AUTOMORPHIC L-FUNCTIONS AND THEIR APPLICATIONS TO NUMBER
THEORY

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
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Abstract

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2012

The main part of the thesis is applications of the Strong Artin conjecture to number theory. We have two applications. One is generating number fields with extreme class numbers. The other is generating extreme positive and negative values of Euler-Kronecker constants.

For each number field K of degree n, there is the Artin L-function $L(s, \rho) = \frac{\zeta_K(s)}{\zeta(s)}$ attached to K. When $L(s, \rho)$ is an entire function and has a zero-free region $[\alpha, 1] \times [-(\log N)^2, (\log N)^2]$ where N is the conductor of $L(s, \rho)$, we can estimate $\log L(1, \rho)$ and $\frac{L'}{L}(1, \rho)$ as a sum over small primes:

$$\log L(1, \rho) = \sum_{p \leq (\log N)^k} \lambda(p) p^{-1} + O_{l,k,\alpha}(1)$$

$$\frac{L'}{L}(1, \rho) = -\sum_{p \leq x^2} \frac{\lambda(p) \log p}{p} + O_{l,x,\alpha}(1).$$

where $0 < k < \frac{16}{1-\alpha}$ and $(\log N)^{\frac{16}{1-\alpha}} \leq x \leq N^{\frac{1}{4}}$. With these approximations, we can study extreme values of class numbers and Euler-Kronecker constants.

Let $\mathfrak{K}(n, G, r_1, r_2)$ be the set of number fields of degree n with signature (r_1, r_2) whose normal clousres are Galois G extension over $\mathbb{Q}$. Let $f(x, t) \in \mathbb{Z}[t][x]$ be a parametric polynomial whose splitting field over $\mathbb{Q}(t)$ is a regular G extension. By Cohen's theorem, most specialization $t \in \mathbb{Z}$ corresponds to a number field K_t in $\mathfrak{K}(n, G, r_1, r_2)$ with signature (r_1, r_2) and hence we have a family of Artin L-functions $L(s, \rho, t)$. By counting

zeros of L-functions over this family, we can obtain L-functions with the zero-free region above.

In Chapter 1, we collect the known cases for the Strong Artin conjecture and prove it for the cases of $G = A_4$ and S_4. We explain how to obtain the approximations of $\log(1, \rho)$ and $\frac{L'}{L}(1, \rho)$ as a sum over small primes in detail. We review the theorem of Kowalski-Michel on counting zeros of automorphic L-functions in a family.

In Chapter 2, we exhibit many parametric polynomials giving rise to regular extensions. They contain the cases when $G = C_n$, $3 \leq n \leq 6$, D_n, $3 \leq n \leq 5$, A_4, A_5, S_4, S_5 and S_n, $n \geq 2$.

In Chapter 3, we construct number fields with extreme class numbers using the parametric polynomials in Chapter 2.

In Chapter 4, We construct number fields with extreme Euler-Kronecker constants also using the parametric polynomials in Chapter 2.

In Chapter 5, we state the refinement of Weil's theorem on rational points of algebraic curves and prove it.

The second topic in the thesis is about simple zeros of Maass L-functions. We consider a Hecke Maass form f for $SL(2, \mathbb{Z})$. In Chapter 6, we show that if the L-function $L(s, f)$ has a non-trivial simple zero, it has infinitely many simple zeros. This result is an extension of the result of Conrey and Ghosh [14].

Acknowledgements

Contents

Chapter 1

Artin L-functions

1.1 The strong Artin Conjecture

Let ρ be an n-dimensional irreducible representation of $G = Gal(K/\mathbb{Q})$ for an Galois extension $K/\mathbb{Q}$. Artin conjectured that the Artin L-function $L(s, \rho)$ is a holomorphic function on the complex plane and it is called the Artin Conjecutre. Langlands conjectured that $L(s, \rho)$ actually is automorphic, i.e. $L(s, \rho) = L(s, \pi)$ for a cuspidal automorphic representation π of $GL(n)/\mathbb{Q}$. Since a cuspidal automorphic L-function is entire, Langlands's prediction implies the Artin Conjecutre so it is called the strong Artin Conjecture.

For several cases, the strong Artin Conjecture is known to be true. For the one-dimensional representations, the strong Artin conjecture is true. By Langlands and Tunnell, the strong Artin Conjecture is true for 2-dimensional representations of solvable groups. When the projective image of two-dimensional odd representation ρ is isomorphic to A_5, then $L(s, \rho)$ is automorphic by the recent work of Khare and Wintenberger [35].

Here we prove that the strong Artin Conjecture is true for the three-dimensional representations for $Gal(K/\mathbb{Q}) = S_4$ and A_4.

Proposition 1.1. *Let K be a number field such that $G = Gal(K/\mathbb{Q})$ is isomorphic*

to S_4 or A_4. Then, for any irreducible representation ρ of $G = Gal(K/\mathbb{Q})$, its Artin L-function $L(s, K/\mathbb{Q}, \rho)$ is automorphic. Especially, the strong Artin Conjecture for S_4 and A_4 case is true.

Before proving Proposition 1.1, we introduce another proposition. We find an idea of this proof from Wang [69].

Proposition 1.2. Let ϕ be an octahedral (resp, a tetrahedral) representation of $G_{\mathbb{Q}}$ and let G denote $\phi(G_{\mathbb{Q}})$. Then G is isomorphic to $(GL(2, F_3) \times C_{2m})/\{\pm(I, 1)\}$, (resp, $(SL(2, F_3) \times C_{2m})/\{\pm(I, 1)\}$) where $Z(G) \cong C_{2m}$, the cyclic group of order $2m$.

Hence each irreducible representation Φ of G can be expressed uniquely as (Φ_0, μ), where $\Phi_0 = \Phi \mid_{GL(2,F_3)}$ is an irreducible representation of $GL(2, F_3)$ (resp, $SL(2, F_3)$) and $\mu = \Phi \mid_{C_{2m}}$ is a character of C_{2m} and such that $\Phi_0(-I) = \mu(-1)I$. Furthermore, each such pair (Φ_0, μ) gives an irreducible representation of G.

Proof. First S_4 case.

We consider the 2-dimensional representation Ψ of $GL(2, F_3) = SL(2, F_3) \rtimes < a >$ defined by

$$\Psi \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \Psi \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -\sqrt{-2} & -1 + \sqrt{-2} \end{pmatrix}.$$

This induces an embedding Σ of $PGL(2, F_3) = PSL(2, F_3) \rtimes < \overline{a} >$ into $PGL(2, \mathbb{C})$, where $\overline{a}$ is the projective image of a in $PGL(2, F_3)$.

Now let $\overline{\phi}$ be the projective representation from an octahedral representation ϕ. Then $\overline{\phi}$ factors through a Galois group isomorphic to $PGL(2, F_3)$. Via isomorphisms, we identify it with $PGL(2, F_3)$ and $\overline{\phi}(PGL(2, F_3))$ with $\Sigma(PGL(2, F_3))$.

The image of $G' = [G, G]$, a commutator subgroup of G, in $PGL(2, \mathbb{C})$ is $\Sigma(PSL(2, F_3))$ because $[S_4, S_4] = A_4$. Since $det(G') = 1$ and A_4 has no irreducible 2-dimensional representation, we have $Z(G') = \{\pm I\}$. So G' is $\Psi(SL(2, F_3))$ which is isomorphic to the central extension $SL(2, F_3)$ of $PSL(2, F_3)$ by $\{\pm I\}$. Then $\{G' \times C_{2m}\}/\pm(I, 1)$, where C_{2m} is

$Z(G)$, is the pre-image of $\Sigma(PSL(2, F_3))$ and it becomes a normal subgroup of G with index 2. Since $PGL(2, F_3) = PSL(2, F_3) \rtimes <\bar{a}>$, $\Sigma(PGL(2, F_3)) = \Sigma(PSL(2, F_3)) \rtimes \Sigma(<\bar{a}>)$. Since the image in $PGL(2, \mathbb{C})$ is determined by modulo $Z(G) \cong C_{2m}$, G also contains $\Psi(<a>)$. So finally we have $G \cong \{G' \times C_{2m}\}/\pm(I, 1) \rtimes \Psi(<a>) \cong \{(\Psi(PGL(2, F_3)) \rtimes \Psi(<a>)) \times C_{2m}\}/\pm(I, 1) \cong (GL(2, F_3) \times C_{2m})/\{\pm(I, 1)\}$.

Second A_4 case.

If $det(g) = 1$ for all $g \in G$, then we have $G \cong SL(2, F_3)$. If not, we claim that $G' \cong Q_8$, where Q_8 is a quaternion group. To see this, its image in $PGL(2, \mathbb{C})$ is isomorphic to V, the Klein 4-group. Since V can not be embedded into $GL(2, \mathbb{C})$ with property $det = 1$, G' is isomorphic to a central extension of V by $\mathbb{Z}/2\mathbb{Z}$. There are 3 non-split central extensions of V by $\mathbb{Z}/2\mathbb{Z}$: $C_4 \times C_2, D_8$ and Q_8. Since $C_4 \times C_2$ is abelian, it should be excluded. And if $D_8 = \{r^4 = 1, s^2 = 1, srs = r^3\}$ is embedded into $GL(2, \mathbb{C})$ with property $det = 1$, it induce a contradiction. Because $det(s) = 1, s^2 = I$ implies $s = -I$, we have $r^2 = I$. So it is a contradiction.

Since $SL(2, F_3) = H \rtimes K$ where $H \cong Q_8$ and $K = <b>$ for some b of order 3, by the similar argument above, we have $G \cong (SL(2, F_3) \times C_{2m})/\{\pm(I, 1)\}$

The other part of the proposition is easily followed.   □

Proof of Proposition 1.1:

Since $Gal(K/\mathbb{Q}) \cong S_4$, using an embedding of S_4 into $PGL(2, \mathbb{C})$, we have a projective representation $\bar{\omega} : G_\mathbb{Q} \longrightarrow PGL(2, \mathbb{C})$. Since $H^2(G_\mathbb{Q}, \mathbb{C}^*) = \{1\}$, there exists a lifting ω of $\bar{\omega}$. Then by Proposition 4, ω factor through $Gal(\widetilde{K}/\mathbb{Q})$ which is isomorphic to $(GL(2, F_3) \times C_{2m})/\{\pm(I, 1)\}$ for some $m \geq 1$.

By Proposition 1.2, the irreducible representations of $Gal(\widetilde{K}/\mathbb{Q})$ are suitable twists of irreducible representations of $GL(2, F_3)$. Since every irreducible representation of $GL(2, F_3)$ is automorphic (See Kim [37] for relevant references), so is that of $Gal(\widetilde{K}/\mathbb{Q})$. Since $(\chi_1, 1), (\chi_1', 1), (\chi_2', 1), (\chi_3, 1), (\chi_3', 1)$ are irreducible representations of $Gal(\widetilde{K}/\mathbb{Q})$ which factor through S_4, Proposition 1.1 follows. The proof of A_4 case is essentially the

same.

Remark 1.3. Proposition 1.1 is well-known to the experts, at least implicitly, under the assumption of the existence of the double cover $\widetilde{K}/K$ where $Gal(\widetilde{K}/\mathbb{Q})$ is $GL(2, F_3)$ and it is stated in Kim ([37], p.100). What is new is the observation that K always has a central extension $\widetilde{K}$ such that $Gal(\widetilde{K}/\mathbb{Q}) \cong (GL(2, F_3) \times C_{2m})/\pm(I, 1)$ and so every 3-dimensional representation attached to K is equivalent to a twist of a symmetric square of 2 dimensional representation of $GL(2, F_3)$.

Recently, Calegari obtained the modularity of S_5 Galois representations for a special case.

Theorem 1.4 (Calegari [5]). *Let $K/\mathbb{Q}$ be a quintic extension with $Gal(\widehat{K}/\mathbb{Q}) = S_5$. Furthermore, we assume that*

1. *Complex conjugation in $Gal(\widehat{K}/\mathbb{Q}) = S_5$ has conjugacy class $(12)(34)$.*

2. *The extension $\widehat{K}/\mathbb{Q}$ is unramified at 5 and the Frobenius element $Frob_5$ has conjugacy class $(12)(34)$.*

If $\rho : Gal(\widehat{K}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$ be an irreducible representation of dimension 4 or 6, then ρ is modular.

Remark 1.5. The modularity of the 4 dimensional representation ρ is considered by Y. Zhang [71] in his PhD thesis. He observed that ρ is equivalent to a character twist of Asai lift of σ, where σ is an icosahedral representation over F, the quadratic subextension over $\mathbb{Q}$. Calegari also proved it and used the result of S. Sasaki [57] on the muldarity of σ to prove the modularity of ρ.

1.2   Approximation of $\log L(1, \rho)$ and $\frac{L'}{L}(1, \rho)$

When an Artin L-function $L(s, \rho)$ has a zero-free region, we can approximate $\log L(1, \rho)$ and $\frac{L'}{L}(1, \rho)$ as a summation over small primes. The following proposition by Daileda is

crucial for these two approximations.

Proposition 1.6 (Daileda, Corolloary 4 in [18])**.** *Let $F/\mathbb{Q}$ be a finite Galois extension and let ρ be an l-dimensional complex representation of $Gal(F/\mathbb{Q})$ of conductor N. Let $6/7 < \alpha < 1$. If $L(s, \rho)$ is entire and is free from zeros in the rectangle $[\alpha, 1] \times [-(\log N)^2, (\log N)^2]$ and N is sufficiently large, then*

$$\frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s+u, \rho)\Gamma(s)x^s \, ds - \frac{L'}{L}(u, \rho) \ll_l \frac{x^2}{(1-\alpha)^2\sqrt{N}} + \frac{(\log N)^2}{(1-\alpha)^3 x^{(1-\alpha)/8}}$$

for $1 \le u \le 3/2$ and $x \ge 1$.

As an application of Proposition 1.6, Daileda approximates $\log L(1, \rho)$. We copy the proof in [18] for the sake of completeness. Let

$$L(s, \rho) = \prod_p L(s, \rho)_p = \sum_{n=1}^{\infty} \lambda(n)n^{-s}, \quad L(s, \rho)_p = \prod_{i=1}^{n}(1 - \alpha_i(p)p^{-s})^{-1}.$$

Then $\lambda(p) = \sum_{i=1}^{n} \alpha_i(p)$, and $|\lambda(p)| \le n$.

By Mellin inversion of $\Gamma(s)$ and logarithmic derivative of $L(s, \rho)$,

$$\frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s+u, \rho)\Gamma(s)x^s ds = -\sum_{i=1}^{l}\sum_p \log p \sum_{k=1}^{\infty} \alpha_i(p)^k p^{-ku} e^{-p^k/x}.$$

Substitute this to Proposition 1.6 and integrate from $u = 1$ to $u = 3/2$. Then we have

$$\sum_p \lambda(p)p^{-1}e^{-p/x} - \log L(1, \rho) + \log L(3/2, \rho) \ll_l \frac{x^2}{(1-\alpha)^2\sqrt{N}} + \frac{(\log N)^2}{(1-\alpha)^3 x^{(1-\alpha)/8}}.$$

Here we used the fact that the terms for $k \ge 2$ converge absolutely.

If $y < x$ then,

$$\sum_{p \le y} p^{-1}(1 - e^{-p/x}) < 1, \quad \sum_{p > x^2} p^{-1}e^{-p/x} \ll 1 \text{ and } \sum_{y \le p \le x^2} p^{-1}e^{-p/x} = \log\left(\frac{2\log x}{\log y}\right) + O(1).$$

Take $x = (\log N)^{16/(1-\alpha)}$ and $y = (\log N)^k$ with $0 < k < 16/(1 - \alpha)$. Since $\log L(3/2, \rho) \ll 1$, we have

$$\log L(1, \rho) = \sum_{p \le (\log N)^k} \lambda(p)p^{-1} + O_{l,k,\alpha}(1)$$

and it is summarized as follows.

