subgroups and $\mathbf{J}$ is a compact-mod-center subgroup.

Now we briefly describe the construction in [6] and [3] of the supercuspidal representation from an admissible character ξ in five steps.

(i) From the Howe factorization (7.1.1) of ξ we can define a character $\theta = \theta(\xi)$ on H^1. This character depends only on the wild component $\Xi_0 \circ N_{E/E_0}$ of ξ. In fact according to the definition of simple characters in [3], there can be a number of such characters associated to ξ. There is a canonical one $\theta(\xi)$ constructed in [24], which is called the simple character of ξ in this article.

(ii) By classical theory of Heisenberg representation, we can extend θ to a unique representation η of J^1 using the symplectic structure of $V = J^1/H^1$ defined by θ.

(iii) There is a unique extension $\Lambda_0 = \Lambda(\Xi_0 \circ N_{E/E_0})$ of η on $\mathbf{J}$ satisfying the conditions in Lemma 1 and 2 of section 2.3 [6]. The restriction $\Lambda_0|_J$ is called a β-extension of η, in the sense of (5.2.1) [3].

(iv) We still need another representation $\Lambda(\xi_{-1})$ on $\mathbf{J}$, which is defined by the tame component ξ_{-1} of ξ. Suppose $\xi_{-1}|_{U_E}$ is the inflation of a character $\bar{\xi}_{-1}$ of $\mathbf{k}_E$. We first apply Green's parametrization to obtain a unique (up to isomorphism) irreducible cuspidal representation $\bar{\lambda}_{-1}$ of $\mathrm{GL}_{|E/E_0|}(\mathbf{k}_{E_0}) = J/J^1$, then inflate $\bar{\lambda}_{-1}$ to a representation λ_{-1} on J, and finally multiply $\xi(\varpi_E)$ to obtain $\Lambda(\xi_{-1})$ on $\mathbf{J} = \langle \varpi_E \rangle J$.

(v) The supercuspidal is defined by $\pi_\xi = \mathrm{cInd}_{\mathbf{J}}^G(\Lambda(\xi_{-1}) \otimes \Lambda_0)$.

Remark 7.2. The wild component ξ_w and tame component ξ_t of ξ is defined alternatively in [6]. We briefly explain that these choices produce the same representation on $\mathbf{J}$. By construction we have $\xi = (\Xi_0 \circ N_{E/E_0})\xi_{-1} = \xi_w \xi_t$. Since $\xi_w|_{U_E^1} = (\Xi_0 \circ N_{E/E_0})|_{U_E^1}$, they induce the same simple character $\theta = \theta(\xi)$. Therefore we have isomorphism of the β-extensions $\Lambda(\xi_w) = \Lambda_0 \otimes \alpha$ where α is a tamely ramified character α on $E^\times U_{\mathfrak{B}_0}/U^1_{\mathfrak{B}_0}$ such that

$$\alpha|_{U_{\mathfrak{B}_0}} = \Xi_0|_{\mu_{E_0}} \circ \det_{\mathbf{k}_{E_0}} \circ (\mathrm{proj}^J_{J/J^1}) \text{ and } \alpha(\varpi_E) = \xi_w^{-1}(\varpi_E)(\Xi_0 \circ N_{E/E_0})(\varpi_E).$$

Here proj^J_{J/J_1} is the natural projection $J \to J/J^1 \cong \mathrm{GL}_{|E/E_0|}(\mathbf{k}_{E_0})$. (Compare this to (5.2.2) of [3] concerning β-extensions.) On the other hand, it can be checked that $\Lambda(\xi_t) = \Lambda(\xi_{-1}) \otimes \alpha^{-1}$. Indeed by construction in [6] ξ_w is trivial on μ_E. This implies that the Green's representations $\bar{\lambda}_t$ and $\bar{\lambda}_{-1} \otimes (\Xi_0 \circ \det_{\mathbf{k}_{E_0}})$ on $\mathrm{GL}_{|E/E_0|}(\mathbf{k}_{E_0})$ are isomorphic. Therefore $\Lambda(\xi_t) \otimes \Lambda(\xi_w) = \Lambda(\xi_{-1}) \otimes \Lambda_0$. With the Howe factorization of ξ in hand, it is more natural to define our wild and tame component of ξ as in the five steps in the preceding paragraph. □

We analysis the group extensions in step (ii) and (iii) in the five steps of constructing supercuspidals. Since the group $\mathbf{J}$ normalizes J^1 and H^1, it acts on the quotient group $V = V(\xi) = J^1/H^1$. We usually regard V as an $\mathbb{F}_p$-vector space. We have a direct sum

$$V = V_{E_0/E_1} \oplus \cdots \oplus V_{E_d/F}, \tag{7.2.5}$$

where $V_{E_i/E_{i+1}} = U^{j_i}_{\mathfrak{B}_{i+1}}/U^{j_i}_{\mathfrak{B}_i}U^{h_i}_{\mathfrak{B}_{i+1}}$. By the definitions in (7.2.3) the module V_i is non-trivial if and only if the jump r_i is even, in which case we have $V_{E_i/E_{i+1}} \cong \mathfrak{B}_{i+1}/\mathfrak{B}_i{+}\mathfrak{P}_{\mathfrak{B}_{i+1}}$. We call this sum the coarse decomposition of V.

Proposition 7.3. *Let $H^1, J^1, \mathbf{J}, V, V_{E_i/E_{i+1}}$, and θ be those previously described.*

(i) The commutator subgroup $[J^1, J^1]$ lies in H^1.

(ii) The group $\mathbf{J}$ normalizes each component $V_{E_i/E_{i+1}}$ and the simple character θ.

(iii) The simple character θ induces a non-degenerated alternating $\mathbb{F}_p$-bilinear form $h_\theta : V \times V \to \mathbb{C}^\times$ such that the coarse decomposition is an orthogonal sum.

Proof. Some of the proofs can be found in [3] chapter 3, for example (i) is in (3.1.15), the non-degeneracy of the alternating form in (iii) is in (3.4), and that $\mathbf{J}$ normalizes θ in (ii) is from (3.2.3). That $\mathbf{J}$ normalizes each $V_{E_i/E_{i+1}}$ in (ii) is clear by definition. That the coarse decomposition is orthogonal in (iii) is from 6.3 of [8]. □

What are we interested in is the conjugate action of $E^\times$ on V restricted form $\mathbf{J}$. By Proposition 7.3(ii), The action of $\mathbf{J}$, and hence that of $E^\times$, preserves the symplectic structure defined by θ. By Proposition 7.3(i), the subgroup J^1 of $\mathbf{J}$ acts trivially on V, so the $E^\times$-action factors through $E^\times/F^\times(E^\times \cap J^1) \cong \Psi_{E/F}$. Hence V is moreover a finite symplectic $\mathbb{F}_p\Gamma$-module for each cyclic subgroup Γ of $\Psi_{E/F}$. By construction the $\mathbf{k}_F\Psi_{E/F}$-module V is always a submodule of the standard one $U = \mathfrak{A}/\mathfrak{P}_{\mathfrak{A}}$. We denote the $U_{[g]}$-isotypic component in V by $V_{[g]}$, and call the decomposition

$$V = \bigoplus_{[g]\in(\Gamma_E\backslash\Gamma_F/\Gamma_E)'} V_{[g]}$$

the complete decomposition of V.

Theorem 7.4. *The complete decomposition of V is an orthogonal sum with respect to the alternating form h_θ.*

Proof. Recall the bijection $\Gamma_F\backslash\Phi \to (\Gamma_E\backslash\Gamma_F/\Gamma_E)'$ in Proposition 2.1 and write $V_{[g]}$ as $V_{[\lambda]}$ for suitable $[\lambda] \in \Gamma_F\backslash\Phi$. For every λ and $\mu \in \Phi$ such that $\lambda \neq \mu$ or μ^{-1}, choose a finite field extension of $\mathbb{F}_p$, for example $\mathbb{F}_{\lambda\mu} = \mathbb{F}_p[\lambda(\Psi_{E/F}), \mu(\Psi_{E/F})]$, such that $V_{[\lambda]} \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda\mu}$ and $V_{[\mu]} \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda\mu}$ decomposes into eigenspaces of $\Psi_{E/F}$. Let $v \in V_{[\lambda]}$ and $w \in V_{[\mu]}$. We can assume that v and w are respectively contained in certain eigenspaces of $V_{[\lambda]} \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda\mu}$ and $V_{[\mu]} \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda\mu}$. There is a unique $\Psi_{E/F}$-invariant alternating bilinear form $\tilde{h}$ of $V \otimes_{\mathbb{F}_p} \mathbb{F}_{\lambda\mu}$ extending $h = h_\theta$. Therefore $\tilde{h}(v, w) = \tilde{h}({}^tv, {}^tw) = \lambda(t)\mu(t)\tilde{h}(v, w)$ for all $t \in \Psi_{E/F}$. The fact that $\lambda \neq \mu^{-1}$ implies $h(v, w) = \tilde{h}(v, w) = 0$. □

For our purpose it is not necessary to know the form h_θ exactly. Indeed by Proposition 6.6 (iii) the symplectic structure of V is determined by its underlying $\mathbb{F}_p\Psi_{E/F}$-module structure. We conclude this section by proving a promised fact, that not all components of U appears in the symplectic module V.

Proposition 7.5. *Let E/F be a tame extension and ξ run through all admissible characters in $P(E/F)$.*

(i) If $[g] \in \Gamma_E \backslash \Gamma_{E_0}/\Gamma_E = \Gamma_{E_0}/\Gamma_E$, then $V_{[g]}$ is always trivial.

(ii) If $e = e(E/F)$ is even, then $V_{[\sigma^{e/2}]}$ is always trivial.

Proof. The first statement is clear from the definition of $J^1(\xi)$ and $H^1(\xi)$ in (7.2.4). For the second statement, let $E_j \supsetneq E_{j+1}$ be the intermediate subfields in (7.1.2) such that $e(E/E_{j+1})$ is even and $e(E/E_j)$ is odd. By Proposition 7.1(iv) the jump r_j must be odd, so $V_{E_j/E_{j+1}}$ is trivial. Since $\sigma^{e/2} \in \Gamma_{E_{j+1}} - \Gamma_{E_j}$, the component $V_{[\sigma^{e/2}]}$ is contained in $V_{E_j/E_{j+1}}$ and hence is also trivial. □

7.3 Explicit values of rectifiers

For each character ξ we give the values of the rectifier ${}_F\mu_\xi$ following [6], [7] and [9]. Recall that each rectifier ${}_F\mu_\xi$ admits a factorization as in (3.4.2) in terms of ν-rectifiers. Since each factor being tamely ramified, it is enough to give its values on μ_E and at ϖ_E.

We would distinguish between the following cases

(I) E/K_l is totally ramified of odd degree,

(II) each K_i/K_{i-1}, $i = 1, \dots, l$, is totally ramified quadratic, and

(III) K_0/F is unramified

as in section 3.3. The case (I) is the easiest. By Theorem 4.4 of [6] we have

$$ {}_{E/K_l}\mu_\xi|_{\mu_E} \equiv 1 \text{ and } {}_{E/K_l}\mu_\xi(\varpi_E) = \left(\frac{q^f}{e(E/K_l)}\right). \tag{7.3.1}$$

We consider case (II). For any field extension K, we denote

$$\mathfrak{G}_K = \mathfrak{G}_K(\varpi_K, \psi_K),\ \mathfrak{K}_K = \mathfrak{K}_K(\alpha_0, \varpi_K, \psi_K) \text{ and } \mathfrak{K}^\kappa_K = \mathfrak{K}^\kappa_K(\alpha_0, \varpi_K, \psi_K)$$

for various Gauss sums and Kloostermann sums in chapter 8 and 9 of [7]. By Theorem 6.6 of [7] we have

$$\begin{aligned} &{}_{K_l/K_{l-1}}\mu_\xi|_{\mu_E} = \left(\frac{\quad}{\mu_E}\right) \text{ and} \\ &{}_{K_l/K_{l-1}}\mu_\xi(\varpi_E) = t_\varpi(V_{K_l/K_{l-1}})\operatorname{sgn}((\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l})(\mathfrak{K}^\kappa_{K_{l-1}}/\mathfrak{K}_{K_{l-1}})) \end{aligned} \tag{7.3.2}$$

and for $j = l-1, \dots, 1$

$$\begin{aligned} &{}_{K_j/K_{j-1}}\mu_\xi|_{\mu_E} \equiv 1 \text{ and} \\ &{}_{K_j/K_{j-1}}\mu_\xi(\varpi_E) = t_\varpi(V_{K_j/K_{j-1}})\operatorname{sgn}((\mathfrak{G}_{K_{j-1}}/\mathfrak{G}_{K_j})(\mathfrak{K}^\kappa_{K_{j-1}}/\mathfrak{K}_{K_{j-1}})). \end{aligned} \tag{7.3.3}$$

Here $\operatorname{sgn}(x) = x/|x|^{-1}$ for all $x \in \mathbb{C}$. We need more notations to compute the explicit values of these signs. For convenience we assume that E/F is totally ramified and K/F is a quadratic sub-extension of E. We further distinguish between the cases of (7.3.2) and (7.3.3) above, i.e. when E/K is of odd or even degree respectively. Suppose the character ξ is admissible over F with jump data $E = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_d \supsetneq F$ and $1 \leq r_0 < \cdots < r_d$. We denote

(i) r_S to be the largest odd jump, and

(ii) T to be the index such that $|E_{T+1}/F|$ is odd.

Lemma 7.6. *(i) $T \geq S$, and $T = S$ if and only if r_T is odd;*

(ii) $|E_i/E_{i+1}|$ is odd for $i < S$, $|E_S/E_{S+1}|$ and r_{S+1} are even;

(iii) $|E_T/E_{T+1}|$ is even, and for $i > T$, we have $|E_i/E_{i+1}|$ is odd and r_i are even.

Proof. We just make use of the divisibility and the gcd condition in Proposition 7.1 (i) and (iv).

(i) If $T < S$, then the odd r_S is divisible by $|E/E_S|$ which is in turn divisible by E/E_{T+1} which is even by definition. It is clear that $T = S$ implies r_T being odd. If r_T is odd, then $|E/E_T|$ is also odd. This forces $S = T$ by definition.

(ii) If $i < S$, then $|E_i/E_{i+1}|$ divides r_S and hence is odd. That r_{S+1} being even is by definition. If $|E_S/E_{S+1}|$ is odd, then applying $\gcd(r_i, |E/E_{i+1}|) = |E/E_i|$ for $i \geq S+1$ to show that all $|E_i/E_{i+1}|$ are odd for $i \geq S$. Hence $|E_S/E_{S+1}|$ must be even.

(iii) Again by definition and divisibility.

□

In [7], the symbols r_S, r_T, $|E_{S+1}/F|$ and ζ_S are denoted by i^+, i_+, d^+ and $\zeta(\varpi) = \zeta(\varpi, \xi)$ respectively. We would follow their notation and state the values of the signs of those quotient-sums in (7.3.2) and (7.3.3). The citations below are all from [7]. (Notice that there is a shift of index between ours and those in [7], e.g. our r_0 is denoted by r_1 in [7].)

(i) If the first jump r_0 is > 1, then $\mathfrak{K}_F^\kappa/\mathfrak{K}_F = 1$ by convention and

$$\mathrm{sgn}(\mathfrak{G}_F/\mathfrak{G}_K) = \begin{cases} \left(\frac{-1}{q}\right)^{\frac{i^+-1}{2}} \left(\frac{d^+}{q}\right) \left(\frac{\zeta(\varpi)}{q}\right) \mathfrak{n}(\psi_F)^{e/2d^+} & \text{if } e/2 \text{ is odd, by Corollary 8.3} \\ \left(\frac{-1}{q}\right)^{ei_+/4} & \text{if } e/2 \text{ is even, by Proposition 8.4} \end{cases}.$$

(ii) When $r_0 = 1$, there are three possible cases $i_+ = i^+ = r_0 = 1$, $i_+ > i^+ = r_0 = 1$ and $i_+ \geq i^+ > r_0 = 1$.