Proposition 1.7 (Daileda, Proposition 2 in [18]). *Let $L(s, \rho)$ and N be as above. Let $\frac{6}{7} < \alpha < 1$. Suppose that $L(s, \rho)$ is zero-free in the rectangle $[\alpha, 1] \times [-(\log N)^2, (\log N)^2]$. If N is sufficiently large, then for any $0 < k < \frac{16}{1-\alpha}$,*

$$\log L(1, \rho) = \sum_{p \leq (\log N)^k} \lambda(p) p^{-1} + O_{l,k,\alpha}(1).$$

Now we want to approximate $\frac{L'}{L}(1, \rho)$. Set $u = 1$ in Proposition 1.6. Then

$$-\frac{L'}{L}(1, \rho) + \frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s + 1, \rho) x^s \Gamma(s) ds \ll \frac{x^2}{(1 - \alpha)^2 \sqrt{N}} + \frac{(\log N)^2}{(1 - \alpha)^3 x^{(1-\alpha)/8}}. \quad (1.8)$$

By taking logarithmic derivative of $L(s, \rho)$ and the Mellin inversion of $\Gamma(s)$, we have

$$-\frac{1}{2\pi i} \int_{(2)} \frac{L'}{L}(s + 1, \rho) \Gamma(s) x^s \, ds = \sum_{i=1}^{l} \sum_{p} \log p \sum_{k=1}^{\infty} \alpha_i(p)^k p^{-k} e^{-p^k/x}.$$

Since the terms for $k \geq 2$ converge absolutely, we only need to estimate

$$\sum_{p} \lambda(p) \frac{\log p}{p} e^{-\frac{p}{x}}.$$

Let x be a constant with $(\log N)^{\frac{16}{1-\alpha}} \leq x \leq N^{\frac{1}{4}}$. Then the error term in (1.8) is $O_{l,x,\alpha}(1)$. On the other hand,

$$\sum_{p \leq x} \frac{\log p}{p}(1 - e^{-p/x}) < 1, \quad \sum_{p > x^2} \frac{\log p}{p} e^{-p/x} \ll 1.$$

Hence we obtain an approximation of $\frac{L'}{L}(1, \rho)$ as a sum over a short interval and it is summarized as follows:

Proposition 1.9. *Suppose that $L(s, \rho)$ is entire and free from zeros in the rectangle $[\alpha, 1] \times [-(\log N)^2, (\log N)^2]$. If N is sufficiently large, then for any constant x with $(\log N)^{\frac{16}{1-\alpha}} \leq x \leq N^{\frac{1}{4}}$,*

$$\frac{L'}{L}(1, \rho) = -\sum_{p \leq x^2} \frac{\lambda(p) \log p}{p} + O_{l,x,\alpha}(1).$$

Due to lack of GRH, we cannot use Proposition 1.7 and 1.9 directly. We extend the result of Kowalski-Michel [39] to isobaric automorphic representations of $GL(n)$.

Let $n = n_1 + \cdots + n_r$, and let $S(q)$ be a set of $(\pi_1, ..., \pi_r)$, where π_i is a cuspidal automorphic representation of $GL(n_i)/\mathbb{Q}$ which satisfies the Ramanujan-Petersson conjecture at the finite places for each i, such that

(1) There exists $e > 0$ such that for $(\pi_1, ..., \pi_r) \in S(q)$, $Cond(\pi_1) \cdots Cond(\pi_r) \leq q^e$;

(2) There exists $d > 0$ such that $|S(q)| \leq q^d$.

(3) The Γ factors of π_i are of the form $\prod_{j=1}^{n_i} \Gamma(\frac{s}{2} + \alpha_j)$, where $\alpha_j \in \mathbb{R}$.

Note that for $(\pi_1, ..., \pi_r) \in S(q)$, $\pi = \pi_1 \boxplus \cdots \boxplus \pi_r$ is an isobaric automorphic representation of $GL(n)/\mathbb{Q}$, and $Cond(\pi) = Cond(\pi_1) \cdots Cond(\pi_r)$. So we can think of $S(q)$ as a finite set of isobaric automorphic representations of $GL(n)/\mathbb{Q}$.

Let, for $\alpha \geq \frac{3}{4}$, $T \geq 2$,

$$N(\pi; \alpha, T) = |\{\rho : \ L(\rho, \pi) = 0, \ Re(\rho) \geq \alpha, \ |Im(\rho)| \leq T\}|.$$

Then clearly, $N(\pi; \alpha, T) = N(\pi_1; \alpha, T) + \cdots + N(\pi_r; \alpha, T)$.

Theorem 1.10. *For some $B \geq 0$,*

$$\sum_{\pi \in S(q)} N(\pi; \alpha, T) \ll T^B q^{c_0 \frac{1-\alpha}{2\alpha - 1}}.$$

One can choose any $c_0 > c_0'$, where $c_0' = \frac{5n'e}{2} + d$ and $n' = max\{n_i\}_{1 \leq i \leq r}$.

Proof. Let $S(q)_i$ be the set of the cuspidal automorphic representation π_i of $GL(n_i)/\mathbb{Q}$ such that there exist $\pi_1, ..., \pi_{i-1}, \pi_{i+1}, ..., \pi_r$ and $(\pi_1, ..., \pi_{i-1}, \pi_i, \pi_{i+1}, ..., \pi_r) \in S(q)$. Then clearly, $Cond(\pi_i) \leq q^e$ and $|S(q)_i| \leq q^d$. So

$$\sum_{\pi \in S(q)} N(\pi; \alpha, T) = \sum_{i=1}^{r} \sum_{\pi_i \in S(q)_i} N(\pi_i; \alpha, T).$$

Now we apply the result of Kowalski-Michel [39] to the inner sum. They assumed that the Gamma factors of π_i are the same. However, the assumption is used only to obtain

the convexity bound (Lemma 10 of [39]), and our Γ-factors provide the same convexity bound. Hence our result follows. $\square$

In the following, we apply the above result to a family of Artin L-functions. In this case, the Γ-factors are a product of $\Gamma(\frac{s}{2})$ and $\Gamma(\frac{s+1}{2})$.

Chapter 2

Regular extensions and Parametric polynomials

This chapter is based on [6], [7] and [8]. We follow closely [6], [7] and [8].

A finite extension E of the rational function field $\mathbb{Q}(t)$ is called regular if $\overline{\mathbb{Q}} \cap E = \mathbb{Q}$. Suppose $f(x, t)$ is an irreducible polynomial of degree n, and gives rise to a regular Galois extension over $\mathbb{Q}(t)$ with the Galois group G. Let K_t be a field obtained by adjoining to $\mathbb{Q}$ a root of $f(x, t)$ with a specialization $t \in \mathbb{Z}$ and let $\widehat{K_t}$ be the Galois closure of K_t. Let C be any conjugacy class of G. Serre observed the following important fact, regarding distribution of Frobenius elements in a regular Galois extension ([63], page 45).

Theorem 2.1. *There is a constant $k > 0$ depending on $f(x, t)$ such that for any prime $p \geq k$, there is $t_C \in \mathbb{Z}$ so that for any $t \equiv t_C \pmod{p}$, p is unramified in $\widehat{K_t}/\mathbb{Q}$, and $Frob_p \in C$.*

We want to construct regular extensions E over $\mathbb{Q}(t)$ as a splitting field of $f(x, t) \in \mathbb{Z}[t][x]$ over $\mathbb{Q}(t)$. Let $\mathfrak{K}(n, G, r_1, r_2)$ be the set of number fields of degree n with signature (r_1, r_2) whose normal closures are G Galois extension over $\mathbb{Q}$. Assume that $f(x, t)$ of degree n gives rise to an G Galois extension over $\mathbb{Q}(t)$ which is also regular. By Hilbert's irreducibility theorem, for infinitely many specialization $t \in \mathbb{Z}$, we have number fields

of degree n whose normal closures are G Galois extensions over Q. Moreover, Cohen obtained a quantitative version of Hilbert's irreducibility theorem. His theorem is valid in a very general setting. We paraphrase it into our special case. For $f(x,t) \in \mathbb{Z}[t][x]$, we define the height $|f|$ of f to be the maximum of absolute values of the integral coefficients unless this quantity is less than 8, in which case put $|f| = 8$ and let $\|f\| = \log |f|$.

Theorem 2.2 (Cohen [12], Theorem 2.1). *Let $f(x,t) \in \mathbb{Z}[t][x]$ be a non-zero polynomial of total degree not exceeding n in the indeterminate x with Galois group G over $\mathbb{Q}(t)$. Then, provided $N > |f|^c$, the number of $\alpha \in \{t \in \mathbb{Z} | |t| \le N\}$ for which $G(\alpha)$, the Galois group of $f(x,\alpha)$ over $\mathbb{Q}$, differs from G does not exceed*

$$|f|^{c/3} N^{1/2} \log N$$

where $c = c(n)$. Indeed, if actually $N > exp(c\|f\|^2)$, then the number of exceptional α does not exceed $cN^{1/2} \log N$.

By Cohen's theorem, once we find a polynomial $f(x,t)$ giving rise to a regular G Galois extension, we also have number fields $K \in \mathfrak{K}(n, G, r_1, r_2)$ for some suitable signature (r_1, r_2).

Remark 2.3. To apply Theorem 2.2, the total degree of $f(x,t)$ should not exceed the degree of x. In section 2.1.3, the total degree of the simplest quintic polynomial is 6 which is bigger than 5. However, we don't need Theorem 2.2 in the case because the Galois group of $f(x,\alpha)$ is C_5 for all $\alpha \in \mathbb{Z}$.

Also we have to compute an upper bound of regulators of number fields we constructed when we estimate the class numbers. In many cases, it is crucial to know the locations of roots of the polynomial $f(x,t)$ to determine independence of units and find an upper bound of regulators. We introduce the following lemma from [52],[60].

Lemma 2.4. *Let f be a polynomial of degree m and $f(\alpha) \neq 0$, $f'(\alpha) \neq 0$. Then for every circle C passing through α, $\alpha - \frac{mf(\alpha)}{f'(\alpha)}$, at least one root of f is inside of C, and one root outside of C.*

2.1   Cyclic groups

Let $f(x,t) = x^n + a_1(t)x^{n-1} + \cdots + a_{n-1}(t)x \pm 1$ be an irreducible polynomial over $\mathbb{Q}(t)$ such that $a_i(t) \in \mathbb{Z}[t]$. Suppose $f(x,t)$ gives rise to a cyclic extension over $\mathbb{Q}(t)$, and if $t \in \mathbb{Z}$, it gives rise to a totally real extension over $\mathbb{Q}$. For each integer $t > 0$, let K_t be the cyclic extension over $\mathbb{Q}$. Let $Gal(K_t/\mathbb{Q}) = \{1, \sigma, \sigma^2, ..., \sigma^{n-1}\}$. Let θ be a root of $f(x,t)$. Then $\theta, \sigma(\theta), ..., \sigma^{n-1}(\theta)$ are roots of $f(x,t)$.

We show that if $n = p$ is a prime, $\sigma(\theta), ..., \sigma^{p-1}(\theta)$ form independent units, and the regulator of K_t is small. By definition, the regulator of $\mathbb{Z}[\theta]$ is

$$R = |\det(\log|\sigma^{i+j}(\theta)|)_{1 \le i,j \le p-1}|.$$

Theorem 2.5. *(1) $R \ne 0$, and (2) $R \ll (\log t)^{p-1}$.*

Proof. By Lemma 5.26 in [70], we have

$$R = \frac{1}{p} \prod_{\chi \ne 1} \left(\sum_{i=0}^{p-1} \chi(\sigma^i) \log|\sigma^i(\theta)| \right),$$

where the product runs over the nontrivial characters of $Gal(K_t/\mathbb{Q})$.

Since $t^{-c} \ll \sigma^i(\theta) \ll t^d$ for some $c, d > 0$ depending only on $f(x,t)$, $|\log|\sigma^i(\theta)|| \ll \log t$. Hence (2) follows.

Since $\theta \cdot \sigma(\theta) \cdots \sigma^{p-1}(\theta) = \pm 1$,

$$\log|\theta| + \log|\sigma(\theta)| + \cdots + \log|\sigma^{p-1}(\theta)| = 0.$$

Hence we write
$$\sum_{i=0}^{p-1} \chi(\sigma^i) \log|\sigma^i(\theta)| = \sum_{i=1}^{p-1} (\chi(\sigma^i) - 1) \log|\sigma^i(\theta)|.$$

Since p is a prime, by [19], $\sigma(\theta), \ldots, \sigma^{p-1}(\theta)$ are multiplicatively independent. Hence $\log|\sigma(\theta)|, \ldots, \log|\sigma^{p-1}(\theta)|$ are linearly independent over $\mathbb{Q}$. By Baker's theorem [2], they are linearly independent over $\overline{\mathbb{Q}}$.

Since $\chi(\sigma^i)$ are roots of unity and $\chi \neq 1$, one of $\chi(\sigma^i) - 1$ is not zero. Hence

$$\sum_{i=1}^{p-1} (\chi(\sigma^i) - 1) \log |\sigma^i(\theta)| \neq 0.$$

$\square$

Remark 2.6. If n is not a prime, we still have $R \ll (\log t)^{n-1}$. However, $R = 0$ for a composite number n. We show this for simplest quartic and sextic fields.

2.1.1 Simplest Cubic Fields

It was Shanks [64] who introduced Simplest cubic fields parameterized by

$$f(x, t) = x^3 - tx^2 - (t+3)x - 1,$$

for $t \in \mathbb{Z}^+$. Its discriminant is $g(t)^2$ with $g(t) = t^2 + 3t + 9$.

2.1.2 Simplest Quartic Fields

Consider totally real cyclic quartic fields K_t generated by a root of

$$f(x, t) = x^4 - tx^3 - 6x^2 + tx + 1, \ t \in \mathbb{Z}^+.$$

Here $Disc(f(x, t)) = 4(t^2 + 16)^3$. This polynomial is studied by Lazarus. We refer to his PhD thesis [41] for the detail.

We can express the 4 roots of $f(x, t)$ explicitly.

$$\theta_{1,2,3,4} = \pm \frac{\sqrt[4]{t^2 + 16}\sqrt{\sqrt{t^2 + 16} \pm t}}{2\sqrt{2}} \pm \frac{\sqrt{t^2 + 16}}{4} + \frac{t}{4}, \ t \neq 0, 3$$

where the second and third ambiguous signs agree.

Let θ_1 be the largest root by choosing $+$ for all signs. The Galois group action on the roots is given by

$$\sigma : \theta_j \mapsto \frac{\theta_j - 1}{\theta_j + 1} = \theta_{j+1}, \ j = 1, 2, 3, 4.$$

Now $\theta_1, \theta_2 = \sigma(\theta_1)$ and $\theta_3 = \sigma^2(\theta_1)$ are not multiplicatively independent. The regulator

$$R = \frac{1}{4} \prod_{\chi \neq 1} \left(\sum_{i=0}^{3} \chi(\sigma^i) \log |\sigma^i(\theta)| \right)$$

vanishes because the term corresponding to $\chi(\sigma) = e^{\pi i}$ is zero.

It is known that θ_1, θ_2 and ϵ_t are independent units where ϵ_t is the fundamental units of $\mathbb{Q}(\sqrt{t^2 + 16})$ (See p.10 in [41]). When t is even, we can find ϵ_t.

Proposition 2.7 (Lazarus). *When t is even, ϵ_t is given by*

$$\epsilon_t = \begin{cases} \frac{t/2 + \sqrt{(t/2)^2 + 4}}{2}, & t \equiv 2 \ mod \ 4 \\ \frac{1 + \sqrt{5}}{2}, & t = 8 \\ (t/4) + \sqrt{(t/4)^2 + 1}, & otherwise. \end{cases}$$

Hence by Proposition 2.7 and the value of θ_1 and θ_2, for even t the regulator R_{K_t} is

$$R_{K_t} = O(\log^3 t) \text{ for even } t. \tag{2.8}$$

Also we can determine the discriminant d_{K_t} with 2-adic valuation $\nu_2(t)$ of t and it is at least $\frac{(t^2+16)^3}{64}$. So the simplest quartic fields are distinct and the regulator is $O((\log d_{K_t})^3)$ for even t.

From the root formula for $f(x, t)$, it is clear that $f(x, t)$ gives rise to a C_4 regular extension over $\mathbb{Q}(t)$.

2.1.3 Simplest Quintic Fields

Emma Lehmer [42] introduced a family of quintic polynomials $f(x, t)$ for $t \in \mathbb{Z}$:

$$f(x, t) = x^5 + t^2 x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 + (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2 + (t^3 + 4t^2 + 10t + 10)x + 1.$$

Here $disc(f(x, t)) = (t^3 + 5t^2 + 10t + 7)^2(t^4 + 5t^3 + 15t^2 + 25t + 25)^4$.

It is easy to show that $f(x, t)$ is irreducible for all $t \in \mathbb{Z}$ when we observe it modulo 2. And it is also known that the zeros of $f(x, t)$ generate a cyclic extension K_t of degree

5 over $\mathbb{Q}$. Let $G = Gal(K_t/\mathbb{Q})$ be the Galois group and σ be a generator of G given by

$$\sigma(\theta) = \frac{(t+2) + t\theta - \theta^2}{1 + (t+2)\theta}$$

for a root θ of $f(x, t)$. For the detail, we refer to [59]. Also it is obvious that the Galois groups of $f(x, t)$ over $\mathbb{Q}(t)$ and over $\overline{\mathbb{Q}}(t)$ are both C_5 generated by σ. Hence $f(x, t)$ gives rise to a regular C_5 extension over $\mathbb{Q}(t)$. Also, Smith [67] showed that $f(x, t)$ gives rise to a C_5 regular extension.

Schoof and Washington studied these simplest quintic fields when $P_t = t^4 + 5t^3 + 15t^2 + 25t + 25$ is a prime number. When P_t is a prime, then the zeros of $f(x, t)$ form a fundamental system of units in K_t. Gaál and Pohst extended this result for square-free P_t (see the proof of Theorem 3.5 in [59]).

In this case, by Theorem 2.5, $R_{K_t} \ll (\log t)^4$. It also follows from [59]: Let U denote the group of units generated by the zeros modulo $\{\pm 1\}$. Define $i_\theta = [O^*_{K_t}/\{\pm 1\} : U]$. Schoof and Washington [59] showed that $i_\theta \leq 11$ if $|t + 1| \geq 20$ and

$$R = |det(\log|\sigma^{i+j}(\theta)|)_{1\leq i,j\leq 4}| \leq \left(71 + \frac{36}{\log|t+1|}\right) \log^4 |t + 1|.$$

Jeannin [33] found the prime factorization of P_t.

Theorem 2.9 (Jeannin). *The number P_t is written in a unique way: $P_t = 5^c q^5 \prod_{i=1}^n p_i^{x_i}, c \in \{0, 2\}, q \in \mathbb{N}, p_i$ distinct primes, $x_i \in [1, 4]$. So the conductor of K_t is $f_t = 5^c \prod_{i=1}^n p_i$.*

Especially if P_t is cubic-free, then $P_t = 5^c \prod_{i=1}^n p_i^{x_i}$ and $x_1 = 1$ or 2 and

$$t^4 \ll P_t \leq 5^c (\prod_{i=1}^n p_i)^2 \leq (D_{K_t})^2.$$

Hence for cubic-free P_t, when we combine all these arguments, we have

$$R_{K_t} \ll \log^4(D_{K_t}) \qquad (2.10)$$

2.1.4   Simplest Sextic Fields

It was Gras [22] who introduced the simplest sextic polynomial $f(x,t)$ first, given by

$$f(x,t) = x^6 - \frac{t-6}{2}x^5 - 5\frac{t+6}{4}x^4 - 20x^3 + 5\frac{t-6}{4}x^2 + \frac{t+6}{2}x + 1$$

and discriminant of $f(x,t)$ is $\frac{3^6}{2^{14}}(t^2+108)^5$.

Let $K_t = \mathbb{Q}(\theta)$, where θ is a root of $f(x,t)$. She showed the following properties:

1. If $t \in \mathbb{Z} - \{0, \pm6, \pm26\}$, then $f(x,t)$ is irreducible in $\mathbb{Q}[X]$, and K_t is a real cyclic sextic field; a generator σ of its Galois group is characterized by the relation $\sigma(\theta) = (\theta - 1)/(\theta + 2)$. We have $K_{-t} = K_t$ for all $t \in \mathbb{Z}$, and we can suppose that $t \in \mathbb{N} - \{0, 6, 26\}$.

2. The quadratic subfield of K_t is $k_2 = \mathbb{Q}(\sqrt{t^2 + 108})$.

3. The cubic field of K_t is $k_3 = \mathbb{Q}(\phi)$, where

$$\phi = \theta^{-1-\sigma^3} = -\frac{2\theta + 1}{\theta(\theta + 2)}$$

   and

$$Irr(\phi, \mathbb{Q}) = x^3 - \frac{t-6}{4}x^2 - \frac{t+6}{4}x - 1;$$

   the discriminant of this polynomial is $((t^2 + 108)/16)^2$. If $t \equiv 2 \pmod 4$, k_3 is the simplest cubic field.

4. The conductor f of K_t is given by the following procedure: Let m be the product of primes, difference from 2 and 3, dividing $t^2 + 108$ with an exponent not congruent to 0 modulo 6; then $f = 4^k 3^l m$, where

$$k = 0 \text{ if } t \equiv 1 \bmod 2 \quad \text{ or } t \equiv \pm6 \bmod 16, \quad k = 1 \text{ if not.}$$
$$l = 0 \text{ if } t \equiv 1 \bmod 3 \quad l = 1 \text{ if } t \equiv 0 \bmod 27, \quad l = 2 \text{ if not .}$$

As in the case of simplest quartic fields, $\theta, \sigma(\theta), \sigma^2(\theta), \sigma^3(\theta)$ and $\sigma^4(\theta)$ do not form independent units. The regulator

$$R = \frac{1}{6} \prod_{\chi \neq 1} \left(\sum_{i=0}^{5} \chi(\sigma^i) \log |\sigma^i(\theta)| \right)$$

vanishes because the term corresponding to $\chi(\sigma) = e^{\pi i}$ is zero.

We need to find a set of 5 independent units in K_t to compute an upper bound of the regulator of K_t. Let E be the group of units in K_t. A unit in K_t is called a relative unit if its norm is ± 1 over k_2 and k_3. The group of the relative units in K_t is $< \pm 1 > \oplus E_*$ where

$$E_* = \{ u \in E \mid u^{1-\sigma+\sigma^2} = 1 \}.$$

and there is a unit ξ such that every unit $u \in E_*$ may be written

$$u = \xi^{\lambda + \mu \sigma} \text{ for some } \lambda, \mu \in \mathbb{Z}.$$

When t is of some specific form, Gras [22] found a generator ξ of E_*. Define $S(X)$ be a finite set of positive numbers.

$$S(X) = \left\{ 0 < r < X \mid 3r^2 + 3r + 1 \text{ and } 12r^2 + 12r + 7 \text{ square-free} \right\}.$$

For all $r \in S(X)$, let $t = (6r + 3)(36r^2 + 36r + 18)$ and we consider fields $L_r = K_t = \mathbb{Q}(w)$ where $w = \theta^{1-\sigma^3} = -\frac{\theta(2\theta+1)}{\theta+2}$. Then there exits a unit v such that $w = v^{1+\sigma}$. Hence $v = \frac{(w+1)-\sqrt{(w+1)^2-8w}}{2}$. Then v is a generator ξ of E_*. Gras also shows that if $r \in S(X)$, then the conductor of k_2 is $f_2 = 36r^2 + 36r + 21$, and the fundamental unit of k_2 is

$$\epsilon_2 = \frac{(12r^2 + 12r + 5) + (2r + 1)\sqrt{36r^2 + 36r + 21}}{2}.$$

Since $t = (6r+3)(36r^2+36r+18) \equiv 2 \bmod 4$, the field k_3 is a simplest cubic field. Hence for $r \in S$, we have an explicit set of independent units:

$$\{ \epsilon_2, \tau, \tau^{\sigma^2}, v, v^\sigma \}$$

where τ is a root of $x^3 - \frac{t-6}{4}x^2 - \frac{t+6}{4}x - 1$. Hence, for $t = (6r+3)(36r^2 + 36r + 18)$ with $r \in S(X)$.

$$R_{K_t} \ll (\log d_{K_t})^5 \tag{2.11}$$

Now we show that $f(x, (6r+3)(36r^2 + 36r + 18))$ gives rise to a regular C_6 extension over $\mathbb{Q}(r)$. If θ_r is a root of $f(x, (6r+3)(36r^2 + 36r + 18))$, then it is clear $\mathbb{Q}(r)(\theta_r)$ is the splitting field of $f(x, (6r+3)(36r^2 + 36r + 18)$ over $\mathbb{Q}(r)$ with Galois group $C_6 = <\sigma>$. By the same argument, the Galois group of $f(x, (6r+3)(36r^2 + 36r + 18))$ over $\overline{\mathbb{Q}}(t)$ is also $C_6 = <\sigma>$. Hence the claim follows.

2.2 Dihedral groups

2.2.1 D_3 extensions with signature $(1, 1)$

Consider a cubic polynomial

$$f(x, t) = x^3 + tx - 1$$

with $disc(f(x, t) = -(4t^3 + 27)$. This polynomial was studied by Ishida [31] first.

Theorem 2.12 (Ishida). *Let $K_t = \mathbb{Q}(\eta)$ be the cubic field of signature (1,1), where η is the real root of the cubic equation*

$$x^3 + tx - 1 = 0, \qquad (t \in \mathbb{Z}, t \geq 2).$$

If $4t^3 + 27$ is square-free or $t = 3m$ and $4m^3 + 1$ is square-free, then η is the fundamental unit of K.

It is easy to show that the real root η is located between $-\frac{1+\epsilon}{t}$ and $-\frac{1}{t}$ for any $\epsilon > 0$. Hence for t with square-free $4t^3 + 27$, the regulator R_{K_t} is

$$\log t < R_{K_t} < (1 + \epsilon) \log t.$$

Since $d_{K_t} = -(4t^3 + 27)$,

$$R_{K_t} \ll \log |d_{K_t}|. \tag{2.13}$$

It is obvious that $f(x, t)$ is irreducible over $\overline{\mathbb{Q}}(t)$ and its discriminant is not a square in $\overline{\mathbb{Q}}(t)$. Hence $f(x, t)$ gives rise to a D_3 regular extension.

2.2.2 D_4 extensions

Nakamula [51] constructed quartic fields with small regulators whose Galois closures have D_4 as the Galois group. Nakamula uses a polynomial with 3 parameters

$$f = x^4 - sx^3 + (t + 2u)x^2 - usx + 1$$

where $(s, t, u) \in \mathbb{N} \times \mathbb{Z} \times \{\pm 1\}$, $(s, t, u) \neq (1, -1, 1)$.

The discriminant D_f of f is given by

$$D_f = D_1^2 D_2 \text{ with } D_1 = s^2 - 4t, \ D_2 = (t + 4u)^2 - 4us^2.$$

For a zero ϵ of f with $|\epsilon| \geq 1$, we define $\alpha := \epsilon + u\epsilon^{-1}$. Put

$$K = \mathbb{Q}(\epsilon), \ F = \mathbb{Q}(\sqrt{D_1}), \ L = \mathbb{Q}(\sqrt{D_2}), \ M = \mathbb{Q}(\sqrt{D_1 D_2})$$

Then $F = \mathbb{Q}(\alpha) \subseteq K = F(\epsilon) = F(\sqrt{\alpha^2 - 4u})$.

With signs of D_1 and D_2 we can determine the signature of K. More precisely,

Lemma 2.14. *[51] Assume $F \neq \mathbb{Q}$ and $L \neq \mathbb{Q}$. Then K is a non-CM quartic field with a quadratic subfield F, and $|\epsilon| > 1$. If $F = L$, then K is cyclic over $\mathbb{Q}$. If $F \neq L$, then K is non-Galois over $\mathbb{Q}$, and the composite MK is dihedral over $\mathbb{Q}$ and cyclic over M. Moreover*

$$\begin{cases} (r_1, r_2) = (0, 2) & if & D_1 < 0 \\ (r_1, r_2) = (2, 1) & if & D_2 < 0 \\ (r_1, r_2) = (4, 0) & otherwise. \end{cases}$$

Moreover, if $F \neq L, D_F = D_1$ and $D_L = D_2$, then $D_K = D_f$.

Note that if K is not totally complex, the quadratic subfield F is real.

D_4 extension with signature (0,2)

We specify that $s = u = 1$. Then we have, for positive integer t,

$$f(x, t) = x^4 - x^3 + (t+2)x^2 - x + 1$$

with $D_1 = 1 - 4t$, $D_2 = t^2 + 8t + 12$. If D_1, D_2 are square-free for odd integer t, D_K equals $(1 - 4t)^2(t+2)(t+6)$. For a positive integer t, D_1 is negative, by Lemma 2.14, $(r_1, r_2) = (0, 2)$ and $MK/\mathbb{Q}$ is a D_4 Galois extension.

Nakamula estimated the regulator R_F of the field F.

$$R_K = \frac{1}{4} \log \frac{D_K}{16} + o(1) \text{ as } D_K \to \infty. \tag{2.15}$$

To show that $f(x, t)$ gives rise to a regular D_4 extension, we briefly recall how to determine the Galois group of a quartic polynomial over an arbitrary field F in [11], page 358. We write a quartic polynomial f in the form

$$f = x^4 - c_1 x^3 + c_2 x^2 - c_3 x + c_4$$

and we define the Ferrari resolvent of f to be

$$\theta_f(y) = y^3 - c_2 y^2 + (c_1 c_3 - 4c_4)y - c_3^2 - c_1^2 c_4 + 4c_2 c_4.$$

Theorem 2.16. *Let F have characteristic $\neq 2$, and $f \in F[x]$ be monic and irreducible of degree 4. Then Galois group of f over F is determined as follows:*

(a) If $\theta_f(y)$ is irreducible over F, then

$$G = \begin{cases} S_4, & \text{if } disc(f) \notin F^2 \\ A_4, & \text{if } disc(f) \in F^2 \end{cases}$$

(b) If $\theta_f(y)$ splits completely over F, then $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(c) If $\theta_f(y)$ has a unique root β in F, then

$$G \text{ is isomorphic to} \begin{cases} D_4, & \text{if } 4\beta + c_1^2 - 4c_2 \neq 0 \text{ and } disc(f)(4\beta + c_1^2 - 4c_2) \notin (F^*)^2 \\ & \quad \text{or } 4\beta + c_1^2 - 4c_2 = 0 \text{ and } disc(f)(\beta^2 - 4c_4) \notin (F^*)^2 \\ C_4, & \text{otherwise.} \end{cases}$$

The Ferrari resolvent of $f(x, t)$ is

$$y^3 - (t+2)y^2 - 3y + 4t + 6 = (y-2)(y^2 - ty - (2t+3)).$$

Then $disc(f(x,t))(4\beta + c_1^2 - 4c_2) = (1 - 4t)^3(t+2)(t+6) \notin (\overline{\mathbb{Q}}(t)^*)^2$ and $(\mathbb{Q}(t)^*)^2$. By Theorem 2.16, the Galois group of $f(x,t)$ over both $\overline{\mathbb{Q}}(t)$ and $\mathbb{Q}(t)$ is D_4. Hence $f(x,t)$ gives rise to a D_4 regular extension over $\mathbb{Q}(t)$.

D_4 extension with signature (2,1)

We specify that $u = 1, t = 1$. Then we have $f(x, s) = x^4 - sx^3 + 3x^2 - sx + 1$ and $D_1 = s^2 - 4$ and $D_2 = 25 - 4s^2 = (5 + 2s)(5 - 2s)$. Assume that D_1 and D_2 are square-free for odd integers s. Then $D_F = D_1$, $D_L = D_2$ and by Lemma 2.14 we have $D_K = D_f = (s^2 - 4)^2(5 + 2s)(5 - 2s)$. For a positive integer s bigger than 3, D_1 is positive and D_2 is negative, by Lemma 2.14, $(r_1, r_2) = (2, 1)$ and $MF/\mathbb{Q}$ is a D_4 Galois extension.

Nakamula showed for the field generated by $f(x,t)$,

$$\frac{QR_K}{R_F} = \frac{1}{3}\log\frac{|D_K|}{4} + o(1)$$
$$R_F = \frac{1}{2}\log D_F + o(1)$$

as $|D_K|$ and $D_F \longrightarrow \infty$. Here Q is 1 or 2 depending on K and F. Hence

$$R_K \ll (\log|D_K|)^2. \tag{2.17}$$

The Ferrari resolvent of $f(x, s)$ is

$$y^3 - 3y^2 + (s^2 - 4)y - 2(s^2 - 6) = (y-2)(y^2 - y + (s^2 - 6)).$$

Then $disc(f(x,s))(4\beta + c_1^2 - 4c_2) = (s^2 - 4)^3(5 + 2s)(5 - 2s) \notin (\overline{\mathbb{Q}}(s)^*)^2$ and $(\mathbb{Q}(s)^*)^2$. Hence $f(x, s)$ gives rise to a D_4 regular extension over $\mathbb{Q}(s)$.