(a) When $i_+ = i^+ = r_0 = 1$, we have $\mathfrak{G}_F = \mathfrak{G}_K = 1$ by Lemma 8.1.(1) and

$$\mathrm{sgn}(\mathfrak{K}_F^\kappa/\mathfrak{K}_F) = \begin{cases} \left(\frac{d^+}{q}\right) \left(\frac{\zeta(\varpi)}{q}\right) \mathfrak{n}(\psi_F)^{e/2d^+} & \text{if } e/2 \text{ is odd} \\ \left(\frac{-1}{q}\right)^{e/4} & \text{if } e/2 \text{ is even} \end{cases}$$

both by section 9.3.

(b) When $i_+ > i^+ = r_0 = 1$, we show that this forces that i_+ and $e/2$ are even. That i_+ being even is from Lemma 1.2.(1). Suppose $e/2$ is odd. Here $i^+ = 1$ implies that the set of jumps equals $\{1, r_1, \dots, r_d\}$, where r_i are all even for $i > 0$. The condition (iv) of Proposition 7.1 implies $e(E/E_i)$ and $e(E/E_{i+1})$ have the same parity for all i. Since $e(E/E_{d+1}) = e(E/F)$ is even, we have that $e(E/E_1)$ is also even. Hence $e(E_1/F)$ divides $e/2$ and is odd. This implies $i_+ = 1$ which is a contradiction. We have

$$\mathfrak{G}_F = \mathfrak{G}_K = 1 \text{ and } \mathfrak{K}_F^\kappa = \mathfrak{K}_F$$

by Lemma 8.1.(1) and Proposition 8.1 respectively.

(c) When $i_+ \geq i^+ > r_0 = 1$, we have

$$\mathrm{sgn}(\mathfrak{G}_F/\mathfrak{G}_K) = \begin{cases} \left(\frac{-1}{q}\right)^{\frac{i^+-1}{2}} \left(\frac{d^+}{q}\right) \left(\frac{\zeta(\varpi)}{q}\right) \mathfrak{n}(\psi_F)^{e/2d^+} & \text{if } e/2 \text{ is odd, by Corollary 8.3} \\ \left(\frac{-1}{q}\right)^{ei_+/4} & \text{if } e/2 \text{ is even, by Proposition 8.4} \end{cases}.$$

and $\mathfrak{K}_F^\kappa/\mathfrak{K}_F = 1$ by Lemma 8.1(3).

Therefore in all cases we have

Proposition 7.7.

$$\mathrm{sgn}((\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)) = \begin{cases} \left(\frac{-1}{q}\right)^{\frac{i^+-1}{2}} \left(\frac{d^+}{q}\right) \left(\frac{\zeta(\varpi)}{q}\right) \mathfrak{n}(\psi_F)^{e/2d^+} & \textit{if } e/2 \textit{ is odd} \\ \left(\frac{-1}{q}\right)^{ei_+/4} & \textit{if } e/2 \textit{ is even} \end{cases}.$$

We refer the reader to chapter 7 of [7] for the explicit values of the t-factors $t_\varpi(V_{K_j/K_{j-1}})$ in (7.3.2) and (7.3.3). We would not need them in this article.

We consider case (III), i.e. when K_0/F is unramified and E/K_0 is totally ramified. Let V^ϖ be the subspace of fixed points of the subgroup ϖ of $\Psi_{E/F}$. We have, by the Main Theorem 5.2 of [9],

$${}_{K_0/F}\mu_\xi|_{\mu_E} \equiv t_\mu^1(V_{K_0/F}) \text{ and } {}_{K_0/F}\mu_\xi(\varpi_E) = (-1)^{e(f-1)} t_\mu^0(V/V^\varpi) t_\varpi(V_{K_0/F}).$$

Again we refer the reader to chapter 7 and 8 of [9] for the explicit values of the above t-factors. We would comment on the one we needs in section 8.5.

If f is even, we denote r_U the jump such that $f(E/E_U)$ is odd and $f(E/E_{U+1})$ is even. To compute the sign $t^0_\mu(V) = t^0_\mu(V_{K_0/F})$, we have to check those anisotropic $\mathbb{F}_p(\mu)$-submodule $V_{[g]}$ in $V_{K_0/F}$, i.e. those of the form $V_{[\sigma^k\phi^{f/2}]}$, such that the corresponding jump is even. In this case, by Proposition 6.10 we have $t^0_\mu(V_{[\sigma^k\phi^{f/2}]}) = -1$. Since either $V_{[\sigma^k\phi^{f/2}]}$ or $W_{[\sigma^k\phi^{f/2}]}$ is non-trivial, and $t(W_{[\sigma^k\phi^{f/2}]}) = -1$ if $W_{[\sigma^k\phi^{f/2}]}$ is symmetric non-trivial, we have

$$t(W_{[\sigma^k\phi^{f/2}]}) = -t^0_\mu(V_{[\sigma^k\phi^{f/2}]}), \text{ if } [\sigma^k\phi^{f/2}] \text{ is symmetric.} \tag{7.3.4}$$

Indeed we have computed the sign $t^0_\mu(V_{K_0/F})$ somewhere else.

Proposition 7.8. *The sign $t^0_\mu(V_{K_0/F})$ is -1 if and only if there exists an even jump r_U such that*

$$f(E/E_U) \text{ is odd, } f(E/E_{U+1}) \text{ is even and } e(E/E_{U+1}) \text{ is odd.} \tag{7.3.5}$$

Proof. We refer to Corollary 8 of 8.3 in [9]. □

We restate the result as follows. We denote $f_0 = f(E/E_0) = |E/E_0|$ and recall the jump r_S defined right before Lemma 7.6.

Proposition 7.9. *(i) If f_0 is even, or if f_0 is odd, e is even and $S \leq U$, then $t^0_\mu(V_{K_0/F}) = 1$.*

(ii) If f_0 and e are odd, or if f_0 is odd, e is even and $S > U$, then $t^0_\mu(V_{K_0/F}) = (-1)^{r_U+1}$.

Proof. If f_0 is even, then all $f(E/E_i)$ are even for $i \geq 0$ and the parity condition (7.3.5) cannot occur. If f_0 is odd and e is even, then the condition $S \leq U$ implies that $e(E/E_{U+1})$ must be even by Proposition 7.1(i) and again (7.3.5) cannot occur. In the remaining cases, (7.3.5) can occur, depending the parity of the jump r_U. The remaining is then direct. □

For the sign $t^0_\mu(V^\varpi) = t^0_\mu(V^\varpi_{K_0/F})$, we have to check those anisotropic $\mathbb{F}_p(\mu)$-submodule $V_{[\sigma^k\phi^{f/2}]}$ with even jump and on which ϖ acts trivially. We give some equivalent conditions for such $\sigma^k\phi^{f/2}$ exists. Recall from (2.2.1) that $\varpi_E^e = \zeta_{E/F}\varpi_F$ and from section 6.2 that ϖ_E acts on $V_{[\sigma^k\phi^{f/2}]}$ by multiplying $\zeta_e^k\zeta_{\phi^{f/2}}$.

Lemma 7.10. *The following are equivalent.*

(i) There exists $\sigma^k\phi^{f/2} \in W_{F[\varpi_E]}$,

(ii) $\zeta_e^k\zeta_{\phi^{f/2}} = 1$ *for some* k, *i.e.* $\zeta_{\phi^{f/2}}$ *is an* e*th root of unity,*

(iii) $\zeta_{E/F} \in K_+$ *where* K_0/K_+ *is quadratic unramified, and*

(iv) $f_\varpi = |E/F[\varpi_E]|$ *is even.*

Proof. (i)⇔(ii) is clear since ${}^{\sigma^k\phi^{f/2}}\varpi_E = \zeta_e^k\zeta_{\phi^{f/2}}\varpi_E$. For (iii)⇔(ii), recall that $\zeta_{\phi^{f/2}}$ is an eth root of $\zeta_{E/F}^{q^{f/2}-1}$. If $\zeta_{E/F} \in K_+$, then $\zeta_{E/F}^{q^{f/2}-1} = 1$ and $\zeta_{\phi^{f/2}}$ is an eth root of unity. The converse is similar. For (iii)⇔(iv), notice that $f(F[\varpi_E]/F) = f(F[\zeta_{E/F}]/F) = f/f_\varpi$. Hence $F[\zeta_{E/F}] \subseteq K_+ \Leftrightarrow f/f_\varpi$ divides $f/2 \Leftrightarrow f_\varpi$ is even. □

Remark 7.11. By (ii) we know that such $\sigma^k\phi^{f/2}$ is unique, if it exists. □

Chapter 8

Comparing character formulae

We proof the first main result Theorem 8.5, that the automorphic induction formula and the spectral transfer relation formula are equal under the Whittaker normalization in the following cases.

(I) $E = K$ and K/F is cyclic totally ramified of odd degree [6],

(II) E/F is totally ramified and K/F is quadratic [7], and

(III) E/K is totally ramified and K/F is unramified [9].

We first explain the normalization for both formulae in section 8.1. We then provide three lemmas and explain the constants which relate the two formulae in section 8.2. We finally compute the value of such constants and deduce the main results in the cases (I)-(III) in section 8.3-8.5 respectively.

8.1 Whittaker normalizations

In this section we make use of the F-Borel subgroup $\mathbf{B}$ in the chosen splitting $\mathbf{spl}_G$. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$, which is also defined over F. The simple roots $\{X_\alpha\}_{\alpha\in\Delta}$

in $\mathbf{spl}_G$ gives rise to a morphism

$$\mathbf{N} \to \bigoplus_{\alpha\in\Delta} \mathbb{G}_a$$

defined only over $\bar{F}$. We then compose this morphism with the summing up

$$\bigoplus_{\alpha\in\Delta} \mathbb{G}_a \to \mathbb{G}_a,\ (c_\alpha)_{\alpha\in\Delta} \mapsto \sum_{\alpha\in\Delta} c_\alpha.$$

Then the composition

$$\mathbf{N} \to \bigoplus_{\alpha\in\Delta} \mathbb{G}_a \to \mathbb{G}_a$$

is then defined over F, and we get a character

$$U := \mathbf{N}(F) \to F.$$

We choose an non-trivial additive character $\psi_F : F \to \mathbb{C}^\times$ and get a non-degenerate

$$\psi : U \to F \to \mathbb{C}^\times$$

by composing the morphisms. We call this pair (U, ψ) a Whittaker-datum for $G(F)$. In case $U = U_0$ the upper triangular unipotent subgroup and

$$\psi_0 : \begin{pmatrix} 1 & x_1 & & * \\ & 1 & \ddots & \\ & & \ddots & x_{n-1} \\ 0 & & & 1 \end{pmatrix} \mapsto \sum_{i=1}^{n-1} x_i,$$

we call (U_0, ψ_0) the standard Whittaker datum for $G(F)$.

Kottwitz and Shelstad defined a normalization of the transfer factor Δ_0 which depends only on the Whittaker datum. We define $\epsilon_{\mathrm{L}}(V_{G/H})$ to be the local constant $\epsilon_{\mathrm{L}}(V_{G/H}, \psi_F)$ in (3.6) of [30], depending on the additive character ψ_F of F. Here $V_{G/H}$ is the virtual W_F-module

$$X^*(\mathbf{T}_G) - X^*(\mathbf{T}_H)$$

of degree 0. When $G = \mathrm{GL}_n$ and $H = \mathrm{Res}_{K/F}\mathrm{GL}_m$ we have

$$X^*(\mathbf{T}_G) \cong (1_{W_F})^{\oplus n} \text{ and } X^*(\mathbf{T}_H) \cong (\mathrm{Ind}_{K/F} 1_{W_K})^{\oplus m}.$$

Hence

$$\epsilon_{\mathrm{L}}(V_{G/H},\psi_F)=\epsilon_{\mathrm{L}}(1_F,\psi_F)^n\epsilon_{\mathrm{L}}(\mathrm{Ind}_{K/F}1_K,\psi_F)^{-m}.$$

Let $\lambda_{K/F}=\lambda_{K/F}(\psi_F)$ be the Langlands constant (see section 2.4 of [24]), depending on the additive character ψ_F of F. Since ϵ_L equals 1 on trivial modules and satisfies the induction property

$$\epsilon_{\mathrm{L}}(\mathrm{Ind}_{K/F}\sigma,\psi_F)=\lambda_{K/F}\epsilon_{\mathrm{L}}(\sigma,\psi_K)$$

for every finite dimensional W_K representation σ, we have

$$\epsilon_{\mathrm{L}}(V_{G/H})=\lambda_{K/F}^{-m}.$$

We need the values of $\lambda_{K/F}$ in cases (I)-(III) when ψ_F is of level 0, i.e. $\psi_F|\mathfrak{p}_F\equiv 1$. They are computed in section 2.5 of [24].

(I) If K/F is totally ramified of odd degree e and $\#\mathbf{k}_F=q$, then $\lambda_{K/F}=\left(\frac{q}{e}\right)$.

(II) If K/F is quadratic totally ramified, then $\lambda_{K/F}=\mathfrak{n}(\psi_F)$ the normalized Gauss sum defined in section 6.3.

(III) If K/F is unramified of degree f, then $\lambda_{K/F}=(-1)^{f-1}$.

Proposition 8.1. *The product $\epsilon_{\mathrm{L}}(V_{G/H})\Delta_0$ depends only on the Whittaker datum (U,ψ) and is independent of the choices of ψ and $\mathbf{spl}_G$ that giving rise to (U,ψ).*

Proof. We refer the proof to section 5.3 of [18]. □

We then introduce the Whittaker normalization on the automorphic induction defined in section 3.3. Suppose $\pi\in\mathcal{A}_n^{\mathrm{et}}(F)$ and V is the vector space realizing π. It is well-known that any supercuspidal π is generic, i.e. there exists a $G(F)$-morphism

$$(\pi,V)\to\mathrm{Ind}_U^{G(F)}\psi$$

which is unique up to scalar. We call this morphism a Whittaker model of π. Equivalently, there exists a U-linear functional $\lambda=\lambda_\psi$ of V, in the sense that

$$\lambda(\pi(n)v)=\psi(n)\lambda(v)\text{ for all }u\in U,v\in V.$$

Suppose (π, V) is automorphically induced and $\Psi : \kappa\pi \to \pi$ be an intertwining operator. We call Ψ Whittaker-normalized, or more precisely (U, ψ)-normalized, if we have

$$\lambda \circ \Psi = \lambda.$$

Remark 8.2. Since all Whittaker data are $G(F)$-conjugate, the genericity of π is independent of the choice of Whittaker datum. If we realize the Whittaker space $\mathrm{Ind}_U^{G(F)}\psi$ consisting of functions $f : G(F) \to \mathbb{C}$ satisfying

$$f(ux) = \psi(u)f(x) \text{ for all } u \in U, x \in G(F),$$

then the Whittaker spaces $\mathrm{Ind}_U^{G(F)}\psi$ and $\mathrm{Ind}_{U^g}^{G(F)}\psi^g$, for some $g \in G(F)$, are isomorphic by the left-translation

$$\mathrm{Ind}_U^{G(F)}\psi \to \mathrm{Ind}_{U^g}^{G(F)}\psi^g, f \mapsto (x \mapsto f(g(x))).$$

□

Proposition 8.3. *For each $\rho \in \mathcal{A}^{\mathrm{et}}_{n/d}(K)$ with automorphic induction π, if the intertwining operator Ψ is (U_0, ψ_0)-normalized, then the constant $c(\rho, \kappa, \Psi)$ is independent of ρ.*

Proof. In 4.11. Théorème of [13]. □

We denote $c(\rho, \kappa, \Psi)$ as $c(\kappa, \psi_0)$ in this case. Using a global method, Henniart and Lemaire has computed the value of $c(\kappa, \psi_0)$ when K/F is unramified of degree d. More precisely, combining 3.10 Proposition and 4.11 Théorème in [13] we can show that

(i) When ψ_F is unramified, i.e. $\psi_F|\mathfrak{o}_F \equiv 1$, we have $c(\kappa, \psi_0) = 1$.

(ii) When ψ_F is of level 0, i.e. $\psi_F|\mathfrak{p}_F \equiv 1$, we have $c(\kappa, \psi_0) = (-1)^{m(d-1)}$, where $m = n/d$.

Proposition 8.4. *When K/F is unramified of degree d and ψ_F is unramified or of level 0, we have $c(\kappa, \psi_0) = \epsilon_{\mathrm{L}}(V_{G/H}, \psi_F)$.*

Proof. When ψ_F is of level 0, we have computed that $\lambda_{K/F} = (-1)^{f-1}$. If ψ_F is unramified, then we can follow the computations in section 2.5 of [24] and show that $\lambda_{K/F} = 1$. □

In this chapter we will prove the following

Theorem 8.5. *In the settings (I)-(III), the automorphic induction character (3.3.1) and the spectral transfer formula (4.1.3) are equal under the Whittaker normalization by* (U_0, ψ_0).

More precisely, we are going to show that the normalized automorphic induction character formula

$$\Theta_\pi^{\kappa,\Psi}(\gamma) = c(\kappa, \psi_0)\Delta^2(\gamma)\Delta^1(\gamma)^{-1} \sum_{g \in \Gamma_{K/F}} \Theta_\rho^g(\gamma)$$

and the normalized spectral transfer formula

$$\Theta_\pi^{\kappa}(\gamma) = \epsilon_{\mathrm{L}}(V_{G/H})\Delta_{\mathrm{I,II,III}}(\gamma)\Delta_{\mathrm{IV}}(\gamma)^{-1} \sum_{g \in \Gamma_{K/F}} \Theta_\rho^g(\gamma)$$

are equal for all γ lies in the elliptic torus $E^\times$ of $H(F)$. Clearly it suffices to show the equality for both transfer factors under the (U_0, ψ_0)-normalization, which is

$$c(\kappa, \psi_0)\Delta^2(\gamma) = \epsilon_L(V_{G/H})\Delta_{\mathrm{II,III}}(\gamma). \tag{8.1.1}$$

We would actually verify a variant (8.2.1) of this formula. Since we already know that $\Delta^2 = \Delta_{\mathrm{II,III}}$ up to a constant (a sign), it is enough to verify this equality on a particular choices of γ. We will specify this γ in the different cases (I)-(III) in the subsequent sections.

8.2 Three Lemmas

We give three known lemmas to see how the above procedure works. These lemmas concern about the relation between the internal structure of our supercuspidal π and various Whittaker data.

Let $(E/F, \xi) \in P_n(F)$ be the class of admissible pair that corresponds to $\pi \in \mathcal{A}_n^{\mathrm{et}}(F)$ and $\mathbf{V}$ be the vector space realizing π. From section 7.2 we know that there is a representation $(\mathbf{J}_\xi, \Lambda)$ compactly inducing π, i.e. $\pi = \mathrm{cInd}_{\mathbf{J}_\xi}^{G(F)}\Lambda$. Let $g \in G$ be the element that intertwines the standard Whittaker datum (U_0, ψ_0) and $(\mathbf{J}_\xi, \Lambda)$. This is equivalent to require that $\Lambda|_{\mathbf{J}_\xi \cap {}^g U_0}$ contains the character $\psi_0|_{\mathbf{J}_\xi \cap {}^g U_0}$. In [9] we call $({}^g U_0, {}^g\psi_0)$ being adopted to $(\mathbf{J}_\xi, \Lambda)$. By the uniqueness of Whittaker model and Frobenius reciprocity, there is a unique double coset in $\mathbf{J}\backslash G/U_0$ whose elements intertwine (U_0, ψ_0) and $(\mathbf{J}_\xi, \Lambda)$.

We now setup the first lemma. Suppose that $\kappa\pi \cong \pi$. Take a Whittaker datum (U, ψ) and a model $(\pi, \mathbf{V}) \hookrightarrow \mathrm{Ind}_U^{G(F)}\psi$. Notice that the morphism is also a model for $(\kappa\pi, \mathbf{V})$. Define a bijective operator $\mathbf{\Psi}$ on $\mathrm{Ind}_U^{G(F)}\psi$ by

$$\mathbf{\Psi} f : g \mapsto \kappa(\det(g))f(g)$$

for all $f \in \mathrm{Ind}_U^{G(F)}\Psi$ and $g \in G$. This operator satisfies

$$\kappa(\det(g))(\mathbf{\Psi} \circ \rho(g)) = \rho(g) \circ \mathbf{\Psi}.$$

Lemma 8.6. *If the intertwining operator* $\Psi : \kappa\pi \to \pi$ *is* (U, ψ)*-normalized, then the diagram*

$$\begin{array}{ccc} (\kappa\pi, \mathbf{V}) & \xrightarrow{\Psi} & (\pi, \mathbf{V}) \\ \downarrow & & \downarrow \\ \mathrm{Ind}_U^{G(F)}\psi & \xrightarrow{\mathbf{\Psi}} & \mathrm{Ind}_U^{G(F)}\psi \end{array}$$

commutes.

Proof. see section 1.5 of [9]. □

To set up the second lemma, we assume from now on that

$$\mathbf{J} \subseteq \ker(\kappa \circ \det).$$

We can check easily that this condition is satisfied in our cases (I)-(III).

Lemma 8.7. *Suppose* (U, ψ) *is adopted to* $(\mathbf{J}, \Lambda)$*. If* $\Psi : \kappa\pi \to \pi$ *is* (U, ψ)*-normalized, then* Ψ *acts on the* Λ*-isotypic component* π^Λ *of* π *as identity.*

Proof. Identify π with its Whittaker model in $\mathrm{Ind}_U^{G(F)}\psi$. If (U,ψ) is adopted to $(\mathbf{J},\Lambda)$, then π^Λ isomorphic to the functions in $\mathrm{Ind}_U^{G(F)}\psi$ whose supports are all lie in the identity double coset $\mathbf{J}U$ of $\mathbf{J}\backslash G/U$. Since $\mathbf{J}U \subseteq \ker\kappa$, the operator Ψ acts on these functions as identity. By Lemma 8.6 we have that $\Psi|_{\pi^\Lambda}$ is also identity. For more details, see Lemma 3.(2) of [9]. □

For the last lemma, let $\alpha \in E^\times$ be the element defined in Proposition 2.4 and Definition 2.7 of [5]. Indeed this α is our $\beta = \beta(\xi)$ defined in (7.1.3) when $E = E_0$, and is $\beta(\xi) + \zeta$ for a chosen regular $\zeta \in \mu_E$ when $E \neq E_0$.

Lemma 8.8. *If (U,ψ) is adopted to $(\mathbf{J},\Lambda)$, then (U,ψ) can be identified to the standard Whittaker datum with respect to the basis $\mathfrak{a} = \{1, \alpha, ..., \alpha^{n-1}\}$.*

Proof. We refer the proof to Proposition 2.4 and Theorem 2.9 of [5]. □

We can now state the main result of this section. Recall that (U_0,ψ_0) is the standard Whittaker datum, the one defined with respect to the basis $\mathfrak{b}$ defined in (4.3.1). Write $\mathfrak{b} = \{b_1, \dots, b_m\}$ in this order. If the 'Galois set' $\Gamma_{E/F} = \{g_1, \dots, g_n\}$ is also ordered, then we write $V(\mathfrak{b})$ the Vandemonde determinant

$$\begin{vmatrix} {}^{g_1}b_1 & \cdots & {}^{g_1}b_n \\ \vdots & & \vdots \\ {}^{g_n}b_1 & \cdots & {}^{g_n}b_n \end{vmatrix}$$

Let (U,ψ) be the Whittaker-datum adopted to $(\mathbf{J},\Lambda)$ and Ψ^ψ be the (U,ψ)-normalized intertwining operator $\kappa\pi \to \pi$.

Proposition 8.9. *Let $c(\kappa,\psi_0)$ and $c(\kappa,\psi)$ be the automorphic induction constant with respect to Ψ^{ψ_0} and Ψ^ψ respectively. Then $c(\kappa,\psi_0) = \kappa(V(\mathfrak{a})V(\mathfrak{b})^{-1})c(\kappa,\psi)$.*

Proof. By Lemma 8.7 Ψ^ψ acts on π^Λ by identity. If we identify π to its model in $\mathrm{Ind}_{U_0}^{G(F)}\psi_0$, then by Lemma 8.6 Ψ^{ψ_0} acts on π^Λ by $\kappa(\det g)$, where $g \in G$ such that the functions in

$(\mathrm{Ind}_{U_0}^{G(F)}\psi_0)^\Lambda$ are supported in $\mathbf{J}gU_0$. By Lemma 8.8 this element defines the change-of-basis isomorphism from $\mathfrak{b}$ to $\mathfrak{a}$, so $\det(g) = V(\mathfrak{a})V(\mathfrak{b})^{-1}$. Since Ψ^ψ and Ψ^{ψ_0} are defined up to scalars, the result follows. □

Remark 8.10. The (U,ψ)-normalized intertwining operator Ψ^ψ is the one used to compare the automorphic induction formula with the Mackey induction formula (3.4.3). We refer to Lemma 3 in section 1.6 of [9] for the proof. □

The element $V(\mathfrak{a})V(\mathfrak{b})^{-1}$ is the element $x \in G(F)$ introduced in the Interlude, which is denoted by $x_{\mathfrak{ab}}$ subsequently. In the setting (I)-(III) above the constants $c(\kappa,\psi)$ are denoted by c_θ in [6], [7], [9] respectively, where we can explicitly compute their values in these cases by comparing the local characters of automorphic induction and the Mackey formulae of the compact induction. Therefore by Proposition 8.9 we would verify the following variant of the equality (8.1.1)

$$\kappa(x_{\mathfrak{ab}})c_\theta\Delta^2(\gamma) = \epsilon_L(V_{G/H})\Delta_{\mathrm{I,II,III}}(\gamma) \tag{8.2.1}$$

for certain choices of γ.

Before we move on we explain how to expand the product

$$V(\mathfrak{a}) = \prod_{\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right] \in \Phi(G,T)_+} ({}^g\alpha - {}^h\alpha).$$

Let E_i be an intermediate subfield in the jump-data of ξ. If $g|_{E_{i+1}} = h|_{E_{i+1}}$, then

$${}^g\alpha - {}^h\alpha = {}^g(\beta_i + \cdots + \beta_0 + \zeta) - {}^h(\beta_i + \cdots + \beta_0 + \zeta).$$

If further $g|_{E_i} \neq h|_{E_i}$, then we make use of the genericity of the Howe-factor ξ_i of ξ in Proposition 7.1(iii), i.e. the first term $\alpha_i = \zeta_i\varpi_E^{-r_i}$ of β_i is regular, such that

$${}^g\alpha - {}^h\alpha = ({}^g\alpha_i - {}^h\alpha_i)u \text{ for some } u \in U_E^1.$$

In case when $\mathfrak{b}$ is also a cyclic base, i.e. $\mathfrak{b} = \{1, y, \dots, y^{n-1}\}$ for some $y \in E^\times$, we have

$$\kappa(x_{\mathfrak{ab}}) = \kappa\left(\prod_{i=-1}^{d} \prod_{\substack{g|_{E_{i+1}} = h|_{E_{i+1}} \\ g|_{E_i} \neq h|_{E_i}}} \frac{{}^g\alpha_i - {}^h\alpha_i}{{}^g y - {}^h y}\right) \tag{8.2.2}$$

since κ is tamely ramified. Notice that for arbitrary bases $\mathfrak{a}$ and $\mathfrak{b}$, the constant $\kappa(x_{\mathfrak{ab}})$ is independent of the order of the Galois set $\Gamma_{E/F}$.

8.3 Case (I)

We first compute the factor $\kappa(x_{\mathfrak{ab}})$ in the formula (8.2.1) of case (I). We claim that

Proposition 8.11. $\kappa(x_{\mathfrak{ab}}) = 1$.

Proof. The factor $\kappa(x_{\mathfrak{ab}})$ equals

$$\kappa\left(\prod_{0\leq i<j\leq e-1} \frac{\sigma^i\beta - \sigma^j\beta}{\sigma^i\varpi_E - \sigma^j\varpi_E}\right) = \kappa\left(\prod_{k=1}^{e-1}\prod_{j=0}^{e-k-1} \frac{\sigma^j\beta - \sigma^{j+k}\beta}{\sigma^j\varpi_E - \sigma^{j+k}\varpi_E}\right).$$

Here the index k stands for the factor corresponds to the roots in the Γ-orbit of $\left[\begin{smallmatrix}1\\ \sigma^k\end{smallmatrix}\right]$. By (8.2.2) the above product equals

$$\kappa\left(\prod_{k=1}^{e-1}\prod_{j=0}^{e-k-1} \zeta^{a_{r_{i(k)}}}\varpi_E^{-(r_{i(k)}+1)}\frac{\zeta_e^{j(-r_{i(k)})} - \zeta_e^{(j+k)(-r_{i(k)})}}{\zeta_e^j - \zeta_e^{j+k}}\right).$$

Here $r_{i(k)}$ is the jump corresponds to $[\sigma^k]$. We consider the kth term and $(e-k)$th term. Notice that $i(k) = i(e-k)$. We can rewrite the quotients for the kth term

$$\frac{\zeta_e^{j(-r_{i(k)})} - \zeta_e^{(j+k)(-r_{i(k)})}}{\zeta_e^j - \zeta_e^{j+k}} = \zeta_e^{-j(r_{i(k)}+1)}\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k} \quad \text{for } j = 0,\dots,e-k-1$$

and the $(e-k)$th term

$$\frac{\zeta_e^{j(-r_{i(k)})} - \zeta_e^{(j+e-k)(-r_{i(k)})}}{\zeta_e^j - \zeta_e^{j+e-k}} = \zeta_e^{-(j+e-k)(r_{i(k)}+1)}\frac{\zeta_e^{k(-r_{i(k)})}-1}{\zeta_e^k-1} \quad \text{for } j = 0,\dots,k-1.$$

Hence by pairing up, we have

$$\begin{aligned}&\kappa\left(\prod_{k=1}^{\frac{e-1}{2}}\prod_{j=0}^{e-1}\zeta^{a_{r_{i(k)}}}\varpi_E^{-(r_{i(k)}+1)}\zeta_e^{-j(r_{i(k)}+1)}\frac{1-\zeta_E^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)\\ =&\kappa\left(\prod_{k=1}^{\frac{e-1}{2}}(\zeta^{a_{r_{i(k)}}})^e(\varpi_F^{-(r_{i(k)}+1)})(\zeta_e^{(e(e-1)/2)(r_{i(k)}+1)})\left(\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)^e\right).\end{aligned}$$

The first three factors are in $N_{K/F}(K^\times)$, so we get

$$\kappa\left(\prod_{k=1}^{\frac{e-1}{2}}\left(\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)^e\right) \tag{8.3.1}$$

The product is in $\mathfrak{o}_F^\times$. To claim that it equals 1, it suffices to show that

$$\prod_{k=1}^{\frac{e-1}{2}}\left(\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)^{e/d} \in F^\times. \tag{8.3.2}$$

We first compute the denominator (8.3.2), starting from the fact that

$$\prod_{k=1}^{e-1}(1-\zeta_e^k)=e.$$

By pairing up the indices, we have

$$\left(\prod_{k=1}^{\frac{e-1}{2}}(1-\zeta_e^k)\right)^2\left(\prod_{k=1}^{\frac{e-1}{2}}\zeta_e^{-k}\right)=e$$

Therefore

$$\left(\prod_{k=1}^{\frac{e-1}{2}}(1-\zeta_e^k)\right)^{e/d}=\left(\pm e^{1/2}\zeta_{2e}^{\frac{1}{2}(\frac{e-1}{2})(\frac{e+1}{2})}\right)^{e/d}=e^{e/2d}\left(\pm\zeta_{2d}^{\frac{1}{2}(\frac{e-1}{2})(\frac{e+1}{2})}\right).$$

The last factor $\left(\pm\zeta_{2d}^{\frac{1}{2}(\frac{e-1}{2})(\frac{e+1}{2})}\right)$ is in $F^\times$. We then compute the factor in the numerator of (8.3.2) for a fixed i, i.e.

$$\prod_{\substack{k=1,\dots,(e-1)/2\\ |E_{i+1}/F|\,|\,k,\ |E_i/F|\nmid k}}\left(1-\zeta_e^{-kr_i}\right)^{e/d}.$$