D_4 extension with signature (4,0)

We specify that $u = -1, t = 1$ and $s > 6$. Then we have $f(x, s) = x^4 - sx^3 - x^2 + sx + 1$ and $D_1 = s^2 - 4$ and $D_2 = 9 + 4s^2$. Assume that D_1 and D_2 are square-free for odd integer s. Then $D_F = D_1$, $D_L = D_2$ and by Lemma 2.14 we have $D_K = D_f = (s^2 - 4)^2(9 + 4s^2)$. For square-free $s^2 - 4$ and $4s^2 + 9$, K, L are always distinct. Hence, by Lemma 2.14, $(r_1, r_2) = (4, 0)$ and $MK/\mathbb{Q}$ is a D_4 Galois extension.

Nakamula showed for the field generated by $f(x, s)$

$$\frac{QR_K}{R_F} = \frac{1}{18} \log \frac{D_K}{4} \log \frac{D_F}{2^{10}} + o(1)$$

$$R_F = \frac{1}{2} \log D_F + o(1)$$

as D_K and $D_F \longrightarrow \infty$. Here Q is 1 or 2 depending on K and F.

Hence

$$R_K \ll (\log |D_K|)^3. \tag{2.18}$$

The Ferrari resolvent of $f(x, s)$ is

$$y^3 + y^2 - (s^2 + 4)y - 2(s^2 + 2) = (y + 2)(y^2 - y - (s^2 + 2)).$$

Then $disc(f(x, s))(4\beta + c_1^2 - 4c_2) = (s^2 - 4)^3(9 + 4s^2) \notin (\overline{\mathbb{Q}}(s)^*)^2$ and $(\mathbb{Q}(s)^*)^2$. Hence $f(x, s)$ gives rise to a D_4 regular extension over $\mathbb{Q}(s)$.

2.2.3 D_5 extensions

Consider the polynomial in [60] for the case of D_5 extension;

$$f(x, t) = x^5 - tx^4 + (2t - 1)x^3 - (t - 2)x^2 - 2x + 1.$$

Its discriminant is $(4t^3 - 28t^2 + 24t - 47)^2$ and the signature is $(1, 2)$ for $t \leq 6$ and $(5, 0)$ for $n \geq 7$. We claim that $f(x, t)$ gives rise to a regular D_5 extension over $\mathbb{Q}(t)$, i.e., if we consider $f(x, t)$ as a polynomial over $\mathbb{Q}(t)$ and E is the splitting field, then $E \cap \overline{\mathbb{Q}} = \mathbb{Q}$. This is equivalent to the fact that $Gal(E\overline{\mathbb{Q}}/\overline{\mathbb{Q}}(t)) \simeq Gal(E/\mathbb{Q}(t))$.

By [34], page 41, the Weber sextic resolvent of $f(x,t)$ is

$$G(z) = (z^3 + b_4 z^2 + b_2 z + b_0)^2 - 2^{10}(4t^3 - 28t^2 + 24t - 47)^2 z,$$

where $b_4 = -4t^2 - 4t + 37, b_2 = 64t^3 - 312t^2 + 328t + 115, b_0 = 64t^3 - 2884t^2 + 4348t - 249$. It factors as

$$G(z) = (z - 4t^2 + 12t - 9)(z^5 - (4t^2 + 20t - 83)z^4 + (128t^3 - 368t^2 - 816t + 2346)z^3$$

$$-(1024t^4 - 7040t^3 + 16536t^2 + 3448t - 29126)z^2$$

$$+(2048t^4 + 51072t^3 - 201328t^2 + 18640t + 256933)z$$

$$-(1024t^4 - 89216t^3 + 1948548t^2 - 231404t + 6889)).$$

Therefore, the Galois group of $f(x,t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ is either D_5 or C_5. In order to distinguish it, we use the criterion in [34], page 42. Namely, the Galois group is C_5 if and only if the resolvent $R(x_1 - x_2, f(x,t))(X)$ factors into irreducible polynomials of degree 5. Here

$$R(x_1 - x_2, f(x,t))(X) = X^{-5}Res(f(Y - X, t), f(Y, t))$$

$$= (X^{10} + X^8(-2t^2 + 12t - 8) + X^6(t^4 - 12t^3 + 46t^2 - 56t - 4)$$

$$+X^4(-2t^4 + 16t^3 - 8t^2 - 112t + 127) + X^2(t^4 - 60t^2 + 128t + 6) - 4t^3 + 28t^2 - 24t + 47)$$

$$(X^{10} + X^8(-2t^2 + 8t - 2) + X^6(t^4 - 8t^3 + 20t^2 - 4t + 7) + X^4(-2t^4 + 36t^2 - 4t + 41)$$

$$+X^2(t^4 - 4t^3 + 66t^2 - 36t + 103) - 4t^3 + 28t^2 - 24t + 47)$$

It is clear that the above factors cannot be factored into irreducible polynomials of degree 5. Hence the Galois group of $f(x,t)$ over both $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ is D_5.

Let θ_t be a root of $f(x,t)$. Schöpp found the fundamental units in the equation order $\mathbb{Z}[\theta_t]$ for $t \leq 6$. More precisely, he shows

Theorem 2.19 (Schöpp)**.** *The elements $\theta_t, \theta_t - 1$ form a system of independent units in the order $\mathbb{Z}[\theta_t]$. Moreover, they are fundamental units in $\mathbb{Z}[\theta_t]$ for $t \leq 6$.*

However, Schöpp could not show that $\mathbb{Z}[\theta_t]$ is the maximal order of $\mathbb{Q}(\theta_t)$. Lavallee, Spearman, Williams and Yang [40] found a parametric family of quintics with a power integral basis. The parametric polynomial $F_b(x)$ is given by

$$F_b(x) := x^5 - 2x^4 + (b+2)x^3 - (2b+1)x^2 + bx + 1, b \in \mathbb{Z}.$$

and its discriminant is $(4b^3 + 28b^2 + 24b + 47)^2$.

They showed that when $4b^3 + 28b^2 + 24b + 47$ is square-free, then the field $Q(\theta_b)$ generated by a root θ_b of $F_b(x)$ has a power integral basis $\mathbb{Z}[\theta_b]$. Since $x^5 F_{-t}\left(\frac{1}{x}\right) = f(x,t)$, this implies that $\theta_t, \theta_t - 1$ are fundamental units of $\mathbb{Q}(\theta_t)$ when $4t^3 - 28t^2 + 24t - 47$ is square-free.

Schöpp found the locations of roots of $f(x,t)$.

Lemma 2.20 (Schöpp). *Let $\theta_t^{(1)}$ be the real root and let $\theta_t^{(2)} = \overline{\theta_t^{(3)}}, \theta_t^{(4)} = \overline{\theta_t^{(5)}}$ be the pairs of complex roots of $f(x,t)$. Then we have the following approximations:*

$$
\begin{array}{rll}
(i) & -t + 2 + \frac{2}{t} < |\theta_t^{(1)}| < -t + 2 + \frac{1}{t} & for \quad t < -4 \\[4pt]
(ii) & -t + 3 + \frac{2}{t} < |\theta_t^{(1)} - 1| < -t + 3 + \frac{1}{t} & for \quad t < -4 \\[4pt]
(iii) & \frac{1}{2\sqrt{-t}} < |\theta_t^{(2)}| < \frac{2}{\sqrt{-t}} & for \quad t < -4 \\[4pt]
(iv) & \sqrt{1 - \frac{3}{4t}} < |\theta_t^{(2)} - 1| < \sqrt{1 - \frac{6}{5t}} & for \quad t < -144 \\[4pt]
(v) & \sqrt{1 + \frac{3}{t}} < |\theta_t^{(4)}| < \sqrt{1 - \frac{1}{t^2}} & for \quad t < -174 \\[4pt]
(v) & \sqrt{-\frac{5}{6t}} < |\theta_t^{(4)} - 1| < \sqrt{-\frac{14}{13t}} & for \quad t < -139.
\end{array}
$$

Let K_t be the quintic field by adjoining $\theta_t^{(1)}$ to $\mathbb{Q}$. Since we know the absolute value of roots, it is easy to show that the regulator R_{K_t} of a quintic field K_t.

Lemma 2.21. *For t with $4t^3 - 28t^2 + 24t - 47$ square-free,*

$$R_{K_t} \ll (\log d_{K_t})^2.$$

2.3 Alternating groups

2.3.1 A_4 extensions

Let

$$f(x,t) = x^4 - 8tx^3 + 18t^2x^2 + 1.$$

Its discriminant is $16^2(1+27t^4)^2$. We claim that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is a regular extension with Galois group A_4. We have to show that the Galois groups of $f(x,t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both A_4.

First, we have to show that $f(x,t)$ is irreducible over $\overline{\mathbb{Q}}(t)$. By Gauss Lemma, it is enough to check the irreducibility of $f(x,t)$ over $\overline{\mathbb{Q}}[t]$. It is easy to check that $f(x,t)$ has no root in $\overline{\mathbb{Q}}[t]$. If $f(x,t)$ is decomposed into a product of two quadratic polynomials, we have

$$x^4 - 8tx^3 + 18t^2x^2 + 1 = (x^2 + bx + c)(x^2 + dx + 1/c)$$

for some $b, d \in \overline{\mathbb{Q}}[t]$ and $c \in \overline{\mathbb{Q}}$ and we can induce a contradiction easily. Hence $f(x,t)$ is irreducible over $\overline{\mathbb{Q}}[t]$.

Its Ferrari resolvent $\theta(y)$ equals $y^3 - 18t^2y^2 - 4y + 8t^2$ and it is irreducible over $\overline{\mathbb{Q}}[t]$. Since the discriminant of $f(x,t)$ is a square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, the Galois groups over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both A_4.

Let θ_t be a root of $f(x,t)$, and $K_t = \mathbb{Q}(\theta_t)$. Then θ_t is a unit in $\mathbb{Z}_{K_t}$. We prove that K_t has the smallest possible regulator. We can see easily that $f(x,t)$ has 4 complex roots.

Proposition 2.22. *The regulator R_{K_t} satisfies $R_{K_t} \ll \log t$.*

Proof. By considering $\alpha = 6t$ in Lemma 2.4, we can see that

$$2t - \frac{1}{54t^3} < |\theta_t| < 6t.$$

Therefore

$$Reg(\mathbb{Z}[\theta_t]) \leq \log |\theta_t| \ll \log t.$$

$\square$

Lemma 2.23. *If $27t^4 + 1$ is cubic-free, then $d_{K_t} \gg t^2$. Hence $R_{K_t} \ll \log(d_{K_t})$.*

Proof. First note that $f(x, t) = (x + t)(x - 3t)^3 + 27t^4 + 1$. Take t such that $27t^4 + 1$ is cubic-free. Let $p| (27t^4 + 1)$, $p > 3$. Then $p \nmid t$ and $f(x, t) \equiv (x + t)(x - 3t)^3 \mod p$. The vertices of the Newton polygon with respect to $x - 3t$ are $(0,0)$, $(1,0)$, $(4, i)$ with $i = 1, 2$. By Cohen [11], page 315, Newton polygon argument shows that

$$p\mathbb{Z}_{K_t} = \mathfrak{p}_1 \mathfrak{p}_2^3$$

with prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$. Hence $p|Disc(K_t)$.

Therefore, $d_{K_t} \geq \displaystyle\prod_{p| (27t^4+1), p \neq 3} p$. But $27t^4 + 1 \leq \left(\displaystyle\prod_{p| (27t^4+1)} p \right)^2$. Hence $d_{K_t} \gg t^2$. $\square$

Consider a quartic polynomial for positive integers $t \in \mathbb{Z}^+$,

$$f_t(x) = x^4 + 18x^2 - 4tx + t^2 + 81.$$

with $disc(f_t(x)) = 2^8 t^2 (t^2 + 108)^2$. This polynomial is studied by Spearman [40] first. Let K_t be the quartic field obtained by adjoining a root θ_t of $f_t(x)$ to the rational numbers. It is easy to check that the signature of K_t is $(0, 2)$. Spearman showed, under some square-free condition, that K_t is monogenic. Hence they are distinct. More precisely;

Theorem 2.24 (Spearman). *Suppose that t is a positive integer and that $t(t^2 + 108)$ is square-free. Then K_t is a monogenic A_4 quartic extensions of $\mathbb{Q}$. Moreover the fields K_t are distinct.*

We claim that the Galois group of $f_t(x)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both A_4. i.e, the splitting field E of $f(x, t)$ over $\mathbb{Q}(t)$ is a regular extension. It is enough to show that the Galois group of $f_t(x)$ over $\overline{\mathbb{Q}}(t)$ is A_4. First we have to show that $f_t(x)$ is irreducible over

$\overline{\mathbb{Q}}[t]$. This can be shown by checking that $f(x, t)$ has no root in $\overline{\mathbb{Q}}[t]$ and it can not be a product of two quadratic polynomials.

The Ferrari resolvent of $f_t(x)$ is

$$y^3 - 18y^2 - (4t^2 + 324)y + (56t^2 + 1296).$$

Since $disc(f_t(x))$ is a square in $\overline{\mathbb{Q}}[t]$, if the Ferrai resolvent has no root in $\overline{\mathbb{Q}}[t]$, the Galois group of $f_t(x)$ is A_4. The only possibilities for a root are of the forms $a, b(t \pm i\sqrt{\frac{162}{7}})$ and $c(t^2 + 162)$ for some $a, b, c \in \overline{\mathbb{Q}}$. We can check that they are not a root of the Ferrai resolvent. Hence we showed that $f_t(x)$ gives rise to an A_4 regular extension.

2.3.2 A_5 extensions

Consider the polynomial $f(x, t) = x^5 + 5(5t^2 - 1)x - 4(5t^2 - 1)$, where $5t^2 - 1$ is square free. Here $disc(f(x, t)) = 2^8 5^6 t^2 (5t^2 - 1)^4$.

For a non-zero integer t, $f(x, t)$ has one real root and four complex roots. Let K_t be the quintic field obtained by adjoining the real root θ_t of $f(x, t)$.

We claim that the Galois groups of $f(x, t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both A_5. First we have to check that $f(x, t)$ is irreducible over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$. This is true because $f(x, t)$ is an Eisenstein polynomial with respect to $5t^2 - 1$ (resp. $\sqrt{5}t + 1$) as a polynomial over $\mathbb{Q}(t)$ (resp. $\overline{\mathbb{Q}}(t)$). Since the discriminant is a square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, the Galois group is a subgroup of A_5. It is enough to show that the below sextic resolvent of $f(x, t)$ has no root in $\overline{\mathbb{Q}}(t)$.

$$\theta(y) = (y^3 + b_2 y^2 + b_4 y + b_6)^2 - 2^{10} disc(f(x, t))y$$

where $b_2 = -100(5t^2 - 1)$, $b_4 = 6000(5t^2 - 1)^2$ and $b_6 = 4000(5t^2 - 1)^3$. If $\theta(y)$ has a root α in $\overline{\mathbb{Q}}(t)$ which is a divisor of b_6^2, then we have

$$(\alpha^3 + b_2 \alpha^2 + b_4 \alpha + b_6)^2 = 2^{10} disc(f(x, t))\alpha \tag{2.25}$$

Hence α should be a square. The possible degrees of α are $0, 2, 4, 6$. When the degree is 0, then such α can not be a root of $\theta(y)$. If the degree is 4 or 6, then the degree of LHS and the degree of RHS in (2.25) does not match. Hence the only possible forms of α are

$$a(\sqrt{5}t + 1)^2 \text{ or } a(\sqrt{5}t - 1)^2$$

for some algeraic number $a \in \overline{\mathbb{Q}}$. By the help of computer algebra such as PARI, we can check that they can not be a root of $\theta(y)$. Hence, we have

Lemma 2.26. $f(x, t)$ *gives rise to an A_5 regular extension.*

Assume that $5t^2 - 1$ is square-free. For every odd prime divisor p of $5t^2 - 1$, $f(x, t)$ is an Eisenstein polynomial with respect to p and p does not divide the index of θ_t. This implies d_{K_t} is at least divided by $(5t^2 - 1)^4$ or $\left(\frac{5t^2-1}{2}\right)^4$.

Lemma 2.27. *For square-free $5t^2 - 1$,*

$$\log d_{K_t} \gg \log |t|.$$

2.4 Symmetric groups

2.4.1 S_4 extensions

S_4 extensions with signature $(2, 1)$

Let $t > 1$ be a positive square-free integer and $f(x, t) = x^2(x - 10t)(x - 18t) + t$. Then the discriminant $Disc(f(x, t))$ of $f(x, t)$ is $-256t^3(12t + 1)(15t - 1)(144t^2 - 12t + 1)(225t^2 + 15t + 1) < 0$. Since $f(x, t)$ is an Eisenstein polynomial for each prime divisor of t, $Disc(f(x, t))$ is divided by t^3. (See [51], page 60.)

Consider $f(x, t) = x^2(x - 10t)(x - 18t) + t$ over $\overline{\mathbb{Q}}(t)$. It is easy to see that the cubic resolvent $y^3 - 180t^2y^2 - 4ty - 64t^3$, is irreducible over $\overline{\mathbb{Q}}[t]$. By Gauss Lemma, it is irreducible over $\overline{\mathbb{Q}}(t)$. Hence the Galois group of $f(x, t)$ over $\overline{\mathbb{Q}}(t)$ is S_4. Therefore, $f(x, t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$.

Note that $f'(x,t) = 4x(x - 6t)(x - 15t)$, and we can see easily that $f(x,t)$ has two real roots θ_1, θ_2 and two complex roots $\theta_3, \theta_4 = \overline{\theta_3}$. For sufficiently large t, we can see that $10t + \frac{1}{t^3} < \theta_1 < 10t + \frac{1}{t^2}, 18t - \frac{1}{t^2} < \theta_2 < 18t - \frac{1}{t^3}$. Also by taking $\alpha = \frac{1}{t}$ and applying Lemma 2.4, we can see that θ_3 and its conjugate θ_4 are inside of the circle of radius $1/9$ centered at the origin.

Let $K_t = \mathbb{Q}[\theta_1]$. Then we can find two independent units in $\mathbb{Z}_{K_t}$.

Lemma 2.28. $\frac{\theta_1^4}{t}$ and $\frac{(\theta_1 - 10t)^4}{t}$ are independent units in $\mathbb{Z}_{K_t}$.

Proof. Since $\theta_1^4 - 28t\theta_1^3 + 180t^2\theta_1^3 + t = 0$, $\frac{\theta_1^4}{t} = 28\theta_1^3 - 180t\theta_1^2 - 1$. Hence $\frac{\theta_1^4}{t}$ is an algebraic integer. Also

$$\frac{(\theta_1 - 10t)^2(\theta_1 - 18t)^2}{t} = \frac{(\theta_1^2 - t(28\theta_1 - 180t))^2}{t} = \frac{\theta_1^4}{t} - 2\theta_1^2(28\theta_1 - 180t)^2 + t(28\theta_1 - 180t)^2.$$

So $\frac{(\theta_1 - 10t)^2(\theta_1 - 18t)^2}{t}$ is an algebraic integer. Now we have

$$\frac{\theta_1^4}{t} \cdot \frac{(\theta_1 - 10t)^2(\theta_1 - 18t)^2}{t} = 1.$$

Hence $\frac{\theta_1^4}{t}$ is a unit. By considering $y = x - 10t$ or $y = x - 18t$, we can see that $\frac{(\theta_1 - 10t)^4}{t}$ and $\frac{(\theta_1 - 18t)^4}{t}$ are algebraic integers. We have

$$\frac{\theta_1^8}{t^2} \cdot \frac{(\theta_1 - 10t)^4}{t} \cdot \frac{(\theta_1 - 18t)^4}{t} = 1.$$

Hence $\frac{(\theta_1 - 10t)^4}{t}$ is a unit.

Assume that $\frac{\theta_1^4}{t}$ and $\frac{(\theta_1 - 10t)^4}{t}$ are dependent. Then

$$\left(\frac{\theta_1^4}{t}\right)^k = \left(\frac{(\theta_1 - 10t)^4}{t}\right)^m$$

for some integers k and m. Without the loss of generality, we can assume that k is positive. When we consider the size of θ_1, m should be a negative integer. But when we replace θ_1 by θ_2, m should be a positive integer and it induces a contradiction. □

Lemma 2.29. *For positive square-free t, $R_{K_t} \ll (\log|d_{K_t}|)^2$.*

Proof. By definition,

$$R_{K_t} \leq \left| det \begin{array}{cc} \log|\frac{\theta_1^4}{t}| & \log|\frac{\theta_2^4}{t}| \\ \log|\frac{(\theta_1-10t)^4}{t}| & \log|\frac{(\theta_2-10t)^4}{t}| \end{array} \right|$$

By the above estimates on θ_1, θ_2, it is clear that $R_{K_t} \leq (\log t)^2$. Since $t^3 \mid d_{K_t}$, we prove the claim. □

S_4 extensions with signature $(0, 2)$

Let $t > 1$ be a positive square-free integer and $f(x, t) = x^4 + tx^2 + tx + t$. Then the discriminant $Disc(f(x, t))$ of $f(x, t)$ is $t^3(12t^2 - 11t + 256)$. Since $f(x, t)$ is an Eisenstein polynomial for each prime divisor of t, $Disc(f(x, t))$ is divided by t^3. (See [51], page 60.)

The cubic resolvent $y^3 - ty^2 - 4ty + 3t^2$ of $f(x, t)$ has three real roots. One of them is located between $t+1$ and $t+2$ hence it is not an integer. So if the cubic resolvent has an integer root, we can show that the integer root should be divided by t. Since $\pm t, \pm 3t, \pm t^2$ and $\pm 3t^2$ are not a root of the cubic resolvent, the cubic resolvent is irreducible. Hence $f(x, t)$ gives rise to an S_4 Galois extension for each positive square-free integer t.

Consider $f(x, t) = x^4 + tx^2 + tx + t$ over $\overline{\mathbb{Q}}(t)$. It is easy to see that the cubic resolvent $y^3 - ty^2 - 4ty + 3t^2$, is irreducible over $\overline{\mathbb{Q}}[t]$. By Gauss Lemma, it is irreducible over $\overline{\mathbb{Q}}(t)$. Hence the Galois group of $f(x, t)$ over $\overline{\mathbb{Q}}(t)$ is S_4. Therefore, $f(x, t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$.

Note that $f'(x, t) = 4x^3 + 2tx + t$ has only one real root x_0, and we can easily check that $f(x_0) > 0$. Hence $f(x, t)$ has four complex roots $\theta_1, \theta_2 = \overline{\theta_1}, \theta_3$ and $\theta_4 = \overline{\theta_3}$. For sufficiently large t, when we apply Lemma 2.4 with $\alpha = i\sqrt{t}$, we can see that one root lies inside of the circle of radius 1 centered at $1 + i\sqrt{t}$. Let θ be the root.

Let $K_t = \mathbb{Q}[\theta]$. Then since $\frac{\theta^4}{t} = -(\theta^2 + \theta + 1)$, $\frac{\theta^4}{t}$ is an algebraic integer. Here $N_{K_t/\mathbb{Q}}(\theta) = t$. Hence $\frac{\theta^4}{t}$ has norm 1, and it is a unit in $\mathbb{Z}_{K_t}$.

Since $|\theta| \ll \sqrt{t}$, $\log|\frac{\theta^4}{t}| \ll \log t$. Since $t^3 | d_{K_t}$, we have

Lemma 2.30. *For positive square-free t, $R_{K_t} \ll \log d_{K_t}$.*

2.4.2 S_5 extensions

Consider a quintic polynomial for positive square-free integers $t \in \mathbb{Z}^+$ with $t \equiv 1 \pmod{5}$,

$$f(x, t) = x^5 + tx + t$$

with the discriminant $disc(f(x, t)) = t^4(256t + 3125)$. We claim that the Galois groups of $f(x, t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both S_5. Since $f_t(x)$ is an Eisenstein polynomial for an irreducible element t, it is irreducible over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ and it is clear that $disc(f(x, t))$ is not an square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$. If the sextic resolvent has no root in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, then the Galois group is S_5 over both fields.

The sextic resolvent of $f(x, t)$ is given by

$$\theta_t(y) = (y^3 + b_2 y^2 + b_4 y + b_6)^2 - 2^{10} disc(f_t(x))y$$

where $b_2 = -20t$, $b_4 = 240t^2$, and $b_6 = 320t^3$.

We have to show that $\theta_t(y)$ does not have a root in $\overline{\mathbb{Q}}(t)$. If α, a divisor of b_6^2, is a root of $\theta_t(y)$, then

$$(\alpha^3 + b_2 \alpha^2 + b_4 \alpha + b_6)^2 = 2^{10} disc(f(x, t))\alpha.$$

Even though $disc(f(x, t))$ and b_6 has a common divisor t^3, but the other factors are coprime, it induces a contradiction.

Let K_t be a quintic field obtained by adjoining a root of $f_t(x)$ to $\mathbb{Q}$. Then the signature of K_t is $(1, 2)$. Since $f_t(x)$ is an Eisenstein polynomial for a square-free integer t, the field discriminant d_{K_t} of K_t is divided by t^4. This polynomial is used to generate extreme values of logarithmic derivatives of Artin L-functions.

Consider, for a positive square-free integer t with $t \equiv 1 \pmod{5}$,

$$f(x, t) = (x + t)(x^2 + 5t)(x^2 + 10t) + t$$

with the discriminant $disc(f(x,t)) = t^4(500000t^{10}+15000000t^9+162350000t^8+746700000t^7+1234759600t^6 + 7714500t^5 - 394744t^4 + 5143500t^3 + 162500t^2 + 3125)$.

We claim that the Galois groups of $f(x,t)$ over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ are both S_5. Since $f(x,t)$ is an Eisenstein polynomial for an irreducible element t, it is irreducible over $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$ and it is clear that $disc(f(x,t))$ is not an square in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$. If the sextic resolvent has no root in $\mathbb{Q}(t)$ and $\overline{\mathbb{Q}}(t)$, then the Galois group is S_5 over both fields.

The sextic resolvent of $f(x,t)$ is given by

$$\theta_t(y) = (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}disc(f_t(x))y$$

where $b_2 = 5t^2(24t - 335)$, $b_4 = t^3(400t^3 - 192000t^2 + 661811t - 2400)$, and $b_6 = 5^2t^3(12400t^5 + 3069000t^4 + 17775t^3 + 168480t^2 - 64t + 2400)$. As in the previous case, we can show that $\theta_t(y)$ does not have a root in $\overline{\mathbb{Q}}(t)$. Hence $f(x,t)$ gives rise to a S_5 regular extension over $\mathbb{Q}(t)$.

Let $K_t = \mathbb{Q}(\theta_1)$ be a quintic field obtained by adjoining the real root θ_1 of $f(x,t)$ to the rational numbers $\mathbb{Q}$. Moreover, $f_t(x) \equiv x^4(x + 1) + 1 \equiv (x + 2)(x^2 + x + 1)(x^2 + 3x + 3)$ (mod 5) and the signature of K_t is $(1, 2)$. Hence the Galois extensions $\widehat{K_t}/\mathbb{Q}$ satisfy the hypothesis of Theorem 1.4, and Artin L-functions $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ are cuspidal automorphic L-functions of $GL(4)/\mathbb{Q}$.

We claim that $\frac{(\theta_1+t)^5}{t}$ and $\frac{(\theta_1^2+5t)^5}{t^2}$ are two independent units in K_t: Since $f_t(x) = x^5 + tx^4 + 15tx^3 + 15t^2x^2 + 50t^2x + 50t^3 + t$,

$$\frac{\theta_1^5}{t} = -(\theta_1^4 + 15\theta_1^3 + 15t\theta_1^2 + 50t\theta_1 + 50t^2 + 1).$$

Hence $\frac{\theta_1^5}{t}$ is an algebraic integer. From this, it is easy to show that $\frac{(\theta_1+t)^5}{t}$ and $\frac{(\theta_1^2+5t)^5}{t^2}$ are algebraic integers. Now we have

$$\frac{(\theta_1 + t)^5}{t} \cdot \frac{(\theta_1^2 + 5t)^5}{t^2} \cdot \frac{(\theta_1^2 + 10t)^5}{t^2} = -1.$$

Hence $\frac{(\theta_1+t)^5}{t}$ and $\frac{(\theta_1^2+5t)^5}{t^2}$ are units. To show that they are independent, we need to know the locations of 5 roots of $f_t(x)$. For the case of the real root θ_1, we have $-t-\frac{1}{t} < \theta_1 < -t$.

For complex roots, we apply Lemma 2.4 to $f_t(x)$ with $\alpha = i\sqrt{5t}$ then we can see that another root θ_2 of $f_t(x)$ converges to $i\sqrt{5t}$ as t increases. More precisely,

$$\left|\theta_2 - i\sqrt{5t}\right| = O\left(\frac{1}{t^{1.5}}\right).$$

Put $\theta_3 = \overline{\theta_2}$. Apply Lemma 2.4 again to $f_t(x)$ with $\alpha = i\sqrt{10t}$. Then we can find the fourth root θ_4 with

$$\left|\theta_4 - i\sqrt{10t}\right| = O\left(\frac{1}{t^{1.5}}\right)$$

and put $\theta_5 = \overline{\theta_4}$.

Assume that $\frac{(\theta_1+t)^5}{t}$ and $\frac{(\theta_1^2+5t)^5}{t^2}$ are dependent. Then $\left(\frac{(\theta_1+t)^5}{t}\right)^k = \left(\frac{(\theta_1^2+5t)^5}{t^2}\right)^l$ for some integers k, l. Then it holds when we replace θ_1 by θ_4, and when we consider the size of θ_1, θ_4, we obtain contradiction. By definition,

$$R_{K_t} \ll \left| \begin{array}{cc} \log\left|\frac{(\theta_1+t)^5}{t}\right| & \log\left|\frac{(\theta_4+t)^5}{t}\right| \\ \log\left|\frac{(\theta_1^2+5t)^5}{t^2}\right| & \log\left|\frac{(\theta_4^2+5t)^5}{t^2}\right| \end{array} \right|.$$

By the above estimates on θ_1, θ_4, it is clear that $R_{K_t} \ll (\log t)^2$. Since $f_t(x)$ is an Eisenstein polynomial, d_{K_t} is divisible by t^4 if t is square-free. Hence $\log d_{K_t} \gg \log t$. We have proved

Lemma 2.31. *For a square-free positive integer t,*

$$R_{K_t} \ll (\log d_{K_t})^2.$$

2.4.3 S_n extensions with signature $(n, 0)$

Duke [16] considered a polynomial of the form

$$f(x, t) = (x - t)(x - 2^2 t)(x - 3^2 t) \cdots (x - n^2 t) - t.$$

Let K_t is a number field obtained by adjoining a root θ_t of $f(x, t)$. He showed that the regulator of K_t is minimal up to a constant.

Proposition 2.32 (Duke)**.** *Let $t \in \mathbb{Z}^+$ be a square-free integer. Then $f(x,t)$ is irreducible. If t is sufficiently large, then K is totally real of degree n and the regulator R_{K_t} of K_t satisfies $R_{K_t} \ll_f (\log d_{K_t})^{n-1}$.*

Moreover, he showed

Proposition 2.33 (Duke)**.** *For $n \geq 1$, the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is a regular S_n extension.*

Chapter 3

Number fields with extreme class numbers

This Chapter is based on Cho [6], Cho and Kim [7] and Cho and Kim [8]. Except two sections 3.3.2 and 3.3.5, they are the joints works with H. Kim. We follow closely [6],[7] and [8].

3.1 Upper bound of class numbers

Let $\mathfrak{K}(n, G, r_1, r_2)$ is the set of number fields of degree n with signature (r_1, r_2) whose normal closures have G as their Galois group. Then by the class number formula, the class number h_K for $K \in \mathfrak{K}(n, G, r_1, r_2)$ is given by

$$h_K = \frac{w_K |d_K|^{\frac{1}{2}}}{2^{r_1}(2\pi)^{r_2} R_K} L(1, \rho),$$

where w_K is the number of roots of unity in K, d_K is the discriminant of K and R_K is its regulator and $L(s, \rho) = \dfrac{\zeta_K(s)}{\zeta(s)}$ is the Artin L-function.

Silverman [66] obtained a lower bound of regulator R_K of number fields K:

$$R_K > c_n (\log \gamma_n |d_K|)^{r - r_0}, \tag{3.1}$$

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where c_n, γ_n are positive constant depending on degree n of K and $r = r_1 + r_2 - 1$ and r_0 is the maximum of unit ranks of subfields of K.