Recall from Proposition 7.1 that we can write kr_i as $\ell|E_{i+1}/F|\cdot s_i|E/E_i|$ where $\ell=1,\dots,|E/E_{i+1}|-1$ but $|E_i/E_{i+1}|\nmid\ell$. We have

$$\prod_{\substack{k=1,\dots,(e-1)/2\\ |E_{i+1}/F|\,|\,k,\ |E_i/F|\nmid k}}\left(1-\zeta_e^{-kr_i}\right)=\prod_{\substack{\ell=1,\dots,|E/E_{i+1}|-1\\ |E_i/E_{i+1}|\nmid\ell}}\left(1-\zeta_{|E_i/E_{i+1}|}^{-\ell s_i}\right)$$

Since $\gcd(s_i,|E_i/E_{i+1}|)=1$, the above equals

$$\prod_{\substack{\ell=1,\dots,|E/E_{i+1}|-1\\ |E_i/E_{i+1}|\nmid\ell}}\left(1-\zeta_{|E_i/E_{i+1}|}^{-\ell}\right)=\prod_{\ell=1}^{|E_i/E_{i+1}|-1}\left(1-\zeta_{|E_i/E_{i+1}|}^{-\ell}\right)^{|E/E_i|}=|E_i/E_{i+1}|^{|E/E_i|}.$$

Therefore using similar method as in he denominator of (8.3.2) we can compute that

$$\prod_{\substack{k=1,\dots,(e-1)/2 \\ |E_{i+1}/F|\,|k,\ |E_i/F|\nmid k}} \left(1-\zeta_e^{-kr_i}\right)^{e/d} = |E_i/E_{i+1}|^{e|E/E_i|/2d}\left(\pm\zeta_{2d}^*\right).$$

for certain power $(\pm\zeta_{2d}^*)$ of ζ_{2d} which is in $F^\times$. Hence the quotient (8.3.2) mod $F^\times$ is

$$e^{e/2d}\left(\prod_{i=0}^{d}|E_i/E_{i+1}|^{e|E/E_i|/2d}\right)^{-1} = \prod_{i=0}^{d}|E_i/E_{i+1}|^{\frac{e}{2d}\left(\frac{1-|E/E_i|}{2}\right)}$$

which is also in $F^\times$. Hence we have checked that $\kappa(x_{\mathfrak{ab}}) = 1$. □

We then check the remaining factors

(i) Using Theorem 4.3 of [6], we have

$$c(\kappa,\Lambda) =_{K/F} \mu_\xi(\varpi_E)t_\varpi(V_{K/F})(\Delta^2(\varpi_E)/\Delta^1(\varpi_E))^{-1}.$$

It turns out that ${}_{K/F}\mu_\xi(\varpi_E) = 1$ by Proposition 4.5 of [6] and $\Delta^1(\varpi_E) = 1$.

(ii) $\epsilon_L(V_{G/H}) = \lambda_{K/F}^{-e/d} = \left(\frac{q}{e(K/F)}\right) = 1$.

(iii) $\Delta_{\mathrm{I,II,III}}(\varpi_E) = 1$ by choosing appropriate a-data, see Proposition 4.14.

Hence all together we have

$$(c(\kappa,\Lambda)\Delta^2(\varpi_E))^{-1}\epsilon_L(V_{G/H})\Delta_{\mathrm{I,II,III}}(\varpi_E) = t_\varpi(V_{K/F}).$$

Now we would show that

Proposition 8.12. $t_\varpi(V_{K/F}) = 1$.

Proof. Recall that

$$V_{K/F} = \bigoplus_{[g]\in W_E\backslash W_F/W_E - W_E\backslash W_K/W_E} V_{[g]}$$

such that $V_{[g]}$ is non-trivial if the corresponding jump r_i for $[g]$ is even, and in which case $t_\varpi(V_{[g]})$ is 1 or -1 depending on whether $[g]$ is asymmetric or symmetric respectively. It is enough to show that, for fixed i, the number of symmetric $[g]$ with corresponding jump r_i is always even.

Recall the following decomposition of q-orbits of $\mathbb{Z}/e$, namely

$$W_F/W_E \xrightarrow{\sim} \mathbb{Z}/e \xrightarrow{\sim} \bigoplus_{a|e}(e/a)(\mathbb{Z}/a)^\times$$

such that W_K/W_E corresponds to

$$W_K/W_E \xrightarrow{\sim} d\mathbb{Z}/e \xrightarrow{\sim} \bigoplus_{a|m}(e/a)(\mathbb{Z}/a)^\times.$$

Hence

$$W_E\backslash W_F/W_E - W_E\backslash W_K/W_E \xrightarrow{\sim} \bigoplus_{a|e,\, a\nmid m} (e/a)\left(q\backslash(\mathbb{Z}/a)^\times\right).$$

For fixed a, the orbits in $(e/a)\left(q\backslash(\mathbb{Z}/a)^\times\right)$ have the same jump r_i. Hence it is enough to show that the cardinality of $q\backslash(\mathbb{Z}/a)^\times$ is even. This number equals $\phi(a)/\mathrm{ord}(q,a)$. Since $a \mid e$ and $a \nmid m$, we have $\gcd(a,d) \neq 1$. Consider the surjective morphism

$$(\mathbb{Z}/a)^\times \xrightarrow{\pi} (\mathbb{Z}/\gcd(a,d))^\times.$$

The subgroup $\langle q\rangle$ in $(\mathbb{Z}/a)^\times$ is in $\ker\pi$ since $d \mid q-1$. Hence $\phi(a)/\mathrm{ord}(q,a)$ is divisible by $\phi(a)/\#\ker\pi = \phi(\gcd(a,d))$ which is always even. □

We have verified (8.2.1) in case (I).

8.4 Case (II)

We would verify (8.2.1) in case (II), i.e. when e is even and $d=2$. We write $m=e/2$. We would verify that

Proposition 8.13. $\kappa(x_{\mathfrak{ab}}) =$

$$\begin{cases} \left(\frac{-1}{q}\right)^{\frac{d^+ + i^+}{2}} \left(\frac{d^+}{q}\right)\left(\frac{\zeta^{a r_S}}{q}\right) & \textit{if } e/2 \textit{ is odd} \\ \left(\frac{-1}{q}\right)^{\frac{e}{4}(i_+ - 1)} & \textit{if } e/2 \textit{ is even} \end{cases}.$$

We refer the notations above to case (II) in section 7.3.

Proof. we proceed as in case (I) up to (8.3.1), in which step we have an extra term

$$\kappa\left(\prod_{j=0}^{e/2-1}\frac{\sigma^j\beta-\sigma^{j+e/2}\beta}{\sigma^j\varpi_E-\sigma^{j+e/2}\varpi_E}\right)$$

corresponding to $k=e/2$. We recall the jump $r_S=i^+$ defined right before Lemma 7.6. The above equals

$$\kappa\left((\zeta^{a_{r_S}})^{e/2}\varpi_E^{-(r_S+1)e/2}\prod_{j=0}^{e/2-1}\zeta_e^{-j(r_S+1)}\frac{1-\zeta_e^{e/2(-r_S)}}{1-\zeta_e^{e/2}}\right)$$

Since r_S is odd and $-\varpi_F\in N_{K/F}(K^\times)$, the above equals

$$\begin{aligned}&\kappa\left((\zeta^{a_{r_S}})^{e/2}\varpi_F^{-\frac{r_S+1}{2}}(-1)^{-(\frac{r_S+1}{2})(e/2-1)}\right)\\ =&\kappa\left(\zeta^{a_{r_S}}(-1)^{(r_S+1)/2}\right)^{e/2}=\begin{cases}\left(\frac{\zeta^{a_{r_S}}}{q}\right)\left(\frac{-1}{q}\right)^{(r_S+1)/2} & \text{if } e/2 \text{ is odd}\\ 1 & \text{if } e/2 \text{ is even}\end{cases}.\end{aligned}\tag{8.4.1}$$

We then compute the remaining terms. If we proceed as in (8.3.1) to get rid of the factors in $N_{K/F}(K^\times)$, we get

$$\prod_{k=1}^{e/2-1}\left(\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)^e.$$

We then separate the product in half and regroup

$$\begin{aligned}&\kappa\left(\left(\prod_{k=1}^{e/2-1}\left(\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)\left(\frac{1-\zeta_e^{-k(-r_{i(k)})}}{1-\zeta_e^{-k}}\right)\zeta^{k(r_{i(k)}+1)}\right)^{e/2}\right)\\ =&\kappa\left(\left(\prod_{k=1}^{e-1}\frac{1-\zeta_e^{k(-r_{i(k)})}}{1-\zeta_e^k}\right)^{e/2}\left(\prod_{k=1}^{e/2-1}(-1)^{k(r_{i(k)}+1)}\right)\right).\end{aligned}\tag{8.4.2}$$

We compute the first product as in case (I) and get

$$\left(\prod_{i=0}^{d}|E_i/E_{i+1}|^{|E/E_i|-1}\right)^{e/2}.$$

When $e/2$ is odd, we check the parity of each power $|E/E_i|-1$. We recall the jump $r_T = i_+$ defined right before Lemma 7.6 and know that $|E/E_i|$ is odd if and only if $i \leq T$. Therefore we have

$$\kappa\left(\prod_{i=0}^{d}|E_i/E_{i+1}|^{|E/E_i|-1}\right)^{e/2} = \begin{cases}\left(\frac{|E_{T+1}/F|}{q}\right) = \left(\frac{d^+}{q}\right) & \text{if } e/2 \text{ is odd}\\ 1 & \text{if } e/2 \text{ is even}\end{cases}. \tag{8.4.3}$$

For the second product, unlike case (I), we now have to take care of $\kappa(-1) = \left(\frac{-1}{q}\right)$. It is better to separate the factors according to the jumps like what we have done to the first product. We then have

$$\prod_{i=0}^{d}\prod_{\substack{k=1,\dots,e/2-1\\ |E_{i+1}/F| | k,\, |E_i/F|\nmid k}}(-1)^{k(r_i+1)}.$$

Hence odd k exists if and only if $i \geq T$. We now compute this sign in different cases.

(i) If $e/2$ is odd, then $S=T$. When $i=T$ we have that r_S is odd. When $i>T$, the cardinality of the set

$$\{k=1,\dots,e/2-1 | k \text{ is odd}, |E_{i+1}/F| | k, |E_i/F| \nmid k\} \tag{8.4.4}$$

is $(|E/E_{i+1}|-|E/E_i|)/4$. Therefore the sign in (8.4.2) is the parity of

$$\sum_{i=T+1}^{d}\left(|E/E_{i+1}|-|E/E_i|\right)/4 = \left(|E/F|-|E/E_{T+1}|\right)/4 = |E/E_{T+1}|\left(|E_{T+1}/F|-1\right)/4.$$

Notice that we have denoted $|E_{T+1}/F|$ by d^+. Also $|E/E_{T+1}|$ is even but not divisible by 4. Hence the parity is the same as $\left(\frac{-1}{q}\right)^{(d^+-1)/2}$.

(ii) If $e/2$ is even and $T=S$, then again r_S is odd. When $i>T$ the cardinality of the set (8.4.4) is then even. Hence the sign in (8.4.2) is 1. The sign in (8.13) is $(-1)^{\frac{e}{4}(i_+-1)} = (-1)^{\frac{e}{4}(r_T-1)} = 1$.

(iii) If $e/2$ is even and $T>S$, then r_T is even. The cardinality of the set (8.4.4) for $i=T$ equals $|E/E_{T+1}|/4$ since the last condition $|E_T/F| \nmid k$ is void if k is odd. The cardinality has the same parity as $e/4$. Hence the sign in (8.4.2) is $(-1)^{e/4}$. The sign in (8.13) is also $(-1)^{e/4}$.

Therefore

$$\kappa\left(\prod_{i=0}^{d}\prod_{\substack{k=1,\dots,e/2-1\\ |E_{i+1}/F||k,\,|E_i/F|\nmid k}}(-1)^{k(r_i+1)}\right)=\begin{cases}\left(\frac{-1}{q}\right)^{(d^+-1)/2} & \text{if } e/2 \text{ is odd}\\ (-1)^{e/4} & \text{if } e/2 \text{ is even}\end{cases}.$$

If we combine this with (8.4.1) and (8.4.3), we have verified Proposition 8.13. □

The next step is to simplify the product $c_\theta\Delta^2(\varpi_E)$.

Proposition 8.14. *We have* $c_\theta\Delta^2(\varpi_E)=\operatorname{sgn}\left((\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)\right)$.

Proof. We follow the notations in chapter 6 of [7] that

$$c_\theta=(\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)(\dim\Lambda/\dim\Lambda_K)\delta(1+\varpi)^{-1}.$$

Since

$$\dim\Lambda/\dim\Lambda_K=(J^1/H^1)^{1/2}/(J^1_K/H^1_K)^{1/2}=q^{(\dim_{\mathbf{k}_F}V_{K/F})/2}$$

and

$$|(\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)|=q^{(\dim_{\mathbf{k}_F}W_{K/F})/2},$$

we have

$$|(\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)(\dim\Lambda/\dim\Lambda_K)|=q^{(\dim_{\mathbf{k}_F}U_{K/F})/2}=q^{m(d-1)/2}.$$

Also

$$\begin{aligned}\delta(1+\varpi)=\frac{\Delta^2(1+\varpi_E)}{\Delta^1(1+\varpi_E)}&=\frac{\Delta^2(\varpi_E)}{|\tilde{\Delta}(\varpi_E)^2|_F^{1/2}|\det_G(1+\varpi_E)|_F^{m(d-1)/2}}\\&=\frac{\Delta^2(\varpi_E)}{|\varpi_E|_F^{m^2d(d-1)/2}}=\Delta^2(\varpi_E)q^{m(d-1)/2}.\end{aligned}$$

Therefore $c_\theta\Delta^2(\varpi_E)=\operatorname{sgn}\left((\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)\right)$. □

Recall that we have computed $\operatorname{sgn}\left((\mathfrak{G}_F/\mathfrak{G}_K)(\mathfrak{K}_F^\kappa/\mathfrak{K}_F)\right)$ in Proposition 7.7. Therefore combining Propositions 8.13 and 8.14 we have computed the left side of the equation (8.2.1)

$$\kappa(x_{\mathfrak{ab}})c_\theta\Delta^2(\varpi_E)=\begin{cases}\left(\frac{-1}{q}\right)^{\frac{d^+-1}{2}}\mathfrak{n}(\psi_F)^{m/d^+} & \text{if } e/2 \text{ is odd}\\ \left(\frac{-1}{q}\right)^{e/4} & \text{if } e/2 \text{ is even}\end{cases}. \tag{8.4.5}$$

Notice that when $e/2$ is odd, we have

$$\left(\frac{-1}{q}\right)^{\frac{d^+-1}{2}} \mathfrak{n}(\psi_F)^{m/d^+} = \mathfrak{n}(\psi_F)^{d^+-1+m/d^+} = \mathfrak{n}(\psi_F)^{-m}$$

using the fact that $\mathfrak{n}(\psi_F)$ is a 4th root of unity. Therefore the right side of (8.4.5) in both cases is

$$\mathfrak{n}(\psi_F)^{-m} = \lambda_{K/F}^{-m} = \epsilon_L(V_{G/H}).$$

Since $\Delta_{\mathrm{I,II,III}}(\varpi_E) = 1$ by Proposition 4.14, we have verified (8.2.1) in case (II).

8.5 Case (III)

We would verify (8.2.1) in case (III). We first compute the first factor

Lemma 8.15. $\kappa(x_{\mathfrak{ab}}) = \kappa(\varpi_F)^{-\left(\frac{f(e-1)}{2}+\sum_{i=0}^{d}\frac{r_i f}{2}(|E/E_{i+1}|-|E/E_i|)\right)}$.