It is easy to prove that under the Generalized Riemann Hypothesis (GRH) for $L(s, \rho)$, $L(1, \rho) \ll (\log \log |d_K|)^{n-1}$. Hence we obtain the upper bound for the class numbers:

$$h_K \ll |d_K|^{\frac{1}{2}} \frac{(\log \log |d_K|)^{n-1}}{(\log |d_K|)^{r-r_0}}.$$

Now the question is whether the upper bound is sharp. Namely, are there number fields with the largest possible class number of the size

$$|d_K|^{\frac{1}{2}} \frac{(\log \log |d_K|)^{n-1}}{(\log |d_K|)^{r-r_0}}?$$

For real quadratic fields, this is a classical result of Montgomery and Weinberger [47]. Ankeny, Brauer, and Chowla [1] constructed unconditionally, for any n, r_1, r_2, number fields with arbitrarily large discriminants and $h_K \gg |d_K|^{\frac{1}{2}-\epsilon}$. Under the GRH and Artin conjecture for $L(s, \rho)$, Duke [16] constructed totally real fields of degree n whose Galois closure has the Galois group S_n with the largest possible class numbers. Daileda [18] showed Duke's result unconditionally when $n = 3$.

We show that the upper bound is sharp up to a constant for many families $\mathfrak{K}(n, G, r_1, r_2)$ of number fields.

3.2 Extreme class numbers

In this section we explain how to generate number fields with extreme class numbers in a very general setting. Consider an irreducible polynomial $f(x, t)$ with parameter t,

$$f(x, t) = x^n + a_{n-1}(t)x^{n-1} + a_{n-2}(t)x^{n-2} + \cdots + a_1(t)x + a_0(t) \in \mathbb{Z}[t][x].$$

Assume that the splitting field E of $f(x, t)$ over $\mathbb{Q}(t)$ is regular with Galois group G for some finite group G. For a specialization $t \in \mathbb{Z}$, let K_t be a number field of degree n

obtained by adjoining a root θ_t of $f(x, t)$ and $\widehat{K_t}$ be the Galois closure of K_t. Then by Cohen's theorem, $Gal(\widehat{K_t}/\mathbb{Q}) = G$ in most cases.

Assume that the regulator R_{K_t} is minimal up to constant. i.e, $R_{K_t} \ll (\log |d_{K_t}|)^{r-r_0}$. We will show that this is true for the cases we consider. However, we could not find number fields with minimal regulators in the cases of $\mathfrak{K}(4, D_4, 2, 1)$ and $\mathfrak{K}(4, D_4, 4, 0)$ due to the presense of subfields.

Since $f(x, t)$ gives rise to a G regular extension, by Theorem 2.1, there is a constant c_f depending on f such that for every $p \geq c_f$, there is a residue class s_p modulo p so that the Frobenius element $Frob(p)$ at p in $Gal(\widehat{K_t}/\mathbb{Q})$ is trivial for all $t \equiv s_p \pmod{p}$.

For given $X > 0$, define $M = \prod_{c_f \leq p \leq y} p$ for $y = \frac{\log X}{\log \log X}$. Let s_M be the residue class modulo M for which $s_M \equiv s_p \bmod p$ for $c_f \leq p \leq y$. We assume the modularity of the $n - 1$ dimensional Galois reprensentation ρ for which $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$.

Then we define a set

$$L(X) = \{\frac{X}{2} < t < X \mid t \equiv s_M \bmod M\}$$

Each $t \in L(X)$ gives rise to an cuspidal automorphic L-functions $L(s, \rho, t)$ of $GL(n-1)/\mathbb{Q}$. There is an important question which we may fail to notice. It is possible that two different t_1 and t_2 correspond to the same L-function. This is equivalent to the fact that $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$. When $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$, we say that K_{t_1} and K_{t_2} are arithmetically equivalent. For number fields of small degree, they should be conjugate. More precisely;

Theorem 3.2 (Klingen [38]). *Let $K|k$ be an extension of number fields of degree $n \leq 11$ and assume there exists some non-conjugate field K' being arithmetically equivalent to K over k. Then up to conjugacy only the following four cases are possible for $G = G(\widetilde{K}|k)$:*

$$n = 7 : \qquad G = GL_3(2)$$
$$n = 8 : \quad G = \mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/8\mathbb{Z})^{\times}, G = GL_2(3)$$
$$n = 11 : \qquad G = PSL_2(11).$$

On the other hand, since the conductor of $L(s, \rho, t)$ is $|d_{K_t}|$, different discriminant d_{K_t} distinguishes the L-functions $L(s, \rho, t)$. To determine the discriminant or its factor, we impose some extra conditions on t which depends on $f(x, t)$. Assume that the discriminant of $f(x, t)$ is a polynomial in t of degree D. Then there is a constant C such that $|d_{K_t}| \leq Ct^D$. For $A = C^{1/D}X$, we define a set $L(A)$ of positive integers

$$L(A) = \{\frac{X}{2} < t < X \mid t \equiv s_M \bmod M, \text{ some condition on } t\}.$$

In the cases in consideration, we will show that $|L(A)| \gg X^{1-\epsilon}$ for any $\epsilon > 0$.

Let $c_0 = \frac{5(n-1)D}{2} + 1$. Or we may replace $(n-1)$ in c_0 by a smaller constant if ρ is not irreducible. Choose α with $c_0\frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = D$, $d = 1$ and $T = (\log A^D)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, (\log |d_{K_t}|)^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.6 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \tag{3.3}$$

$$= (n-1) \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n,\alpha}(1) = (n-1)\log\log\log|d_{K_t}| + O_{n,\alpha}(1).$$

where we use the fact that $\sqrt{\log|d_{K_t}|} < y = \frac{\log X}{\log\log X}$ for large X.

By the class number formula and the size of regulator R_{K_t}, we have the required result

$$h_{K_t} \gg |d_{K_t}|^{1/2} \frac{(\log\log|d_{K_t}|)^{n-1}}{(\log|d_{K_t}|)^{r-r_0}}.$$

3.3 Symmetric Groups

3.3.1 S_5 Quintic Extensions with signature $(1, 2)$

In section 2.4.2, we showed that

$$f(x, t) = (x + 5)(x^2 + 5t)(x^2 + 10t) + t$$

gives rise to an S_5 regular extension. Since $disc(f(x,t))$ is a polynomial of degree 14, there is a constant C with $disc(f(x,t)) \leq Ct^{14}$. For given $X > 0$, let $A = C^{1/14}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \{\frac{X}{2} < t < X \mid t : \text{ square-free } t \equiv 1 \bmod 5, t \equiv s_M \bmod M, Gal(\widehat{K_t}/\mathbb{Q}) = S_5\}$$

where s_M and M are defined as in section 3.2. Then we have

$$|L(A)| = \frac{6}{\pi^2}\frac{25}{24} \prod_{p|M} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{2M} + O(X^{1/2} \log X).$$

Every t in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ which is actually an cuspidal automorphic L-function of $GL(4)/\mathbb{Q}$ since ρ is modular by Theorem 1.4.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that K_t are not conjugate each other.

Lemma 3.4. *For a square-free t with $t \equiv 1 \bmod 5$,*

$$p \text{ totally ramifies in } K_t \text{ if and only if } p \text{ divides } t.$$

Proof. Since $f(x,t)$ is an Eisenstein polynomial, if $p \mid t$, p ramifies totally. See Corollary 6.2.4 in [11]. If $p = 5$, then $f(x,t) \equiv (x+1)^5 \bmod 5$. However, by Newton polygon method, $5O_{K_t} = \mathfrak{p}_1^4\mathfrak{p}_2$.

Now assume that p ramifies totally and does not divide 5 and t. Then

$$f(x,t) \equiv (x+a)^5 \bmod p.$$

By comparing coefficients of $f(x,t)$ and $(x+a)^5 \bmod p$, we can induce a contradiction and we finish the proof.

□

Let $c_0 = 141$. Choose α with $c_0\frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 14$, $d = 1$ and $T = (\log A^{14})^2$, every automorphic L-function excluding exceptional

$O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, (\log |d_{K_t}|)^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \\
&= 4 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n,\alpha}(1) = 4 \log \log \log |d_{K_t}| + O_{n,\alpha}(1).
\end{aligned}
\tag{3.5}
$$

By the class number formula and Lemma 2.31,

$$
h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^4}{(\log d_{K_t})^2}.
$$

We summarize as follows:

Theorem 3.6. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(5, S_5, 1, 2)$ with arbitrary large discriminant d_K for which*

$$
h_K > c d_K^{1/2} \frac{(\log \log d_K)^4}{(\log d_K)^2}.
$$

3.3.2 S_4 Quartic Extensions with signature $(4, 0)$

In section 2.4.3, we stated the fact that

$$
f(x, t) = (x - t)(x - 2^2 t)(x - 3^2 t)(x - 4^2 t) - t
$$

gives rise to an S_4 regular extension. Since $disc(f(x, t))$ is a polynomial of degree 12, there is a constant C with $disc(f(x, t)) \leq C t^{12}$. For given $X > 0$, let $A = C^{1/12} X$. We define a set $L(A)$ of square-free integers

$$
L(A) = \{ \frac{X}{2} < t < X \mid t : \text{ square-free }, t \equiv s_M \bmod M, Gal(\widehat{K_t}/\mathbb{Q}) = S_4 \}
$$

where s_M and M are defined as in section 3.2. Then we have

$$
|L(A)| = \frac{6}{\pi^2} \prod_{p \mid M} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{2M} + O(X^{1/2} \log X).
$$

Every t in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ which is actually an cuspidal automorphic L-function of $GL(3)/\mathbb{Q}$ since ρ is modular by Proposition 1.1.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that K_t are not conjugate each other.

Lemma 3.7. *For a square-free t,*

$$p \text{ totally ramifies in } K_t \text{ if and only if } p \text{ divides } t.$$

Proof. Since $f(x, t)$ is an Eisenstein polynomial, if $p \mid t$, p ramifies totally. See Corollary 6.2.4 in [11]. Now assume that p ramifies totally and does not divide t. If t is even, then p is not equal to 2. If t is odd, then p can be even. However, if $p = 2$, then $f_t(x) \equiv (x+a)^4$ (mod 2) and we have $6a^2 \equiv 273t^3$ mod 2 hence it induces a contradiction. So we can assume that $p \neq 2$. Also we can see easily that $a \not\equiv 0$ mod p for totally ramified p.

If $p = 3$, we have $a \equiv 0$ mod p, hence 3 should be excluded. Since $p \neq 2, 3$, we have the following system of equations mod p.

$$a \equiv -\frac{15}{2}t, \ a^2 \equiv \frac{91}{2}t^2, \ a^3 \equiv -205t^3 \text{ mod } p$$

and we can check that this system is inconsistent and we finish the proof.

$\square$

Let $c_0 = 91$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 12$, $d = 1$ and $T = (\log A^{12})^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n, \alpha}(1) \\
&= 3 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n, \alpha}(1) = 3 \log \log \log d_{K_t} + O_{n, \alpha}(1).
\end{aligned}
\tag{3.8}
$$

By the class number formula and Lemma 2.32,

$$h_{K_t} \gg d_{K_t}^{1/2} \left(\frac{\log\log d_{K_t}}{\log d_{K_t}} \right)^3.$$

We summarize as follows:

Theorem 3.9. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, S_4, 4, 0)$ with arbitrary large discriminant d_K for which*

$$h_K > cd_K^{1/2} \left(\frac{\log\log d_K}{\log d_K} \right)^3.$$

3.3.3 S_4 Quartic Extensions with signature $(2, 1)$

In section 2.4.1, we showed that

$$f(x, t) = x^2(x - 10t)(x - 18t) + t$$

gives rise to an S_4 regular extension. Since $disc(f(x, t))$ is a polynomial of degree 9, there is a constant C with $|disc(f(x, t))| \leq Ct^9$. For given $X > 0$, let $A = C^{1/14}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \{ \frac{X}{2} < t < X \mid t : \text{ square-free } t \equiv 1 \bmod 2, \, t \equiv s_M \bmod M, Gal(\widehat{K_t}/\mathbb{Q}) = S_4 \}$$

where s_M and M are defined as in section 3.2. Then we have

$$|L(A)| = \frac{6}{\pi^2} \frac{4}{3} \prod_{p|M} \left(1 - \frac{1}{p^2} \right)^{-1} \frac{X}{2M} + O(X^{1/2}\log X).$$

Every t in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ which is actually an cuspidal automorphic L-function of $GL(3)/\mathbb{Q}$ since ρ is modular by Proposition 1.1.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that K_t are not conjugate to each other.

Lemma 3.10. *For a square-free t,*

p totally ramifies in K_t if and only if p divides t or $p = 2$.

Proof. Since $f(x, t)$ is an Eisenstein polynomial, if $p \mid t$, p ramifies totally. See Corollary 6.2.4 in [11]. When $p = 2$, we can not determine the prime decomposition of 2 in K_t. Now assume that p ramifies totally and does not divide 2 and t.

Then

$$f(x, t) \equiv (x + a)^4 \bmod p.$$

By comparing coefficients of $f(x, t)$ and $(x + a)^4 \bmod p$, we can induce a contradiction and we finish the proof.

$\square$

Let $c_0 = 68$. Choose α with $c_0 \frac{1-\alpha}{2\alpha - 1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 9$, $d = 1$ and $T = (\log A^9)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, (\log |d_{K_t}|)^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \\
&= 3 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n,\alpha}(1) = 3 \log \log \log |d_{K_t}| + O_{n,\alpha}(1).
\end{aligned}
\tag{3.11}
$$

By the class number formula and Lemma 2.29,

$$h_{K_t} \gg |d_{K_t}|^{1/2} \frac{(\log \log |d_{K_t}|)^3}{(\log |d_{K_t}|)^2}.$$

We summarize as follows:

Theorem 3.12. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, S_4, 2, 1)$ with arbitrary large discriminant d_K for which*

$$h_K > c|d_K|^{1/2} \frac{(\log \log |d_K|)^3}{(\log |d_K|)^2}.$$

3.3.4 S_4 **Quartic Extensions with signature** $(0, 2)$

In section 2.4.1, we showed that

$$f(x, t) = x^4 + tx^2 + tx + t$$

gives rise to an S_4 regular extension. Since $disc(f(x,t))$ is a polynomial of degree 5, there is a constant C with $disc(f(x,t)) \leq Ct^5$. For given $X > 0$, let $A = C^{1/5}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \{\frac{X}{2} < t < X \mid t: \text{ square-free } , t \equiv s_M \text{ mod } M, Gal(\widehat{K_t}/\mathbb{Q}) = S_4\}$$

where s_M and M are defined as in section 3.2. Then we have

$$|L(A)| = \frac{6}{\pi^2} \prod_{p|M} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{M} + O(X^{1/2} \log X).$$

Every t in $L(A)$ corresponds an Artin L-function $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ which is actually an cuspidal automorphic L-function of $GL(3)/\mathbb{Q}$ since ρ is modular by Proposition 1.1.

We claim that every $L(s, \rho, t)$ for $t \in L(A)$ is distinct. It follows from the fact that K_t are not conjugate to each other.

Lemma 3.13. *For a square-free t,*

$$p \text{ totally ramifies in } K_t \text{ if and only if } p \text{ divides } t.$$

Proof. Since $f(x,t)$ is an Eisenstein polynomial, if $p \mid t$, p ramifies totally. See Corollary 6.2.4 in [11].

Now assume that p ramifies totally and does not divide t.

Then

$$f(x, t) \equiv (x + a)^4 \text{ mod } p.$$

By comparing coefficients of $f(x,t)$ and $(x+a)^4$ mod p, we can see that the only possibility is $p = 3$ and $t \equiv 1$ mod 3. However, $x^4 + x^2 + 1$ is irreducible mod 3, this case is also excluded and we finish the proof.   □

Let $c_0 = 39$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 5$, $d = 1$ and $T = (\log A^5)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, (\log |d_{K_t}|)^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \le \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \qquad\qquad (3.14) \\
&= 3 \sum_{c_f \le p \le \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = 3 \log\log\log d_{K_t} + O_{n,\alpha}(1).
\end{aligned}
$$

By the class number formula and Lemma 2.30,

$$
h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log\log d_{K_t})^3}{(\log d_{K_t})}.
$$

We summarize as follows:

Theorem 3.15. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, S_4, 0, 2)$ with arbitrary large discriminant d_K for which*

$$
h_K > c d_K^{1/2} \frac{(\log\log d_K)^3}{(\log d_K)}.
$$

3.3.5 S_n number fields of degree n with $(n, 0)$

In section 2.4.3, we stated the fact that

$$
f_t(x) = (x - t)(x - 2^2 t)(x - 3^2 t) \cdots (x - n^2 t) - t
$$

gives rise to an S_n regular extension. Since $disc(f_t(x)) = t^{n-1} g(t)$ is a polynomial of degree $n^2 - n$, there is a constant C with $disc(f_t(x)) \le C t^{n^2 - n}$. For given $X > 0$, let $A = C^{\frac{1}{n^2-n}} X$.