Proof. Notice that if K/F unramified then $\kappa|_{U_F} = 1$, i.e. κ unramified, and $\kappa(\varpi_F) = \zeta_d$. Hence it is enough to check the v_F-order of $x_{\mathfrak{ab}}$. Recall that $V(\mathfrak{b}) = \det\mathbf{g}$ defined in section 4.3 and is equal to

$$\prod_{k=0}^{f-1}\prod_{0\leq i<_\phi j<e}({}^{\phi^k\sigma^i}\varpi_E - {}^{\phi^k\sigma^j}\varpi_E) = \prod_{k=0}^{f-1}\prod_{0\leq i<_\phi j<e}(\zeta_{\phi^k}(\zeta_e^{iq^k} - \zeta_e^{jq^k})\varpi_E).$$

Each factor has v_E-order 1. Hence $V(\mathfrak{b})$ has v_E-order $ef(e-1)/2$, or v_F-order $f(e-1)/2$. We then compute $V(\mathfrak{a}) = V(\{1,\alpha,...,\alpha^{n-1}\})$ whose v_E order is half of that of

$$\prod_{g\neq h}({}^g\alpha - {}^h\alpha) = \prod_{i=0}^{d}\prod_{\substack{g|_{E_{i+1}}=h|_{E_{i+1}}\\ g|_{E_i}\neq h|_{E_i}}}({}^g\alpha - {}^h\alpha) = \prod_{i=0}^{d}\prod_{\substack{g|_{E_{i+1}}=h|_{E_{i+1}}\\ g|_{E_i}\neq h|_{E_i}}}\varpi_E^{-r_i}.$$

Here we use that $\#\{[{}^g_h] \in \Phi,\ [{}^g_h]\,|_{E_i} \equiv 1\} = |E/F|(|E/E_i|-1)$, then the above product equals

$$\prod_{i=0}^{d}\varpi_E^{-r_i|E/F|(|E/E_{i+1}|-|E/E_i|)} = \prod_{i=0}^{d}\varpi_F^{-r_i f(|E/E_{i+1}|-|E/E_i|)}.$$

Therefore all together we have $\kappa(x_{\mathfrak{ab}}) = \kappa(\varpi_F)^{-\left(\frac{f(e-1)}{2}+\sum_{i=0}^{d}\frac{r_i f}{2}(|E/E_{i+1}|-|E/E_i|)\right)}$. $\square$

We consider the remaining factors in (8.2.1).

(i) $c_\theta = (-1)^{f_0-1} t^0_\mu(V_{K/F})$ by Theorem 5.2.(3) of [9],

(ii) $\Delta^2(\varpi_E u) = (-1)^{e(f-1)+f_\varpi-1}$ by (6.5.6) of [9] (still true even for $E \neq E_0$),

(iii) $\epsilon_L(V_{G/H}) = \lambda^{-e}_{K/F} = (-1)^{e(f-1)}$, and

(iv) the product

$$\Delta_{\mathrm{II,III}_2}(\varpi_E u) = \prod_{\alpha \in \mathcal{R}(G)_{\mathrm{sym}} - \Phi(H)} \chi_g \left(\frac{\varpi_E u - {}^g(\varpi_E u)}{a_\alpha} \right)$$

where $\alpha = \left[\begin{smallmatrix}1\\g\end{smallmatrix}\right]$ with $g = \sigma^k \phi^{f/2}$.

We now distinguish between the cases that f is odd and f is even. If $f =$ odd, then

(i) $c_\theta = t^0_\mu(V_{K/F}) = 1$ since $V_{K/F}$ has no anisotropic component,

(ii) $\Delta^2(\varpi_E u) = (-1)^{|E/F[\varpi_E]|-1} = 1$ since $|E/F[\varpi_E]|$ divides f,

(iii) $\epsilon_L(V_{G/H}) = 1$, and

(iv) $\Delta_{\mathrm{II,III}_2}(\varpi_E u) = 1$ since $\Phi(G)_{\mathrm{sym}} - \Phi(H) = \emptyset$.

It remains to show the following.

Proposition 8.16. *If f is odd, then $\kappa(x_{\mathfrak{ab}})$ in (8.15) equals 1.*

Proof. If e is odd, then the right side of (8.15) equals 1 since all $|E/E_{i+1}| - |E/E_i|$ are even. If e is even, then again all $|E/E_{i+1}| - |E/E_i|$ are even except at the place $i = S$ when $|E/E_S|$ is odd and $|E/E_{S+1}|$ is even. Using Proposition 7.1(iv) we have that r_S is odd. Actually this is the largest odd jump. The right side of (8.15) equals

$$\kappa(\varpi_F)^{f\left(\frac{e-1+r_S(|E/E_{S+1}|-|E/E_S|)}{2}\right)}$$

which is again 1 . □

We have verified (8.2.1) in the case when f is odd. We now assume that f is even. To compute $\kappa(x_{\mathfrak{ab}})$ we denote $f_0 = f(E/E_0) = |E/E_0|$ and recall the following jumps,

(i) r_S is the largest odd jump, and

(ii) r_U is the jump such that $f(E/E_U)$ is odd and $f(E/E_{U+1})$ is even.

Proposition 8.17. *If f is even, then $\kappa(x_{\mathfrak{ab}})$ in (8.15) equals*

$$\begin{cases} \left(\frac{-1}{q}\right)^{\frac{d^+-1}{2}} \mathfrak{n}(\psi_F)^{m/d^+} & \textit{if } e/2 \textit{ is odd} \\ \left(\frac{-1}{q}\right)^{e/4} & \textit{if } e/2 \textit{ is even} \end{cases}.$$

Proof. We distinguish the case when e is odd and when e is even.

(i) First we consider when e is odd. If f_0 is odd, then because $|E/E_{i+1}| - |E/E_i|$ is odd if and only if $i = U$, we have

$$\kappa(x_{\mathfrak{ab}}) = \kappa(\varpi_E)^{\frac{r_U f}{2}} = (-1)^{r_U}.$$

If f_0 is even, then $|E/E_{i+1}| - |E/E_i|$ is always even and $\kappa(x_{\mathfrak{ab}}) = 1$.

(ii) When e is even, we have to make more analysis. If f_0 is even, then all $|E/E_{i+1}| - |E/E_i|$ are even. Hence

$$\kappa(x_{\mathfrak{ab}}) = \kappa(\varpi_E)^{f(e-1)/2} = -1.$$

We then assume that f and e are even and f_0 odd. The parity of $|E/E_{i+1}| - |E/E_i|$ is odd when

$$e(E/E_i) \text{ is odd and } e(E/E_{i+1}) \text{ is even} \tag{8.5.1}$$

or

$$f(E/E_i) \text{ is odd and } f(E/E_{i+1}) \text{ is even.} \tag{8.5.2}$$

We know that (8.5.1) happens when $i = S$ by Proposition 7.1(iv), and (8.5.2) happens at $i = U$. Therefore $\kappa(x_{\mathfrak{ab}})$ equals $(-1)^{r_U+1}$ if $S > U$, and equals $(-1)^{r_S+1} = 1$ if $S \leq U$.

□

To compute c_θ, we have to compute $t^0_\mu(V_{K/F})$ and we just recall Proposition 7.9. It would be better to tabulate the values of the constants on the left side of (8.2.1).

	$\kappa(x_{\mathfrak{ab}})$	c_θ	$\Delta^2(\varpi_E u)$
e is odd, f_0 is odd	$(-1)^{r_U}$	$(-1)^{r_U+1}$	$(-1)^{f_\varpi}$
e is odd, f_0 is even	1	-1	$(-1)^{f_\varpi}$
e is even, f_0 is odd, $S > U$	$(-1)^{r_U+1}$	$(-1)^{r_U+1}$	$(-1)^{f_\varpi+1}$
e is even, f_0 is odd, $S \leq U$	1	1	$(-1)^{f_\varpi+1}$
e is even, f_0 is even	-1	-1	$(-1)^{f_\varpi+1}$

Using this table, we can show that the left side of (8.2.1) is always $(-1)^{f_\varpi+1}$. We now compute the factors on the right side of (8.2.1) when f be even. We have

$$\epsilon_L(V_{G/H}) = (-1)^e.$$

It remains to show that

Proposition 8.18. *If f is even, then $\Delta_{\mathrm{II,III}_2} = (-1)^{e+f_\varpi-1}$.*

Proof. Be definition,

$$\begin{aligned}\Delta_{\mathrm{II,III}_2}(\varpi_E u) &= \prod_{[\sigma^k\phi^{f/2}]\text{ sym}} \chi_{\sigma^k\phi^{f/2}}\left(\frac{\varpi_E u - {}^{\sigma^k\phi^{f/2}}(\varpi_E u)}{a_{\sigma^k\phi^{f/2}}}\right)\\ &= \prod_{[\sigma^k\phi^{f/2}]\text{ sym}} \chi_{\sigma^k\phi^{f/2}}\left(\frac{\varpi_E(u-\zeta_{\phi^{f/2}}\zeta_e^k\cdot {}^{\sigma^k\phi^{f/2}}u)}{\zeta - {}^{\sigma^k\phi^{f/2}}\zeta}\right).\end{aligned}$$

Recall that $E_{\sigma^k\phi^{f/2}}/E_{\pm\sigma^k\phi^{f/2}}$ is unramified and

$$N : E_{\sigma^k\phi^{f/2}} \to E_{\pm\sigma^k\phi^{f/2}},\ \varpi_E \mapsto \zeta_{\phi^{f/2}}\zeta_e^k\varpi_E^2.$$

Since $\sigma^k\phi^{f/2} \in W_{F[\varpi_E]}$ if and only if $\zeta_{\phi^{f/2}}\zeta_e^k = 1$, we have

$$\prod_{[\sigma^k\phi^{f/2}]\text{ sym}} \chi_{\sigma^k\phi^{f/2}}\left(\frac{\varpi_E u - {}^{\sigma^k\phi^{f/2}}\varpi_E u}{\zeta - {}^{\sigma^k\phi^{f/2}}\zeta}\right)$$
$$= \begin{cases} \chi_{\sigma^k\phi^{f/2}}\left(\varpi_E^2(\text{unit})\right) = 1 & \text{if } \sigma^k\phi^{f/2} \in W_{F[\varpi_E]} \\ \chi_{\sigma^k\phi^{f/2}}\left(\varpi_E(\text{unit})\right) = -1 & \text{otherwise} \end{cases}.$$

Hence

$$\Delta_{\mathrm{II},\mathrm{III}_2} = \prod_{[\sigma^k\phi^{f/2}]\text{ sym},\ \sigma^k\phi^{f/2}\notin W_{F[\varpi_E]}} (-1).$$

Recall

(i) by Lemma 7.10 that there exists $\sigma^k\phi^{f/2} \in W_{F[\varpi_E]}$ if and only if f_ϖ is even, and

(ii) by Proposition 2.5 that the parity of the number of symmetric $[\sigma^k\phi^{f/2}]$ is the same as that of e.

Combining these two facts we have $\Delta_{\mathrm{II},\mathrm{III}_2} = (-1)^{e+f_\varpi-1}$. □

Therefore we have verified (8.2.1) in case (III). Notice in this case we have also shown that

$$\kappa(x_{\mathfrak{ab}})c_\theta = \epsilon_L(V_{G/H}) \text{ and } \Delta^2 = \Delta_{\mathrm{II},\mathrm{III}}.$$

The first statement is the same as Proposition 8.4, a fact proved in [13].

Chapter 9

Rectifier as transfer factor

We prove the second main result Theorem 9.1 that for each admissible character $\xi \in P(E/F)$ the rectifier ${}_F\mu_\xi$ is a product of canonically chosen χ-data, and therefore express the ν-rectifier of a cyclic extension K/F as a Δ_{III_2} for the group $G = \mathrm{GL}_n$ and its endoscopic group $H = \mathrm{Res}_{K/F}\mathrm{GL}_{n/|K/F|}$. In this way we can interpret the essentially tame local Langlands correspondence in terms of the local bijection Π_n of tame supercuspidals and Langlands-Shelstad admissible embeddings.

9.1 Main results

Let's recall the preliminary setup. For any $[g] = [\sigma^k\phi^i] \in (W_E\backslash W_F/W_E)'$ let $U_{[g]}$ be the standard $\mathbf{k}_F\Psi_{E/F}$-module defined in section 6.1. For $\xi \in P(E/F)$ let $V = V_\xi$ be the $\mathbf{k}_F\Psi_{E/F}$-module defined by ξ. Let $V_{[g]}$ be the $U_{[g]}$-isotypic part of V so that $V_{[g]}$ is either trivial or isomorphic to $U_{[g]}$. Let $W_{[g]}$ be the complementary subspace of $V_{[g]}$ of $U_{[g]}$. We write $\mathbf{V}_{[g]}$ as $V_{[g]} \oplus V_{[g^{-1}]}$ if $[g]$ is asymmetric, $V_{[g]}$ if $[g]$ is symmetric, and similarly for $\mathbf{W}_{[g]}$. The t-factors

$$t^i_\mu(\mathbf{V}_{[g]}) \text{ and } t^i_\varpi(\mathbf{V}_{[g]}),\ i = 0, 1, \text{ and } t(\mathbf{W}_{[g]})$$

are computed in Proposition 6.10 and Proposition 6.11. We write

$$t_\Gamma(\mathbf{V}_{[g]}) = t^0_\Gamma(\mathbf{V}_{[g]})t^1_\Gamma(\mathbf{V}_{[g]})(\gamma)$$

for arbitrary generator γ in the cyclic group $\Gamma = \mu$ or ϖ.

Theorem 9.1. *Let $\xi \in P(E/F)$.*

(i) Let V be the $\mathbf{k}_F\Psi_{E/F}$-module defined by ξ. The following characters define a collection of χ-data $\{\chi_{g,\xi}\}$.

(a) All $\chi_{g,\xi}$ are tamely ramified.

(b) If $[g]$ is asymmetric, then we assign

$$\chi_{g,\xi}|_{\mu_{E_g}} = \mathrm{sgn}_{\mu_{E_g}}(V_{[g]}) \text{ and } \chi_{g,\xi}(\varpi_E) \text{ to be anything appropriate.}$$

(c) If $[g]$ is symmetric, then we assign

$$\chi_{g,\xi}|_{\mu_E} = \begin{cases} \left(\frac{}{\mu_E}\right), & \text{if } [g] = [\sigma^{e/2}], \\ t^1_\mu(V_{[g]}), & \text{otherwise,} \end{cases}$$

and

$$\chi_{g,\xi}(\varpi_E) = t^0_\mu(V^{\varpi}_{[g]})t_\varpi(V_{[g]})t(W_{[g]}).$$

(ii) Let ${}_F\mu_\xi$ be the rectifier of $\xi \in P(E/F)$ and $\{\chi_{g,\xi}\}$ be the χ-data defined above, then

$${}_F\mu_\xi = \prod_{[g]\in(W_E\backslash W_F/W_E)'} \chi_{g,\xi}|_{E^\times}.$$

Remark 9.2. (i) In the case $[g]$ is asymmetric, that $\chi_{g,\xi}(\varpi_E)$ is assigned to any appropriate value is a natural result. Since $\chi_{g,\xi}$ and $\chi_{g^{-1}}$ must come together, by the condition i of χ-data we have

$$\chi_{g,\xi}\chi_{g^{-1}}(\varpi_E) = \chi({\varpi_E}^g\varpi_E^{-1}) \text{ and } {\varpi_E}^g\varpi_E^{-1} \in \mu_{E_g}.$$

Hence we only need the values of $\chi_{g,\xi}$ on units.

(ii) In fact $V_{[g]}^{\varpi}$ consists of a unique isotypic component, namely $V_{\sigma^k\phi^{f/2}}$ such that $({}^{\sigma^k\phi^{f/2}}\varpi_E)\varpi_E^{-1} = \zeta_e^k\zeta_{\phi^{f/2}} = 1$.

□

The proof occupies the remaining of this section. The structure of the proof is as follows. We recall the factorization (3.4.2) of the rectifier into ν-rectifiers

$$ {}_F\mu_\xi = ({}_{K_l}\mu_\xi)({}_{K_l/K_{l-1}}\mu_\xi)\cdots({}_{K_1/K_0}\mu_\xi)({}_{K_0/F}\mu_\xi), $$

where each factor is explicitly expressed in [6], [7] and [9] and restated in Proposition 6.10. We then decompose the double cosets $(W_E\backslash W_F/W_E)'$ into subsets

$$ \begin{aligned} \mathcal{D}_{l+1} &= \mathcal{D}_{E/K_l} = W_E\backslash W_{K_l}/W_E - [1], \\ \mathcal{D}_l &= \mathcal{D}_{K_l/K_{l-1}} = W_E\backslash W_{K_{l-1}}/W_E - W_E\backslash W_{K_l}/W_E, \\ &\vdots \\ \mathcal{D}_1 &= \mathcal{D}_{K_1/K_0} = W_E\backslash W_{K_0}/W_E - W_E\backslash W_{K_1}/W_E, \\ \mathcal{D}_0 &= \mathcal{D}_{K_0/F} = W_E\backslash W_F/W_E - W_E\backslash W_{K_0}/W_E, \end{aligned} $$

Let's call $\mathcal{D}_{l+l}\sqcup\cdots\sqcup\mathcal{D}_1$ the totally ramified part and $\mathcal{D}_0$ the unramified part. We will prove the following

(i) For each $[g]\in\mathcal{D}_j$, we prove that if we define $\chi_{g,\xi}$ by the assigned values in Theorem 9.1, then it satisfies the conditions of χ-data.

(ii) We show that as $[g]$ runs through $\mathcal{D}_j$, the product of the χ-data equals to the ν-rectifier, i.e.

$$ \chi_{\mathcal{D}_j}|_{E^\times} = \prod_{[g]\in\mathcal{D}_j}\chi_{g,\xi}|_{E^\times} = {}_{K_j/K_{j-1}}\mu_\xi. $$

We start proving Theorem 9.1 in case when $\mathrm{char}(\mathbf{k}_F) = p \neq 2$. We first consider those asymmetric $[g]$ and then those symmetric $[g]\in\mathcal{D}_j$ for each j.

9.2 The asymmetric case

For $[g] = [\sigma^k\phi^i]$ asymmetric, we denote the modules $V_{\pm[\sigma^k\phi^i]} = V_{[\sigma^k\phi^i]} \oplus V_{[(\sigma^k\phi^i)^{-1}]}$, which is hyperbolic if it is non-trivial. Since $\chi_{(\sigma^k\phi^i)^{-1}} = (\chi^{\sigma^k\phi^i}_{\sigma^k\phi^i})^{-1}$, we are going to compute the restriction to $E^\times$ of the product

$$\chi_{\sigma^k\phi^i}(\chi^{\sigma^k\phi^i}_{\sigma^k\phi^i})^{-1}|_{E^\times} = \chi^{-1}_{\sigma^k\phi^i} \circ \lambda_{ki}.$$

Our assignment gives

$$(\chi^{-1}_{\sigma^k\phi^i} \circ \lambda_{ki})|_{\mu_E} = \mathrm{sgn}_{\mu_{E_{\sigma^k\phi^i}}}(V_{[\sigma^k\phi^i]}) = t^1_\mu(V_{\pm[\sigma^k\phi^i]}) \tag{9.2.1}$$

and

$$(\chi^{-1}_{\sigma^k\phi^i} \circ \lambda_{ki})(\varpi_E) = \chi_{[\sigma^k\phi^i]}(\lambda_{ki}(\varpi_E)) = \mathrm{sgn}_{\lambda_{ki}(\varpi_E)}(V_{[\sigma^k\phi^i]}) = t^1_\varpi(V_{\pm[\sigma^k\phi^i]})(\varpi_E). \tag{9.2.2}$$

Notice that the relation in (9.2.2) does not depend on the exact value of $\chi_{\sigma^k\phi^i}(\varpi_E)$ and $\chi_{(\sigma^k\phi^i)^{-1}}(\varpi_E)$, therefore we can assign them to anything appropriate. We will use the t-factors on both right sides of (9.2.1) and (9.2.2) later in the symmetric cases when we compare the χ-data with the rectifier.

9.3 The symmetric ramified case

We now consider those $[g] \in (W_E\backslash W_F/W_E)_{\text{sym-ram}}$. They are of the form $[g] = [\sigma^k]$.

9.3.1 The case $[\sigma^k] \in \mathcal{D}_{l+1}$

We first consider those symmetric $[\sigma^k] \in \mathcal{D}_{l+1} = W_E\backslash W_{K_l}/W_E - [1]$, i.e. k has odd order in $\mathbb{Z}/e$. Let t, s be positive integers and minimum so that $e|(1+q^{ft})k$ and $e|(1+p^s)k$. We recall that we have assigned

$$\chi_{\sigma^k}|_{\mu_E} = t^1_\mu(V_{[\sigma^k]}) \tag{9.3.1}$$

which is 1 since μ_E acts on each $V_{[\sigma^k]}$ trivially, and

$$\chi_{\sigma^k}(\varpi_E) = t_\varpi(V_{[\sigma^k]})t(W_{[\sigma^k]}). \tag{9.3.2}$$

If $V_{[\sigma^k]}$ is non-trivial, then we have

$$t^0_\varpi(V_{[\sigma^k]}) = -1 \text{ and } t^1_\varpi(V_{[\sigma^k]})(\varpi_E) = \left(\frac{\zeta_e^k}{\mu_{p^s+1}}\right) = 1$$

since ζ_e^k has odd order and p^s+1 is even. Therefore the right side of (9.3.2) is -1. If $V_{[\sigma^k]}$ is trivial, so that $W_{[\sigma^k]}$ is non-trivial, then

$$t_\varpi(V_{[\sigma^k]}) = 1 \text{ and } t(W_{[\sigma^k]}) = -1.$$

Hence we always have $\chi_{\sigma^k}(\varpi_E) = t_\varpi(U_{[\sigma^k]}) = -1$.

We have to check that such χ_{σ^k} in (9.3.1) and (9.3.2) satisfies the condition of χ-data

$$\chi_{\sigma^k}|_{E_{\pm\sigma^k}} = \delta_{E_{\sigma^k}/E_{\pm\sigma^k}}. \tag{9.3.3}$$

Since $E_{\sigma^k}/E_{\pm\sigma^k}$ is quadratic unramified, the norm group $N_{E_{\sigma^k}/E_{\pm\sigma^k}}(E^\times_{\sigma^k})$ has a decomposition

$$\mu_{E_{\pm\sigma^k}} \times \left\langle \zeta_e^k \varpi_E^2 \right\rangle \times U^1_{E_{\pm\sigma^k}}$$

and let $\zeta_0 \in \mu_{E_{\sigma^k}}$ so that

$$\zeta_0\varpi_E \in E_{\pm\sigma^k} - N_{E_{\sigma^k}/E_{\pm\sigma^k}}(E^\times_{[\sigma^k]}).$$

Therefore the condition (9.3.3) is explicitly

$$\chi_{\sigma^k}|_{\mu_{E_{\pm\sigma^k}}} \equiv 1,\ \chi_{\sigma^k}(\zeta_e^k\varpi_E^2) = 1 \text{ and } \chi_{\sigma^k}(\zeta_0)\chi_{\sigma^k}(\varpi_E) = -1. \tag{9.3.4}$$

Since $\mu_E \subseteq \mu_{E_{\pm\sigma^k}}$, the first condition of (9.3.4) implies that the quadratic character $\chi_{E_{\sigma^k}/E_{\pm\sigma^k}}$ extends χ_{σ^k} from μ_E to $\mu_{E_{\pm\sigma^k}}$. Therefore our assignment (9.3.1) satisfies the first condition in (9.3.4). If we assume $\chi_{\sigma^k}(\varpi_E) = \pm 1$, then the second condition of (9.3.4) gives $\chi(\zeta_e^k) = 1$. Finally we use the third condition of (9.3.4) that $\chi_{\sigma^k}(\varpi_E) = -\chi_{\sigma^k}(\zeta_0)$.

Since $\zeta_0^{1-q^{ft}} = \zeta_e^k = \zeta_N$ for N being odd (indeed if $k = 2^s k'$ for some odd k' and $e = 2^s d$ for some odd d, then $N = d/\gcd(k', d)$ is odd), we have $\zeta_0 \in \mu_{(1-q^{ft})N}$. Hence we have $\chi_{\sigma^k}(\zeta_0^N) = 1$ as $\zeta_0^N \in \mu_{q^{ft}-1} = \mu_{E_{\pm\sigma^k}}$. We assumed that $\chi_{\sigma^k}(\zeta_0) = \pm 1$, and so it must be 1. Therefore we have shown that our assignment (9.3.2) satisfies the conditions in (9.3.4). In other words, χ_{σ^k} is a well-defined character in χ-data.

Finally we have to show that the product

$$\chi_{\mathcal{D}_{l+1}} = \prod_{[\sigma^k]\in\mathcal{D}_{l+1}} \chi_{\sigma^k} = \prod_{[\sigma^k]\in(\mathcal{D}_{l+1})_{sym}} \chi_{\sigma^k} \prod_{[\sigma^k]\in(\mathcal{D}_{l+1})_{asym/\pm}} \chi_{\sigma^k}^{-1} \circ \lambda_{k0}$$

equals ${}_{K_l/E}\mu_\xi$. We first compute its value on μ_E. For asymmetric $[\sigma^k]$ we have right side of (9.2.1) being 1 since μ_E acts trivially on $U_{[\sigma^k]}$. Combining these with the symmetric ones in (9.3.1) and using the explicit value in (7.3.1) we have

$$\chi_{\mathcal{D}_{l+1}}|_{\mu_E} = 1 = \delta_{E/K_l}|_{\mu_{K_l}} = {}_{K_l/E}\mu_\xi|_{\mu_E}.$$

Next we compute the value on ϖ_E. Combining (9.2.2) and (9.3.2) we have

$$\chi_{\mathcal{D}_{l+1}}(\varpi_E) = t_\varpi(U_{E/K_l}).$$

By Lemma 4.2(ii) of [26] we have the product

$$t_\varpi(U_{E/K_l}) = \left(\frac{q^f}{e(E/K_l)}\right)$$

which equals ${}_{E/K_l}\mu_\xi(\varpi_E)$ as in (7.3.1). We have proved Theorem 9.1 for symmetric $[\sigma^k] \in \mathcal{D}_{l+1}$.

9.3.2 The case $[\sigma^k] \in \mathcal{D}_l$

We next study those $[\sigma^k] \in \mathcal{D}_l = \mathcal{D}_{K_l/K_{l-1}}$. We first consider the distinguished element $[g] = [\sigma^{e/2}]$. By Proposition 7.5 the module $V_{[\sigma^{e/2}]}$ is always trivial. Recall that $E_{\sigma^{e/2}} = E$ and $E/E_{\pm\sigma^{e/2}}$ is quadratic totally ramified. We assign

$$\chi_{\sigma^{e/2}}|_{\mu_E} : \zeta \mapsto \left(\frac{\zeta}{\mu_E}\right). \tag{9.3.5}$$

Since $N_{E/E_{\pm\sigma^{e/2}}}(\varpi_E) = -\varpi_{E_{\pm\sigma^{e/2}}} = -\varpi_E^2$, it remains to assign $\chi_{\sigma^{e/2}}(\varpi_E)$ which satisfies

$$\chi_{\sigma^{e/2}}(\varpi_E)^2 = \chi_{\sigma^{e/2}}(\varpi_{E_{\pm\sigma^{e/2}}}) = \chi_{\sigma^{e/2}}(-1) = \left(\frac{-1}{\mu_E}\right)$$

so that it satisfies the condition of χ-data

$$\chi_{\sigma^{e/2}}|_{E^\times_{\pm\sigma^{e/2}}} = \delta_{E/E_{\pm\sigma^{e/2}}}.$$

If we assign $\chi_{\sigma^{e/2}}(\varpi_E) = t(W_{[\sigma^{e/2}]})$ then

$$\chi_{\sigma^{e/2}}(\varpi_E)^2 = \mathfrak{n}(\psi_{K_{l-1}})^2 = \left(\frac{-1}{\mu_E}\right).$$

Therefore we have shown that $\chi_{\sigma^{e/2}}$ is a character in χ-data.

For $[\sigma^k] \in \mathcal{D}_l$ symmetric, $k \neq e/2$, since μ_E acts trivially on $V_{[\sigma^k]}$, we can assign

$$\chi_{\sigma^k}|_{\mu_E} = t^1_\mu(V_{[\sigma^k]}) = 1. \tag{9.3.6}$$

To obtain the value of $\chi_{\sigma^k}(\varpi_E)$ so that χ_{σ^k} defines a character in a χ-data, i.e.

$$\chi_{\sigma^k}|_{E_{\pm\sigma^k}} = \delta_{E_{\sigma^k}/E_{\pm\sigma^k}},$$

we recall the explicit χ-data conditions in (9.3.4) that

$$\chi_{\sigma^k}|_{\mu_{E_{\pm\sigma^k}}} \equiv 1,\ \chi_{\sigma^k}(\zeta_e^k\varpi_E^2) = 1 \text{ and } \chi_{\sigma^k}(\zeta_0)\chi_{\sigma^k}(\varpi_E) = -1. \tag{9.3.7}$$

Our assignment (9.3.6) satisfies the first condition since $\mu_E \subseteq \mu_{E_{\pm\sigma^k}}$ and

$$\chi_{\sigma^k}(\mu_{E_{\pm\sigma^k}}) = \delta_{E_{\sigma^k}/E_{\pm\sigma^k}}\left(N_{E_{\sigma^k}/E_{\pm\sigma^k}}(\mu_{E_{\sigma^k}})\right) = 1.$$

If we assume $\chi_{\sigma^k}(\varpi_E) = \pm 1$, then the second condition of (9.3.7) implies that $\chi_{[\sigma^k]}(\zeta_e^k) = 1$. Let $\zeta_e^k = \zeta_N$ for $N = 2M$ and M being odd. (Indeed for $[\sigma^k] \in \mathcal{D}_l$ we have

$$k \in q^f \backslash (2^{s-1}\mathbb{Z}/e - 2^s\mathbb{Z}/e).$$

Hence if $k = 2^{s-1}k', e = 2^s d$, then $\zeta_e^k = \zeta_{2d}^{k'} = \zeta_N$ for $N = 2d/\gcd(k',d)$.) Recall that $\zeta_0^{1-q^{ft}} = \zeta_e^k = \zeta_N$, hence $\zeta_0^N \in \mu_{1-q^{ft}} = \mu_{E_{\pm\sigma^k}}$. Since N is even and $\chi_{\sigma^k}|_{\mu_{E_{\pm\sigma^k}}} = 1$, we can assign $\chi_{\sigma^k}(\zeta_0)$ to be either ± 1, and from the third condition of (9.3.7) we can assign $\chi_{\sigma^k}(\varpi_E) = -\chi_{\sigma^k}(\zeta_0)$ to be either ± 1. As in Theorem 9.1 we assign

$$\chi_{\sigma^k}(\varpi_E) = t_{\varpi}(V_{[\sigma^k]})t(W_{[\sigma^k]})$$

and χ_{σ^k} is hence a character in χ-data.

We are ready to compute $\chi_{\mathcal{D}_l}$. On μ_E we combine (9.3.5), (9.3.6) and (9.2.1) to obtain

$$\chi_{\mathcal{D}_l}|_{\mu_E} = \left(\frac{}{\mu_E}\right) = \left(\frac{}{\mu_{K_{l-1}}}\right) = \delta_{K_l/K_{l-1}},$$

which equals ${}_{K_l/K_{l-1}}\mu_\xi|_{\mu_E}$ by (3.1.1) of [7].

To compare $\chi_{\mathcal{D}_l}(\varpi_E)$ with ${}_{K_l/K_{l-1}}\mu_\xi(\varpi_E)$, we recall from (7.3.2) that

$${}_{K_l/K_{l-1}}\mu_\xi(\varpi_E) = t_{\varpi}(V_{\mathcal{D}_l})\mathrm{sgn}((\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l})(\mathfrak{K}^{\kappa}_{K_{l-1}}/\mathfrak{K}_{K_{l-1}}))$$

Regard ξ as a admissible character over K_{l-1} and let $E = E_0 \supsetneq E_1 \cdots \supsetneq E_t \supsetneq E_{t+1} = K_{l-1}$ and $\{r_0, \dots, r_t\}$ be the jump data of ξ. By genericity of ξ we always have $i_+ \geq i^+$. We first distinguish between $r_0 > 1$ and $r_0 = 1$.