We define a set $L(A)$ of square-free integers

$$
L(A) = \{\frac{X}{2} < t < X \mid t : \text{square-free}, \nu(t) \le 2, t \equiv s_M \bmod M, Gal(\widehat{K_t}/\mathbb{Q}) = S_n\}
$$

where $\nu(t)$ is the number of distinct prime divisors of t.

Here the reason why we impose the extra condition $\nu(t) \leq 2$ in $L(A)$ is to distinguish $L(s, \rho, t)$ for $t \in L(A)$. It would be possible to prove p ramifies totally in K_t if and only if $p \mid t$.. However, we don't know how to prove it. We explain below how to distinguish $L(s, \rho, t)$ with the condition $\nu(t) \leq 2$ in detail.

The following Proposition makes us able to estimate $|L(A)|$.

Proposition 3.16. *Let x be sufficiently large positive integer and l be a positive integer of the size of x^u for some real number $0 < u \leq 0.3$. For any integer b with $(b, l) = 1$,*

$$| \{0 < n < x \mid n : \text{square-free } n \equiv b \bmod l, \nu(n) \leq 2\} | \gg \frac{x}{\log x \phi(l)}$$

where ϕ is the Euler phi-function.

The proof of Proposition 3.16 can be found in the appendix.

By Theorem 2.2, we have

$$| \{\frac{X}{2} < t < X \mid Gal(\widehat{K_t}/\mathbb{Q}) \cong S_4\} | = \frac{X}{2} + O(X^{1/2} \log X).$$

By Proposition 3.16, we have

$$| \{\frac{X}{2} < t < X \mid t : \text{square-free}, \nu(t) \leq 2, t \equiv t_M \bmod M\} | \gg X^{1-\epsilon}.$$

hence we have that $| L(A) | \gg X^{1-\epsilon}$.

We must assume that the $n - 1$ dimensional representation ρ is modular because we don't have the modularity theorem for S_n with $n \geq 5$ except some special cases in S_5.

Now there is possibility that different square-free integers in $L(A)$ may correspond to an same automorphic L-function. Since the conductor of $L(s, \rho, t)$ is the discriminant of a number field K_t, distinct discriminants distinguish L-functions in $L(A)$. Recall that $disc(f_t(s)) = t^{n-1} g(t)$. Assume that K_{t_1} and K_{t_2} have the same discriminant for $t_1, t_2 \in L(A)$. If they are co-prime, $g(t_1)$ is divided by $t_2{}^3$. If they have a common prime divisor, then $g(t_1)$ is divided by $\left(\frac{t_2}{(t_1,t_2)}\right)^3$. The number of all possible repetition of the

same L-function in $L(A)$ is at most $\ll \nu(g(t_1))^2$. This means that K_t has at most $\nu(g(t))^2$ number fields which have the same discriminant. It is well-known that $\nu(n) < \log n$. (See p. 167 in [56]) Hence $\nu(g(t))^2 = O(\log^2 X)$ for all t in $L(A)$. Hence $L(A)$ has at least $\gg X^{1-\epsilon}$ distinct L-functions. Let $\widetilde{L}(A)$ be the set of distinct L-functions coming from $L(A)$.

Let $c_0 = \frac{5(n-1)(n^2-n)}{2} + 1$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $\widetilde{L}(A)$ with $e = n^2 - n$, $d = 1$ and $T = (\log A^{n^2-n})^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) & (3.17) \\
&= (n-1) \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = (n-1)\log\log\log d_{K_t} + O_{n,\alpha}(1).
\end{aligned}$$

By the class number formula and Lemma 2.32,

$$h_{K_t} \gg d_{K_t}^{1/2} \left(\frac{\log\log d_{K_t}}{\log d_{K_t}} \right)^{n-1}.$$

We summarize as follows:

Theorem 3.18. *Assume the Strong Artin conjecture for S_n, $n \geq 5$ is true. Fix n. There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(n, S_n, n, 0)$ with arbitrary large discriminant d_K for which*

$$h_K > cd_K^{1/2} \left(\frac{\log\log d_K}{\log d_K} \right)^{n-1}.$$

3.4   Alternating Groups

3.4.1 A_4 quartic extension with $(0,2)$

In section 2.3.1, we showed that

$$f(x,t) = x^4 - 8tx^3 + 18t^2x^2 + 1$$

gives rise to an A_4 regular extension. Since $disc(f(x,t))$ is a polynomial of degree 8, there is a constant C with $disc(f(x,t)) \leq Ct^8$. For given $X > 0$, let $A = C^{1/8}X$. We define a set $L(A)$ of cubic-free integers:

$$L(A) = \{0 < t < X : 27t^4 + 1 \text{ cubic-free and } t \equiv s_M \text{ mod } M\}.$$

We prove the following lemma, which is a direct consequence of [28], page 69.

Lemma 3.19. *Let $f(x)$ be an irreducible polynomial of degree $d \geq 3$ in $\mathbb{Z}[x]$. Let M be a positive integer and $gcd(a, M) = 1$. Suppose that if $p|M$, then $f(a) \not\equiv 0 \pmod{p}$. Let $N(X, f, M)$ be the number of integers $1 \leq n < X$ and $n \equiv a \pmod{M}$, with the property that $f(n)$ is $(d-1)$-free. Then*

$$N(X, f, M) = C(M)\frac{X}{M} + O\left(\frac{X}{M}(\log \frac{X}{M})^{\frac{2}{d+1}-1}\right),$$

where $C(M) = \prod_{p\nmid M}(1 - \frac{\rho(p^{d-1})}{p^{d-1}})$, and $\rho(p^k)$ is the number of solutions for $f(x) \equiv 0 \pmod{p^k}$.

Proof. Let $n = Mm + a$, and $g(m) = f(Mm + a)$. Then $1 \leq m < \frac{X}{M}$, and $g(x)$ is an irreducible polynomial of degree d. Hooley [28], page 69 showed that the number of $1 \leq m < \frac{X}{M}$ with the property that $g(m)$ is $(d-1)$-free, is

$$C(M)\frac{X}{M} + O\left(\frac{X}{M}(\log \frac{X}{M})^{\frac{2}{d+1}-1}\right),$$

where $C(M) = \prod_p(1 - \frac{\rho'(p^{d-1})}{p^{d-1}})$, and $\rho'(p^{d-1})$ is the number of solutions for $g(m) \equiv 0 \pmod{p^{d-1}}$. If $p|M$, $g(m) \equiv f(a) \not\equiv 0 \pmod{p}$. Hence $\rho'(p^{d-1}) = 0$. If $p \nmid M$, then since

$Mm + a \equiv 0 \pmod{p^{d-1}}$ has a unique solution mod p^{d-1}, $\rho'(p^{d-1}) = \rho(p^{d-1})$. Our result follows. $\qquad\square$

Note that by the definition of M, t_M, if $p|M$, then $c \leq p \leq y$, $t_M \equiv t_p \pmod{p}$, and p splits completely in $\widehat{K_{t_p}}$. Then $27(Mm + t_M)^4 + 1 \equiv 27t_p^4 + 1 \not\equiv 0 \pmod{p}$. Hence $\rho(p) = 0$. This implies $\rho(p^3) = 0$. If $p \nmid M$, by Nagell [49], page 87, $\rho(p^3) = \rho(p) \leq 4$. So

$$\prod_{p \nmid M}(1 - \frac{\rho(p^3)}{p^3}) \gg \prod_{p \nmid M}(1 - 4p^{-3}) \geq \prod_{p \nmid M}(1 - p^{-3})^5 \geq \zeta(3)^{-5}.$$

Hence by the above lemma, $|L(A)| = C(M)\frac{X}{M} + O\left(\frac{X}{M(\log \frac{X}{M})^{\frac{3}{5}}}\right)$, and $|L(A)| \gg \frac{X}{M}$. In the remark below, we use the recent result of Heath-Brown [27] to obtain a better error term in $|L(A)|$.

Here different $t_1, t_2 \in L(A)$ may give rise to a same L-function. We claim that at most 32 different t's correspond to the same L-function. We owe Ankeny, Brauer and Chowla [1] for this idea.

First, we need to know the locations of the roots more precisely. By applying Lemma 2.4 with $\alpha = 4t + 1.4ti$, for sufficiently large t, we find a complex root inside of the circle of radius $0.03\sqrt{2}t$ centered at $4.015t + 1.385ti$. Again by applying Lemma 2.4 with $\alpha = \frac{0.23i}{t}$, for sufficiently large t, we find a complex root inside of the circle of radius $\frac{0.0115}{t}$ centered at $\frac{0.2415i}{t}$. Here we use PARI to choose a suitable α.

We need to order the roots of $f(x, t)$ in the following way. Let θ_t^1 be the root near the origin whose imaginary part is positive and $\theta_t^2 = \overline{\theta_t^1}$. Let θ_t^3 be the other root whose imaginary part is positive and $\theta_t^4 = \overline{\theta_t^3}$. Let ρ be the complex embedding of K_t which maps θ_t^1 to θ_t^3.

If $t_1, t_2 \in L(A)$ correspond to the same L-function, $\mathbb{Q}(\theta_{t_1}^1)$ and $\mathbb{Q}(\theta_{t_2}^1)$ are isomorphic, since they are quartic fields ([38], page 93). Hence $\mathbb{Q}(\theta_{t_1}^1) = \mathbb{Q}(\theta_{t_2}^j)$ for some $1 \leq j \leq 4$. Assume that 33 different $t_1, t_2, \cdots, t_{33}$ give rise to the same L-function. Then we can see that there are at least nine $t_{i_1}, t_{i_2}, \cdots, t_{i_9}$ with $\mathbb{Q}(\theta_{t_{i_1}}^k) = \mathbb{Q}(\theta_{t_{i_2}}^k) = \cdots = \mathbb{Q}(\theta_{t_{i_9}}^k)$ for some $1 \leq k \leq 4$. Without the loss of the generality, we assume $k = 1$. Then there are at

least two t_{i_l}, t_{i_m} such that $\rho : \theta^1_{t_{i_l}} \to \theta^3_{t_{i_l}}$; $\rho : \theta^1_{t_{i_m}} \to \theta^3_{t_{i_m}}$. Now we further assume that $0.55X < t < X$. Then

$$N(\theta^1_{t_{i_l}} - \theta^1_{t_{i_m}}) < \left(\frac{0.253}{0.55X}\right)^2 \times (4.2896X - 4.204 \times 0.55X)^2 \approx 0.82737943 < 1.$$

Since $\theta^1_{t_{i_l}} - \theta^1_{t_{i_m}} \neq 0$, it induces a contraction. So there are at most 32 t's corresponding to the same L-function.

Let $\widetilde{L}(A)$ be the set of distinct L-functions coming from $L(A)$. Then, we have

$$A^{1-\epsilon} \ll | L(A) | \ll A.$$

Let $c_0 = 61$. Choose α with $c_0\frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $\widetilde{L}(A)$ with $e = 8$, $d = 1$ and $T = (\log A^8)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) &\qquad (3.20)\\
&= 3 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n,\alpha}(1) = 3 \log \log \log |d_{K_t}| + O_{n,\alpha}(1).
\end{aligned}
$$

By the class number formula and Lemma 2.23,

$$h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^3}{(\log d_{K_t})}.$$

We summarize as follows:

Theorem 3.21. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, A_4, 0, 2)$ with arbitrary large discriminant d_K for which*

$$h_K > cd_K^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)}.$$

Remark 3.22. Let $M \sim X^{\delta'}$ with $0 < \delta' < \delta$, where δ is the constant in [27]. Then we have

$$|L(A)| = \prod_{p \nmid M} \left(1 - \frac{\rho(p^3)}{p^3} \right) \frac{X}{M} + O\left(X^{1-\delta}\right).$$

where $\rho(p^3)$ is the number of solutions to $27t^4 + 1 \equiv 0 \pmod{p^3}$.

Proof. We have

$$|L(A)| = N(X, t_M, M) = \sum_{b,\,(b,M)=1} \mu(b) N(b; X, t_M, M)$$

with $N(b; X, t_M, M) = \#\{0 < t < X : b^3 \mid 27t^4 + 1 \text{ and } t \equiv t_M \bmod M\}$. Note that if $p|M$, by the definition of t_M, $27t^4 + 1 \equiv 27t_M^4 + 1 \not\equiv 0 \pmod{p}$.

Denote the solutions to $27t^4 + 1 \equiv 0 \pmod{b^3}$ by $n_1, n_2, \cdots, n_r$. Then

$$
\begin{aligned}
N(b; X, t_M, M) &= \sum_{i \leq r} \#\{0 < t < X : t \equiv n_i \bmod b^3 \text{ and } t \equiv t_M \bmod M\} \\
&= \sum_{i \leq r} (\frac{X}{b^3 M} + O(1)) = \frac{X}{b^3 M} \rho(b^3) + O(\rho(b^3)).
\end{aligned}
$$

It is easy to show

$$\sum_{\substack{b \leq X^{1/2} \\ (b,M)=1}} \frac{\mu(b)\rho(b^3)}{b^3} = \prod_{p \nmid M} \left(1 - \frac{\rho(p^3)}{p^3} \right) + O(X^{-1+\epsilon})$$

and

$$\sum_{b \leq X^{1/2}} \rho(b^3) = O(X^{1/2+\epsilon}).$$

Hence

$$\sum_{\substack{b \leq X^{1/2} \\ (b,M)=1}} \mu(b) N(b; X, t_M, M) = \prod_{p \nmid M} \left(1 - \frac{\rho(p^3)}{p^3} \right) \frac{X}{M} + O(X^{1/2+\epsilon}).$$

Note that $27t^4 + 1 = (27)^{-3}((27t)^4 + 27^3)$. Hence by [27], we obtain $\sum_{b \geq X^{1/2}} \mu(b) N(b; X, t_M, M) = O\left(X^{1-\delta}\right)$, where δ is the constant in [27]. This proves our assertion. □

3.5 Dihedral Groups

3.5.1 Representations of Dihedral Groups

Let's review irreducible representations of D_n: If n is odd, $D_n =< a, x : a^n = x^2 = e, xax = a^{-1} >$. Let $H = \{1, x\}$. Then irreducible representations of D_n are: 2 one-dimensional representations $1, \chi$, and $\frac{n-1}{2}$ two-dimensional representations $\rho_1, ..., \rho_{\frac{n-1}{2}}$, where χ is the character of H. We have $Ind_H^G 1 = 1 + \rho_1 + \cdots + \rho_{\frac{n-1}{2}}$.

If n is even, $D_n =< a, x : a^n = x^2 = e, xax = a^{-1} >$. Let $H_1 = \{1, x\}$, $H_2 = \{1, a^{\frac{n}{2}}\}, H_3 = \{1, a^{\frac{n}{2}}x\}$ be three order 2 subgroups. Then irreducible representations of D_n are: 4 one-dimensional representations $1, \chi_1, \chi_2, \chi_3$, and $\frac{n-2}{2}$ two-dimensional representations $\rho_1, ..., \rho_{\frac{n-2}{2}}$, where χ_i is the character of H_i. We have, for each i, $Ind_{H_i}^G 1 = 1 + \chi_i + \rho_1 + \cdots + \rho_{\frac{n-2}{2}}$.

Let $K/\mathbb{Q}$ be a degree n extension and $\hat{K}/\mathbb{Q}$ be the Galois closure such that $Gal(\hat{K}/\mathbb{Q}) \simeq D_n$. Then if n is odd,

$$\frac{\zeta_K(s)}{\zeta(s)} = L(s, \rho_1) \cdots L(s, \rho_{\frac{n-1}{2}}).$$

If n is even,

$$\frac{\zeta_K(s)}{\zeta(s)} = L(s, \chi)L(s, \rho_1) \cdots L(s, \rho_{\frac{n-2}{2}}),$$

where $H = Gal(\hat{K}/K)$ is one of the order 2 subgroup of D_n, and χ is the non-trivial character of H.