1. If $r_0 > 1$, then $\chi_{\mathcal{D}_l} = t_{\varpi}(V_{\mathcal{D}_l})t(W_{\mathcal{D}_l})$. There is a ϖ-invariant quadratic form on $W^{K_{l-1}}$ defined in section 5.7 of [7], where on each jump component W_j, for $j = 0, \dots, t$, is denoted by $\zeta_j(\varpi_E)\mathbf{q}^j_{K_l}$. By definition in section 6.3 we have

$$t(W_{\mathcal{D}_l}) = \prod_{j=0}^{t} \mathfrak{n}(\zeta_j(\varpi_E)\mathbf{q}^j_{K_l}, \psi_{K_l}).$$

   This equals $\mathrm{sgn}(\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l})$ by section 8.1 of [7]. Also by convention $\mathfrak{K}^{\kappa}_{K_{l-1}}/\mathfrak{K}_{K_{l-1}} = 1$ in the case $r_0 > 1$. Therefore we have checked that

$$\chi_{\mathcal{D}_l}(\varpi_E) = t_{\varpi}(V_{\mathcal{D}_l})t(W_{\mathcal{D}_l}) = t_{\varpi}(V_{\mathcal{D}_l})(\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l})(\mathfrak{K}^{\kappa}_{K_{l-1}}/\mathfrak{K}_{K_{l-1}}) = {}_{K_l/K_{l-1}}\mu_\xi(\varpi_E).$$

2. If $r_0 = 1$, we separate into three possible subcases, namely

$$i_+ = i^+ = r_0 = 1,\ i_+ > i^+ = r_0 = 1 \text{ and } i_+ \geq i^+ > r_0 = 1.$$

   For the second case $i_+ > i^+ = r_0 = 1$, we have shown in section 7.3 that it does not exist when $|E/K_{l-1}|/2$ is odd.

(a) If $i_+ = i^+ = r_0 = 1$, then the jumps are of the form $\{1, 2s_1, ..., 2s_t\}$ and $d^+ = |E_1/K_{l-1}|$ is odd, we have

$$\bigoplus_{j\geq 1} W^{K_{l-1}}_{E_j/E_{j+1}} = 0 \text{ and } W^{K_{l-1}} = W^{K_{l-1}}_{E/E_1}. \tag{9.3.8}$$

The first statement of (9.3.8) gives

$$t\left(\bigoplus_{j\geq 1} W^{K_{l-1}}_{E_j/E_{j+1}}\right) = 1$$

which equals $\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l}$ by Lemma 8.1(1) of [7]. The second statement of (9.3.8) is indeed

$$W^{K_{l-1}}_{E/E_1} = \bigoplus_{k=1}^{|E/E_1|} W_{[\sigma^{ke(E_1/F)}]}.$$

Therefore we have

$$W_0 = (W_{E/E_1})_{\mathcal{D}_l} = \bigoplus_{k \text{ is odd}} W_{[\sigma^{ke(E_1/F)}]}. \tag{9.3.9}$$

This module is denoted by $\mathcal{Y}$ in section 8.2 of [7], whose degree $\dim_{\mathbf{k}_E} \mathcal{Y} = |E/E_1|/2$ is odd. There is a ϖ-invariant bilinear form $\zeta_0(\varpi_E)\mathbf{q}_{K_{l-1}}$ on W_0, then by the proof of [7] Proposition 8.3 (still goes for $s = 0$ verbatim) we have $\left(\frac{\mathbf{q}_{K_{l-1}}|_{\mathcal{Y}}}{q^f}\right) = \left(\frac{|E_1/K_{l-1}|}{q^f}\right)$ and hence

$$\begin{aligned} t(\mathcal{Y}) = \mathfrak{n}(\zeta_0(\varpi_E)\mathbf{q}_{K_{l-1}}|_{\mathcal{Y}}, \psi_{K_{l-1}}) &= \left(\frac{\zeta_0(\varpi_E)}{q^f}\right)^{\dim \mathcal{Y}} \left(\frac{\mathbf{q}_{K_{l-1}}|_{\mathcal{Y}}}{q^f}\right) \mathfrak{n}(\psi_{K_{l-1}})^{\dim \mathcal{Y}} \\ &= \left(\frac{\zeta_0(\varpi_E)}{q^f}\right) \left(\frac{|E_1/K_{l-1}|}{q^f}\right) \mathfrak{n}(\psi_{K_{l-1}})^{|E/E_1|/2}. \end{aligned}$$

This equals $\mathfrak{K}^{\kappa}_{K_{l-1}}/\mathfrak{K}_{K_{l-1}}$ as computed in 7.3 (or by Proposition 9.3 of [7]). Therefore

$$\begin{aligned} &\chi_{\mathcal{D}_l}(\varpi_E) = t_{\varpi}(V_{\mathcal{D}_l}) t(\mathcal{Y}) t\left(\bigoplus_{j\geq 1} W^{K_{l-1}}_{E_j/E_{j+1}}\right) \\ =& t_{\varpi}(V_{\mathcal{D}_l})(\mathfrak{K}^{\kappa}_{K_{l-1}}/\mathfrak{K}_{K_{l-1}})(\mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l}) = {}_{K_l/K_{l-1}}\mu_{\xi}(\varpi_E). \end{aligned}$$

(b) If $i_+ \geq i^+ > r_0 = 1$, then we recall that $i^+ = r_S$ is the smallest jump such that $e(E/E_S)$ is odd and $e(E/E_{S+1})$ is even. Hence

$$W^{K_{l-1}} = \bigoplus_{r_j \leq i^+,\, r_j \text{ is odd}} W_j$$

which contains W_{E/E_1} and $W_{E_S/E_{S+1}}$ with $W_{[\sigma^{e/2}]} \subseteq W_{E_S/E_{S+1}}$. Also in fact $i_+ \gneq i^+$ in this case, and since $|E_{i^+}/E_{i_+}|$ is even, we can check $W^{K_{l-1}}_{\mathcal{D}_l} = W^{K_{l-1}}_{E_S/E_{S+1}}$. In particular $\mathcal{Y} = (W_{E/E_1})_{\mathcal{D}_l} = 0$ and $t(\mathcal{Y}) = 1$, which equals $\mathfrak{K}^{\kappa}_{K_{l-1}}/\mathfrak{K}_{K_{l-1}}$ by Lemma 8.1(3) of [7]. Therefore $t(W_{\mathcal{D}_l}) = \mathfrak{G}_{K_{l-1}}/\mathfrak{G}_{K_l}$ and $\chi_{\mathcal{D}_l}(\varpi_E) = {}_{K_l/K_{l-1}}\mu_\xi(\varpi_E)$.

Combining all cases, we have proved Theorem 9.1 for $[\sigma^k] \in \mathcal{D}_l \sqcup \mathcal{D}_{l-1}$.

9.3.3 The case $[\sigma^k] \in \mathcal{D}_{l-1} \sqcup \cdots \sqcup \mathcal{D}_1$

For $[\sigma^k] \in \mathcal{D}_{l-1}$ we first consider another distinguished element $[g] = [\sigma^{e/4}]$, in case if it is symmetric.

Lemma 9.3. *The following are equivalent.*

(i) $[\sigma^{e/4}]$ *is symmetric.*

(ii) $(1 + q^{ft})/2$ *is even for some* t.

(iii) $q^f \equiv 3 \mod 4$.

(iv) $\left(\frac{-1}{q^f}\right) = -1$.

(v) The degree 4 totally ramified extension K_l/K_{l-2} *is non Galois.*

Proof. By the definition of symmetry, (i) is equivalent to the statement that e divides $(1 + q^{ft})e/4$ for some t, which is clearly equivalent to (ii). That (ii)⇔(iii) and (iii)⇔(iv) are clear. A totally ramified extension E/F of degree 4 is non-Galois if and only if 4 does not divide $q_f - 1$. Hence (v) is equivalent to (iii). □

Consider ξ as a admissible character of $E^\times$ over K_{l-2} and let $E = E_0 \supsetneq E_1 \cdots \supsetneq E_t \supsetneq E_{t+1} = K_{l-2}$ and $\{r_0, \dots, r_t\}$ be the corresponding jump data. Define i^+ and i_+ as the jumps r_T and r_S, and recall that $i_+ \geq i^+$ always. We can characterize $V_{[\sigma^{e/4}]}$ by the following.

Lemma 9.4. *$V_{[\sigma^{e/4}]}$ is non-trivial if and only if $i_+ > i^+$.*

Proof. Let r_j be the jump that $V_{[\sigma^{e/4}]} \subseteq V_j$. The sufficient condition in the Lemma is equivalent to that r_j is even, and the necessary condition in the Lemma is equivalent to that i_+ is even. Hence $4|e(E/E_j)$ and $e(E_j/K_{l-2})$ is odd. We have $r_j = i_+$. □

We assign

$$\chi_{\sigma^{e/4}}|_{\mu_E} = t^1_\mu(V_{[\sigma^{e/4}]}) \equiv 1 \tag{9.3.10}$$

and

$$\chi_{\sigma^{e/4}}(\varpi_E) = t_\varpi(V_{[\sigma^{e/4}]})t(W_{[\sigma^{e/4}]}) = \begin{cases} t_\varpi(V_{[\sigma^{e/4}]}), & \text{if } V_{[\sigma^{e/4}]} \text{ is non-trivial} \\ t(W_{[\sigma^{e/4}]}) = -1, & \text{if } V_{[\sigma^{e/4}]} \text{ is trivial} \end{cases}.$$

The verification of $\chi_{\sigma^{e/4}}$ being a χ-data, no matter $V_{[\sigma^{e/4}]}$ is trivial or not, is similar to the statement following (9.3.7).

Remark 9.5. We can compute the value of $t_\varpi(V_{[\sigma^{e/4}]})$ if $V_{[\sigma^{e/4}]}$ is non-trivial, i.e. $V_{[\sigma^{e/4}]} = \mathbf{k}_E(\zeta_4)$. By Lemma 9.3 $q^f \equiv 3 \mod 4$, which implies also $p \equiv 3 \mod 4$. We have $|\mathbb{F}_p(\zeta_4)/\mathbb{F}_p| = 2$ and that $r = |\mathbf{k}_E(\zeta_4)/\mathbb{F}_p(\zeta_4)| = |\mathbf{k}_E/\mathbb{F}_p|$ must be odd by Lemma 6.9. Therefore

$$t_\varpi(V_{[\sigma^{e/4}]}) = \left(-\left(\frac{\zeta_4}{\mu_{p+1}}\right)\right)^r = -\left(\frac{\zeta_4}{\mu_{p+1}}\right) = \begin{cases} 1 & \text{if } \ p \equiv 3 \mod 8 \\ -1 & \text{if } \ p \equiv 7 \mod 8 \end{cases}.$$

□

For other symmetric $[\sigma^k]$, $k \neq e/4$, we proceed as in the case $[\sigma^k] \in \mathcal{D}_l$, $k \neq e/2$, and obtain

$$\chi_{\sigma^k}|_{\mu_E} = t^0_\mu(V_{[\sigma^k]}) = 1 \text{ and } \chi_{\sigma^k}(\varpi_E) = t_\varpi(V_{[\sigma^k]})t(W_{[\sigma^k]}). \tag{9.3.11}$$

Again we can use similar statements following (9.3.7) to verify that χ_{σ^k} is a character in a χ-data.

On μ_E we combine (9.3.10), (9.3.11) and (9.2.1) to obtain $\chi_{\mathcal{D}_{l-1}}|_{\mu_E} \equiv 1$, which equals ${}_{K_{l-1}/K_{l-2}}\mu_\xi|_{\mu_E}$ by Theorem 3.2 of [7]. To compare $\chi_{\mathcal{D}_{l-1}}(\varpi_E)$ with ${}_{K_{l-1}/K_{l-2}}\mu_\xi(\varpi_E)$, again we distinguish between cases. When $r_0 > 1$, the situation is similar to the case $[g] \in \mathcal{D}_{K_l/K_{l-1}}$. When $r_0 = 1$ we distinguish between $i_+ = i^+ = r_0 = 1$, $i_+ > i^+ = r_0 = 1$ and $i^+ > r_0 = 1$.

(i) If $i_+ = i^+ = r_0 = 1$, then $|E_1/K_{l-2}|$ is odd. Let

$$\mathcal{Y} = (W_{E/E_1})_{\mathcal{D}_{l-1}} = \bigoplus_{k\in\mathbb{Z}/|E/E_1|,\, k \text{ is odd}} W_{\sigma^{ke(E_1/F)|}}$$

defined similarly as in (9.3.9). Then $\dim\mathcal{Y} = |E/E_1|/2$ is even. As in the proof of [7] Proposition 8.4 (which is still true for $r_0 = 1$), we have $\left(\frac{\det \mathbf{q}_{K_{l-1}}|_{\mathcal{Y}}}{q^f}\right) = 1$ and so

$$t(\mathcal{Y}) = \left(\frac{\zeta_0(\varpi_E)}{q^f}\right)^{\dim\mathcal{Y}} \left(\frac{\det \mathbf{q}_{K_{l-1}}|_{\mathcal{Y}}}{q^f}\right) \mathfrak{g}(\psi_{K_{l-2}})^{\dim\mathcal{Y}} = \left(\frac{-1}{q^f}\right)^{|E/E_1|/4}.$$

This equals $\mathfrak{K}^{\kappa}_{K_{l-2}}/\mathfrak{K}_{K_{l-2}}$ by 9.3 of [7]. Moreover $i^+ = 1$ implies that $\bigoplus_{j\geq 1}(W_{E_j/E_{j+1}})_{\mathcal{D}_{l-1}} = 0$. Hence

$$t\left(\bigoplus_{j\geq 1}(W_{E_j/E_{j+1}})_{\mathcal{D}_{l-1}}\right) = 1 = \mathfrak{G}_{K_{l-2}}/\mathfrak{G}_{K_{l-1}}$$

by Lemma 8.1(1) of [7]. Therefore $\chi_{K_{K_{l-1}/K_{l-2}}}(\varpi_E) = {}_{K_{l-1}/K_{l-2}}\mu_\xi(\varpi_E)$.

(ii) For $i_+ > i^+ = r_0 = 1$, we must have that $|E_1/K_{l-2}|$ is even, otherwise $i_+ = 1$. If $[\sigma^{kE(K_{l-2}/F)|}] \in \mathcal{D}_{l-1} \cap W_E \backslash W_{E_1}/W_E$, then k is odd and also a multiple of $|E_1/K_{l-2}|$. No such k exists. Therefore $\mathcal{Y}$ is trivial and $t(\mathcal{Y}) = 1 = \mathfrak{K}^{\kappa}_{K_{l-2}}/\mathfrak{K}_{K_{l-2}}$ by Proposition 8.1 of [7]. Hence again $\chi_{K_{K_{l-1}/K_{l-2}}}(\varpi_E) = {}_{K_{l-1}/K_{l-2}}\mu_\xi(\varpi_E)$.

(iii) The case $i^+ > r_0 = 1$ is similar to the case $i_+ > i^+ = r_0 = 1$.

In all case we have proved that $\chi_{K_{K_{l-1}/K_{l-2}}}(\varpi_E) = {}_{K_{l-1}/K_{l-2}}\mu_\xi(\varpi_E)$. For $[\sigma^k] \in \mathcal{D}_{l-2} \sqcup \cdots \sqcup \mathcal{D}_1$, the proof that χ_{σ^k} defines a character in χ-data and that $\chi_{\mathcal{D}_j} = {}_{K_j/K_{j-1}}\mu_\xi$

are very similar to the previous cases. Notice in this case $e/4$ is even, so by Proposition 7.7 we have

$$\mathfrak{G}_{K_{l-2}}/\mathfrak{G}_{K_{l-1}} = \mathfrak{K}^{\kappa}_{K_{l-2}}/\mathfrak{K}_{K_{l-2}} = 1.$$

We have proved Theorem 9.1 for all symmetric $[g] = [\sigma^k]$.

9.4 The symmetric unramified case

We now consider those $[g] \in (W_E\backslash W_F/W_E)_{\text{sym-unram}}$. They are of the form $[g] = [\sigma^k\phi^{f/2}]$. If $\lambda_{k,f/2}$ is the corresponding root $x \mapsto {}^{\sigma^k\phi^{f/2}}xx^{-1}$, then we distinguish the following three cases

$$\lambda_{k,f/2}(\varpi_E) = 1,\ \lambda_{k,f/2}(\varpi_E) = -1, \text{ and } \lambda_{k,f/2}(\varpi_E) \neq \pm 1.$$

9.4.1 The case $\lambda_{k,f/2}(\varpi_E) = 1$

When $\lambda_{k,f/2}(\varpi_E) = 1$ we have $E_g = E$. If we denote $E_{\pm g} = E_{\pm}$, then the χ-data conditions are

$$\chi_{g,\xi}|_{\mu_{E_\pm}} \equiv 1,\ \chi_{g,\xi}(\varpi_E^2) = 1 \text{ and } \chi_{g,\xi}(\zeta_0\varpi_E) = -1, \tag{9.4.1}$$

where $\zeta_0 \in \mu_E$ so that $\zeta_0^{q^{f/2}-1} = 1$, i.e. $\zeta_0 \in \mu_{E_\pm} \subseteq \mu_E$. If $V_{[g]}$ is trivial, then we have assigned

$$\chi_{g,\xi}|_{\mu_E} = t^1_\mu(V_{[g]}) \equiv 1 \text{ and } \chi_{g,\xi}(\varpi_E) = t^0_\mu(V^{\varpi}_{[g]})t_{\varpi}(V_{[g]})t(W_{[g]}) = (1)(1)(-1) = -1.$$

which satisfies the conditions in (9.4.1). If $V_{[g]}$ is non-trivial, then $V^{\varpi} = V_{[g]}$. We have assigned

$$\chi_{g,\xi}|_{\mu_E} = t^1_\mu(V_{[g]}) : \zeta \mapsto \left(\frac{\zeta^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right)$$

and

$$\chi_{g,\xi}(\varpi_E) = t^0_\mu(V^{\varpi}_{[g]})t_{\varpi}(V_{[g]})t(W_{[g]}) = (-1)(1)(1) = -1.$$

Since $\mu_{E_\pm} = \mu_{q^{f/2}-1}$, we have $t^1_\mu(V_{[g]})(\mu_{E_\pm}) = 1$. The conditions in (9.4.1) are readily satisfied. Therefore $\chi_{g,\xi}$ is a character of χ-data.

9.4.2 The case $\lambda_{k,f/2}(\varpi_E) = -1$

For $\lambda_{k,f/2}(\varpi_E) = -1$, again $E_g = E$ and is quadratic unramified over $E_\pm = E_{\pm g}$. The χ-data conditions are

$$\chi_{g,\xi}|_{\mu_{E_\pm}} \equiv 1,\ \chi_{g,\xi}(-\varpi_E^2) = 1 \text{ and } \chi_{g,\xi}(\zeta_0\varpi_E) = -1. \tag{9.4.2}$$

Here $\zeta_0 \in \mu_E$ satisfies $\zeta_0^{q^{f/2}-1} = -1$, i.e. ζ_0 is a primitive root in $\mu_{2(q^{f/2}-1)}$. If $V_{[g]}$ is trivial, we have assigned

$$\chi_{g,\xi}|_{\mu_E} = t^1_\mu(V_{[g]}) \equiv 1 \text{ and } \chi_{g,\xi}(\varpi_E) = t_\varpi(V_{[g]})t(W_{[g]}) = (-1)(1) = -1.$$

By noticing that $\zeta_0 \in \mu_{2(q^{f/2}-1)} \subset \mu_{q^f-1} = \mu_E$, the verification is an easy analogue of the previous case.

When $V_{[g]}$ is non-trivial, we have assigned

$$\chi_{g,\xi}|_{\mu_E} = t^1_\mu(V_{[g]}) : \zeta \mapsto \left(\frac{\zeta^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right) \tag{9.4.3}$$

and

$$\chi_{g,\xi}(\varpi_E) = t_\varpi(V_{[g]})t(W_{[g]}) = (-1)^{\frac{1}{2}(q^{f/2}-1)}(1) = (-1)^{\frac{1}{2}(q^{f/2}-1)}. \tag{9.4.4}$$

Since $\mu_{E_\pm} = \mu_{q^{f/2}-1}$, our assignment (9.4.3) satisfies the first condition of (9.4.2). Also the second condition of (9.4.2) is satisfied if we assume $\chi_{g,\xi}(\varpi_E) = \pm 1$. Since

$$\chi_{g,\xi}(\zeta_0) = t^1_\mu(V_{[g]})(\zeta_0) = \left(\frac{\zeta_0^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right) = \left(\frac{-1}{\mu_{q^{f/2}+1}}\right) = (-1)^{\frac{1}{2}(q^{f/2}+1)},$$

with (9.4.4) we have

$$\chi_{g,\xi}(\zeta_0\varpi_E) = (-1)^{\frac{1}{2}(q^{f/2}+1)}(-1)^{\frac{1}{2}(q^{f/2}-1)} = -1.$$

Hence the third condition of (9.4.2) is satisfied, and $\chi_{g,\xi}$ is a character of χ-data.

9.4.3 The case $\lambda_{k,f/2}(\varpi_E) \neq \pm 1$

For other $[g] = [\sigma^k\phi^{f/2}]$ symmetric with $\lambda_{k,f/2}(\varpi_E) = \zeta_e^k\zeta_{\phi^{f/2}} \neq \pm 1$, we assign

$$\chi_{g,\xi}|_{\mu_E} = t^1_\mu(V_{[g]}) \tag{9.4.5}$$

and

$$\chi_{g,\xi}(\varpi_E) = t_\varpi(V_{[g]})t(W_{[g]}). \tag{9.4.6}$$

We have to show that it satisfies the condition of χ-data

$$\chi_{g,\xi}(\mu_{E_\pm}) = 1,\ \chi_{g,\xi}(\lambda_{k,f/2}(\varpi_E)\varpi_E^2) = 1 \text{ and } \chi_{g,\xi}(\zeta_0\varpi_E) = -1, \tag{9.4.7}$$

where $\zeta_0 \in \mu_{E_g}$ so that $\zeta_0\varpi_E \in E_{\pm g}$. Since $\mu_E \cap \mu_{E_\pm} = \mu_{q^f-1} \cap \mu_{q^{f(2t+1)/2}-1} = \mu_{q^{f/2}-1}$, which is mapped by

$$t_\mu^1(V_{[g]}) : \zeta \mapsto \begin{cases} 1, & \text{if } V_{[g]} \text{ is trivial} \\ \left(\frac{\zeta^{q^{f/2}-1}}{\mu_{q^{f/2}+1}}\right), & \text{if } V_{[g]} \text{ is non-trivial} \end{cases}$$

into 1, we can extend $\chi_{g,\xi}$ so that $\chi_{g,\xi}|_{E_{\pm g}^\times} \equiv 1$ and hence (9.4.5) satisfies the first condition of (9.4.7). Now notice that

$$\lambda_{k,f/2}(\varpi_E)^{1+q^{f(2t+1)/2}} = {}^{\sigma^k\phi^{f(2t+1)/2}}(\lambda_{k,f/2}(\varpi_E))\lambda_{k,f/2}(\varpi_E) = 1$$

and so

$$\lambda_{k,f/2}(\varpi_E) \in \mu_{1+q^{f(2t+1)/2}} = \ker(N_{E_g/E_{\pm g}}).$$

Since

$$\langle \mu_E, \mu_{E_\pm} \rangle \cap \ker(N_{E_g/E_{\pm g}}) = \mu_{(q^{f/2}+1)(q^{f(2t+1)/2}-1)} \cap \mu_{1+q^{f(2t+1)/2}} = \mu_{q^{f/2}+1}$$

and $t_\mu^1(V_{[g]})|_{\mu_{q^{f/2}+1}} \equiv 1$, we can extend $\chi_{g,\xi}$ so that $\chi_{g,\xi}|_{\ker(N_{E_g/E_{\pm g}})} \equiv 1$ and hence (9.4.5) satisfies the second condition of (9.4.7). It remains to show that (9.4.5) satisfies the last condition of (9.4.7) and hence determines $\chi_{g,\xi}(\varpi_E)$ as in (9.4.6). The proof is the same as that in the cases $[g] = [\sigma^k] \in (W_E \backslash W_F / W_E)_{\text{sym-ram}}$. In case $\lambda_{k,f/2}(\varpi_E)$ has odd order, we can show that $\chi_{g,\xi}(\zeta_0)$ must be 1 and also $\left(\frac{\lambda_{k,f/2}(\varpi_E)}{\mu_{p^s+1}}\right) = 1$. We have that

$$t_\varpi(V_{[g]})t(W_{[g]}) = \begin{cases} (1)(-1) = -1 & \text{if } V_{[g]} \text{ is trivial} \\ -\left(\frac{\lambda_{k,f/2}(\varpi_E)}{\mu_{p^s+1}}\right)(1) = -1 & \text{if } V_{[g]} \text{ is non-trivial} \end{cases}.$$

Hence we should assign $\chi_{g,\xi}(\varpi_E) = -1$. In case $\lambda_{k,f/2}(\varpi_E)$ has even order, we can assign $\chi_{g,\xi}(\zeta_0)$, hence $\chi_{g,\xi}(\varpi_E)$, to be either ± 1. We then assign $\chi_{g,\xi}(\varpi_E)$ as $t_\varpi(V_{[g]})t(W_{[g]})$.

We have checked that each $\chi_{g,\xi}$ is a character in χ-data for symmetric unramified $[g]$. Together with the asymmetric $[g]$ and those $[\sigma^k\phi^i] \in \mathcal{D}_0$ which are all automatically asymmetric, we can check whether

$$\chi_{\mathcal{D}_0}(\varpi_E) = {}_{K_0/F}\mu_\xi(\varpi_E);$$

that is to check whether

$$t^0_\mu(V^\varpi_{[g]})t_\varpi(V_{[g]})t(W_{[g]}) = (-1)^{e(f-1)}t^0_\mu(V_{\mathcal{D}_0})t^0_\mu(V^\varpi_{\mathcal{D}_0})t_\varpi(V_{\mathcal{D}_0}).$$

By applying the fact (7.3.4) that $t(W_{[g]}) = -t^0_\mu(V_{[g]})$ if $[g]$ is symmetric unramified and canceling the t-factors, it suffices to show that on the left side the parity of $\#(W_E\backslash W_F/W_E)_{\text{sym-unram}}$ equals on the right side $(-1)^{e(f-1)}$. This is just Proposition 2.5.

We have proved Theorem 9.1 for $[g] \in \mathcal{D}_0$. Hence we have finished the proof of Theorem 9.1 for the residual characteristic p being odd.

9.5 Towards the end of the proof

The case when the residual characteristic $p = 2$ is much simpler. When E/F is tamely ramified, we have the sequence of subfields $F \subseteq K \subseteq E$ where K/F is unramified and E/K is totally ramified and has odd degree. Since the order of $\Psi_{E/F}$ is odd, all sign characters and Jacobi symbols, and so all t-factors for $V_{[g]}$ are trivial. Hence Theorem 9.1 reduces to the following.

(i) If $[g] \in (W_E\backslash W_F/W_E)_{\text{asym}}$, then

$$\chi_{g,\xi}|_{\mu_{E_g}} \equiv 1 \text{ and } \chi_{g,\xi}(\varpi_E) = \text{anything appropriate}.$$

(ii) If $[g] \in (W_E\backslash W_F/W_E)_{\text{sym}}$, then

$$\chi_{g,\xi}|_{\mu_{E_g}} \equiv 1 \text{ and } \chi_{g,\xi}(\varpi_E) = t^0_\varpi(V_{[g]})t(W_{[g]}) = -1.$$

The proof is just a simpler analog of the odd residual characteristic cases. We have completed the proof of Theorem 9.1. Therefore we can express the essentially tame local Langlands correspondence by the inverse

$$\mathcal{L}_n^{-1} : \pi_\xi \to \chi_{\{\chi_{\lambda,\xi}^{-1}\}} \circ \tilde{\xi}$$

where the notation is referred to Theorem 1.3.

Remark 9.6. One of the conditions of Langlands Correspondence, namely $\omega_\pi = \det \sigma$ if $\mathcal{L}(\pi) = \sigma$, implies that ${}_F\mu_\xi|_{F^\times} = \delta_{E/F}$. This is a general fact about the restriction of the product of the characters in any χ-data, as established in Proposition 4.18. □

Remark 9.7. If we examine the proof of Theorem 9.1 carefully, we see that the χ-data depend only on the W_F-equivalence of the jump data of ξ in almost all g, except in the case when e is even and $[g] = [\sigma^{e/2}]$. In this exceptional case $\chi_{g,\xi}$ depends additionally on the additive character ψ_F of F. □

Let K be a intermediate subfield between E/F. We regard an F-admissible character ξ as being admissible over K and compute the corresponding the rectifier ${}_K\mu_\xi$.

Corollary 9.8. *Let $\{\chi_{g,\xi}\}_{[g]\in(W_E\backslash W_F/W_E)'}$ be the collection of χ-data corresponding to the F-admissible character ξ. Let K be a subfield between E/F. Then we have*

$$ {}_K\mu_\xi = \prod_{[g]\in(W_E\backslash W_K/W_E)'} \chi_{g,\xi}|_{E^\times}, \tag{9.5.1}$$

i.e. the sub-collection $\{\chi_{g,\xi}\}_{[g]\in(W_E\backslash W_K/W_E)'}$ is the χ-data corresponding to ξ as an K-admissible character.

Proof. We first compute (9.5.1) when restricted to μ_E. If $e(E/K)$ is odd, then

$$\prod_{[g]\in(W_E\backslash W_K/W_E)'} \chi_{g,\xi}|_{\mu_E} = \prod_{[g]\in(W_E\backslash W_K/W_E)'} t^1_\mu(V_{[g]}) = t^1_\mu(V^K).$$

which equals ${}_K\mu_\xi|_{\mu_E}$ in this case. If $e(E/K)$ is even, then

$$\prod_{[g]\in(W_E\backslash W_K/W_E)'} \chi_{g,\xi}|_{\mu_E} = \left(\prod_{\substack{[g]\in(W_E\backslash W_K/W_E)' \\ [g]\neq[\sigma^{e/2}]}} t^1_\mu(V_{[g]}) \right) \left(\frac{}{\mu_E} \right) = t^1_\mu(V^K) \left(\frac{}{\mu_E} \right).$$

which also equals ${}_K\mu_\xi|_{\mu_E}$ in this case. We then compute (9.5.1) when restricted to ϖ_E. Let L be the maximal unramified extension in E/K and $V_{L/K}$ be the complementary module of V^L in V^K, then

$$\prod_{[g]\in(W_E\backslash W_K/W_E)'} \chi_{g,\xi}(\varpi_E) = \left(\prod_{[g]\in(W_E\backslash W_K/W_E)_{\mathrm{asym}/\pm}} t_{\varpi_E}(V_{\pm[g]})t(W_{\pm[g]}) \right) \left(\prod_{[g]\in(W_E\backslash W_K/W_E)_{\mathrm{sym}}} t^0_\mu(V^{\varpi}_{[g]})t_\varpi(V_{[g]})t(W_{[g]}) \right).$$

We re-group the t-factors and obtain

$$\left(t_\varpi(V^L)t(W^L)\right)\left((-1)^{e(E/K)(f(E/K)-1)}t^0_\mu(V_{L/K})t^0_\mu(V^{\varpi}_{L/K})t_\varpi(V_{L/K})\right).$$

The first factor is ${}_L\mu_\xi(\varpi_E)$, which the second factor is ${}_{L/K}\mu_\xi(\varpi_E)$. Therefore the product is ${}_K\mu_\xi(\varpi_E)$. □

9.6 Rectifiers in the theory of endoscopy

Let K/F be a cyclic sub-extension of E/F of degree d. Using Corollary 9.5.1 we can extend the definition of ν-rectifiers by

$${}_{K/F}\mu_\xi = {}_F\mu_\xi\, {}_K\mu_\xi^{-1}$$

Let Δ_{III_2} be the transfer factor defined for the groups $(G,H) = (\mathrm{GL}_n, \mathrm{Res}_{K/F}\mathrm{GL}_{n/d})$ and by the χ-data attached to an admissible character ξ of $E^\times$ over F.

Corollary 9.9. *The transfer factor Δ_{III_2} is equal to the ν-rectifier ${}_{K/F}\mu_\xi$. In particular, if E/F is cyclic, then the rectifier ${}_F\mu_\xi$ is exactly Δ_{III_2}.*

Proof. By Corollary 9.8 the ν-rectifier ${}_{K/F}\mu_\xi$ is equal to

$$ {}_F\mu_\xi\, {}_K\mu_\xi^{-1} = \prod_{[g]\in W_E\backslash W_F/W_E - W_E\backslash W_K/W_E} \chi_{g,\xi}|_{E^\times}. $$

Then the first assertion is just a consequence of Corollary 4.13. The second assertion is clear. □

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