3.5.2 D_5 quintic extensions with signature $(1, 2)$

In section 2.2.3, we showed that

$$f(x, t) = x^5 - tx^4 + (2t - 1)x^3 - (t - 2)x^2 - 2x + 1$$

gives rise to an D_5 regular extension. When $t \leq 6$, the signature of K_t is $(1, 2)$. We prefer that t varies in positive integers, we use $f(x, t) = x^5 + tx^4 - (2t+1)x^3 + (t+2)x^2 - 2x + 1$ from now on. Then its discriminant is $(4t^3 + 28t^2 + 24t + 47)^2$. Since $disc(f(x, t))$ is a

polynomial of degree 6, there is a constant C with $disc(f_t(x)) \leq Ct^6$. For given $X > 0$, let $A = C^{1/6}X$. We define a set $L(A)$ of square-free integers for $A = C^{1/6}X$,

$$L(A) = \{\frac{X}{2} < t < X \mid 4t^3 - 28t^2 + 24t - 47 \text{ square-free}, \ t \equiv s_M \bmod M\}.$$

By Lemma 3.19, $|L(A)| = \beta\frac{X}{2M} + O(\frac{X}{M(\log\frac{X}{M})^{\frac{1}{2}}})$ for some constant β. Hence $|L(A)| \gg A^{1-\epsilon}$. Note that every $t \in L(A)$ corresponds to a distinct automorphic L-function $L(s, \rho_t) = \sum_{n=1}^{\infty} \lambda_t(n)n^{-s}$ of $GL(4)/\mathbb{Q}$ with $\lambda_t(q) = 4$ for all $k \leq q \leq y$.

Let $c_0 = 31$. Choose α with $c_0\frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 6$, $d = 1$ and $T = (\log A^6)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \tag{3.23}$$

$$= 4 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = 4\log\log\log d_{K_t} + O_{n,\alpha}(1).$$

By the class number formula and Lemma 2.21,

$$h_{K_t} \gg d_{K_t}^{1/2}\frac{(\log\log d_{K_t})^4}{(\log d_{K_t})^2}.$$

We summarize as follows:

Theorem 3.24. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(5, D_5, 1, 2)$ with arbitrary large discriminant d_K for which*

$$h_K > cd_K^{1/2}\frac{(\log\log d_K)^4}{(\log d_K)^2}.$$

3.5.3 D_4 quartic extensions with signature $(0, 2)$

In section 2.2.2, we showed that

$$f(x, t) = x^4 - x^3 + (t + 2)x^2 - x + 1$$

gives rise to a D_4 regular extension. Since $disc(f(x,t)) = (t+2)(t+6)(1-4t)^2$ is a polynomial of degree 4, there is a constant C with $disc(f_t(x)) \le Ct^4$. For given $X > 0$, let $A = C^{1/4}X$. We define a set $L(A)$ of square-free integers for $A = C^{1/4}X$,

$$L(A) = \{\frac{X}{2} < t < X \mid 1 - 4t \text{ square-free}, t \equiv s_M \bmod M\}.$$

If $(1 - 4t)$ is square-free, then $(1 - 4t)$ is the discriminant of the unique quadratic subfield of K_t. Hence each t in $L(A)$ corresponds to a distinct L-function $L(s, \rho, t)$ and d_{K_t} is divided at least by $(1 - 4t)$.

It is well-known that $|L(A)| = \frac{6}{\pi^2} \prod_{p|M} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{X}{2M} + O(X^{1/2})$. Hence $|L(A)| \gg A^{1-\epsilon}$.

Let $c_0 = 21$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 4$, $d = 1$ and $T = (\log A^4)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\begin{aligned} \log L(1, \rho, t) &= \sum_{p \le \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \qquad (3.25) \\ &= 3 \sum_{c_f \le p \le \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = 3 \log\log\log d_{K_t} + O_{n,\alpha}(1). \end{aligned}$$

By the class number formula and (2.15),

$$h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log\log d_{K_t})^3}{(\log d_{K_t})}.$$

We summarize as follows:

Theorem 3.26. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, D_4, 0, 2)$ with arbitrary large discriminant d_K for which*

$$h_K > c d_K^{1/2} \frac{(\log\log d_K)^3}{(\log d_K)}.$$

3.5.4 D_4 quartic extensions with signature $(2, 1)$

In section 2.2.2, we showed that

$$f(x, s) = x^4 - sx^3 + 3x^2 - sx + 1$$

gives rise to a D_4 regular extension. Since $disc(f(x, s)) = (s^2 - 4)^2(25 - 4s^2)$ is a polynomial of degree 6, there is a constant C with $disc(f(x, s)) \leq Cs^6$. For given $X > 0$, let $A = C^{1/6}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \{\frac{X}{2} < s < X \mid s^2 - 4 \text{ square-free } , s \equiv s_M \text{ mod } M\}.$$

If $(s^2 - 4)$ is square-free, then $(s^2 - 4)$ is the discriminant of the unique quadratic subfield of K_s. Hence each s in $L(A)$ corresponds to a distinct L-function $L(s, \rho, t)$ and d_{K_s} is divided at least by $(s^2 - 4)$.

To estimate $|L(A)|$, we introduce Nair's work.[50]. For an polynomial $f(x) \in \mathbb{Z}[x]$ of degree d, we define,

$$N_k(f, x, h) = N_k(x, h) = |\{n : x < n \leq x + h | f(n) : \text{k-free}\}|.$$

He showed

Theorem 3.27 (Nair). *If*

$$f(x) = \prod_{i=1}^{m}(a_i x - b_i)^{\alpha_i} \text{ and } \alpha = \max_i \alpha_i,$$

then

$$N_k(x, h) = \prod_p \left(1 - \frac{\rho(p^2)}{p^2}\right) h + O\left(\frac{h}{(\log h)^{k-1}}\right)$$

for $h = x^{(\alpha/2k)+\epsilon}$ if $k > \alpha$ and $\epsilon > 0$.

Theorem 3.27 implies

$$|L(A)| = \left(1 - \frac{1}{4}\right) \prod_{p \nmid M}\left(1 - \frac{2}{p^2}\right) \frac{X}{2M} + O\left(\frac{X}{M(\log \frac{X}{M})}\right) \gg A^{1-\epsilon}.$$

Let $c_0 = 31$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 6$, $d = 1$ and $T = (\log A^6)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log|d_{K_s}|)^2, (\log|d_{K_s}|)^2]$ L-function $L(s, \rho, s)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, s) &= \sum_{p \leq \sqrt{\log|d_{K_s}|}} \frac{\lambda(p, s)}{p} + O_{n,\alpha}(1) \qquad\qquad (3.28)\\
&= 3 \sum_{c_f \leq p \leq \sqrt{\log|d_{K_s}|}} \frac{1}{p} + O_{n,\alpha}(1) = 3 \log\log\log|d_{K_s}| + O_{n,\alpha}(1).
\end{aligned}
$$

By the class number formula and (2.17),

$$
h_{K_s} \gg |d_{K_s}|^{1/2} \frac{(\log\log|d_{K_s}|)^3}{(\log|d_{K_s}|)^2}.
$$

We summarize as follows:

Theorem 3.29. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, D_4, 2, 1)$ with arbitrary large discriminant d_K for which*

$$
h_K > c|d_K|^{1/2} \frac{(\log\log|d_K|)^3}{(\log|d_K|)^2}.
$$

3.5.5 D_4 quartic extensions with signature $(4, 0)$

In section 2.2.2, we showed that

$$
f(x, s) = x^4 - sx^3 - x^2 + sx + 1
$$

gives rise to a D_4 regular extension. Since $disc(f(x, s)) = (s^2 - 4)^2(4s^2 + 9)$ is a polynomial of degree 6, there is a constant C with $disc(f(x, s)) \leq Cs^6$. For given $X > 0$, let $A = C^{1/6}X$. We define a set $L(A)$ of square-free integers

$$
L(A) = \{\frac{X}{2} < s < X \mid s^2 - 4 \text{ square-free}, s \equiv s_M \bmod M\}.
$$

If $(s^2 - 4)$ is square-free, then $(s^2 - 4)$ is the discriminant of the unique quadratic subfield of K_s. Hence each s in $L(A)$ corresponds to a distinct L-function $L(s, \rho, t)$ and d_{K_s} is divided at least by $(s^2 - 4)$. As in the previous section, we can show that $\mathfrak{K}(4, D_4, 4, 0)$ has infinitely many number fields with extreme class numbers and it is summarized as follows:

Theorem 3.30. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, D_4, 4, 0)$ with arbitrary large discriminant d_K for which*

$$h_K > c d_K^{1/2} \frac{(\log \log d_{K_t})^3}{(\log d_{K_t})^3}.$$

3.5.6 D_3 cubic extensions with signature $(1, 1)$

In section 2.2.1, we showed that

$$f(x, t) = x^3 + tx - 1$$

gives rise to a D_3 regular extension. Since $disc(f(x,t)) = -(4t^3 + 27)$ is a polynomial of degree 3, there is a constant C with $|disc(f_t(x))| \leq Ct^3$. For given $X > 0$, let $A = C^{1/3}X$. We define a set $L(A)$ of square-free integers

$$L(A) = \{\frac{X}{2} < t < X \mid 4t^3 + 27 \text{ square-free }, t \equiv s_M \bmod M\}.$$

Note that each t in $L(A)$ corresponds to a distinct L-function $L(s, \rho, t)$.

By Lemma 3.19, $|L(A)| = \beta \frac{X}{2M} + O\left(\frac{X}{M(\log \frac{X}{M})^{1/2}}\right)$ for some β, $|L(A)| \gg A^{1-\epsilon}$.

Let $c_0 = 16$. Choose α with $c_0 \frac{1-\alpha}{2\alpha - 1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 3$, $d = 1$ and $T = (\log A^3)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_s}|)^2, (\log |d_{K_s}|)^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, s)$ in $\widehat{L}(A)$, we have

$$\log L(1, \rho, s) \quad = \sum_{p \leq \sqrt{\log |d_{K_t}|}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \tag{3.31}$$

$$= \quad 2 \sum_{c_f \leq p \leq \sqrt{\log |d_{K_t}|}} \frac{1}{p} + O_{n,\alpha}(1) = 2 \log \log \log |d_{K_t}| + O_{n,\alpha}(1).$$

By the class number formula and (2.13),

$$h_{K_s} \gg |d_{K_s}|^{1/2} \frac{(\log \log |d_{K_s}|)^2}{(\log |d_{K_s}|)}.$$

We summarize as follows:

Theorem 3.32. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(3, D_3, 1, 1)$ with arbitrary large discriminant d_K for which*

$$h_K > c|d_K|^{1/2} \frac{(\log \log |d_K|)^2}{(\log |d_K|)}.$$

3.6   Cyclic Groups

3.6.1   Simplest Sextic Fields

For $f(x, t) = f(x, t) = x^6 - \frac{t-6}{2}x^5 - 5\frac{t+6}{4}x^4 - 20x^3 + 5\frac{t-6}{4}x^2 + \frac{t+6}{2}x + 1$, we checked that $f(x, (6r + 3)(36r^2 + 36r + 18))$ gives rise to a C_6 regular extension.

Since $disc(f(x, (6r + 3)(36r^2 + 36r + 18)) = 2^6 3^{21}(3r^2 + 3r + 1)^{10}(12r^2 + 12r + 7)^5$, there is a constant C with $disc(f(x, (6r + 3)(36r^2 + 36r + 18)) \leq Cr^{30}$. Now define a set $L(A)$ for $A = C^{1/30}X$.

$$L(A) = \left\{ \frac{X}{2} < r < X \mid 3r^2 + 3r + 1 \text{ and } 12r^2 + 12r + 7 \text{ square-free}, r \equiv s_M \bmod M \right\}.$$

we can show that $A^{1-\epsilon} \ll |L(A)| \ll A$. Note that each r in $L(A)$ corresponds to a distinct L-function.

Let $c_0 = 76$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 30$, $d = 1$ and $T = (\log A^{30})^2$, every automorphic L-function excluding exceptional

$O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \le \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \tag{3.33}$$

$$= 5 \sum_{c_f \le p \le \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = 5 \log \log \log d_{K_t} + O_{n,\alpha}(1).$$

By the class number formula and (2.11),

$$h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^5}{(\log d_{K_t})^5}.$$

We summarize as follows

Theorem 3.34. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(6, C_6, 6, 0)$ with arbitrary large discriminant d_K for which*

$$h_K > c d_K^{1/2} \frac{(\log \log d_K)^5}{(\log d_K)^5}.$$

3.6.2   Simplest Quintic fields

Smith [67] showed that Lehmer's polynomial $f(x, t) = x^5 + t^2 x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 + (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2 + (t^3 + 4t^2 + 10t + 10)x + 1$ generates a C_5 regular extension. Since the discriminant is a polynomial of degree 22, there is a constant C with $disc(f(x, t)) \le Ct^{22}$.

For $A = C^{1/22}X$, let $L(A)$ be a finite set given by

$$L(A) = \{\frac{X}{2} < t < X \mid t^4 + 5t^3 + 15t^2 + 25t + 25 \text{ cubic-free and } t \equiv t_M \text{ mod} M\}.$$

Then by Lemma 3.19, we have $A^{1-\epsilon} \ll L(A) \ll A$.

Let $c_0 = 56$. Choose α with $c_0 \frac{1-\alpha}{2\alpha - 1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 22$, $d = 1$ and $T = (\log A^{22})^2$, every automorphic L-function excluding exceptional

$O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$.   Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$
\begin{aligned}
\log L(1, \rho, t) &= \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \quad\quad\quad (3.35) \\
&= 4 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = 4 \log \log \log d_{K_t} + O_{n,\alpha}(1).
\end{aligned}
$$

By the class number formula and (2.10),

$$
h_{K_t} \gg d_{K_t}{}^{1/2} \frac{(\log \log d_{K_t})^4}{(\log d_{K_t})^4}.
$$

We summarize as follows

Theorem 3.36. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(5, C_5, 5, 0)$ with arbitrary large discriminant d_K for which*

$$
h_K > c d_K^{1/2} \frac{(\log \log d_K)^4}{(\log d_K)^4}.
$$

3.6.3   Simplest Quartic fields

In section 2.1.1, we showed that $f(x, t) = x^4 - tx^3 - 6x^2 + tx + 1$ generates a C_4 regular extension. Since the discriminant $disc(f(x, t)) = 4(t^2 + 16)^3$ is a polynomial of degree 6, there is a constant C with $disc(f(x, t)) \leq Ct^6$.

For $A = C^{1/6}X$, let $L(A)$ be a finite set given by

$$
L(A) = \{\frac{X}{2} < t < X \mid t \text{ even}, t \equiv t_M \bmod M\}.
$$

Then we have $A^{1-\epsilon} \ll L(A) \ll A$. We can determine the discriminant d_{K_t} with 2-adic valuation $\nu_2(t)$ of t and it is at least $\frac{(t^2+16)^3}{64}$.(See [41]) So every $t \in L(A)$ corresponds to a distinct L-function.

Let $c_0 = 16$. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $L(A)$ with $e = 6$, $d = 1$ and $T = (\log A^6)^2$, every automorphic L-function excluding exceptional $O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log d_{K_t})^2, (\log d_{K_t})^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.7 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$, we have

$$\log L(1, \rho, t) = \sum_{p \leq \sqrt{\log d_{K_t}}} \frac{\lambda(p, t)}{p} + O_{n,\alpha}(1) \tag{3.37}$$

$$= 3 \sum_{c_f \leq p \leq \sqrt{\log d_{K_t}}} \frac{1}{p} + O_{n,\alpha}(1) = 3 \log \log \log d_{K_t} + O_{n,\alpha}(1).$$

By the class number formula and (2.8),

$$h_{K_t} \gg d_{K_t}^{1/2} \frac{(\log \log d_{K_t})^3}{(\log d_{K_t})^3}.$$

We summarize as follows

Theorem 3.38. *There is a constant $c > 0$ such that there exist $K \in \mathfrak{K}(4, C_4, 4, 0)$ with arbitrary large discriminant d_K for which*

$$h_K > c d_K^{1/2} \frac{(\log \log d_K)^3}{(\log d_K)^3}.$$

Chapter 4

Logarithmic derivatives of Artin

L-functions

This chapter is written based on the joint work [9] with H. Kim. We follow [9] closely.

4.1 Euler-Kronecker Constants

Let K be a number field of degree n with discriminant d_K and $\zeta_K(s)$ be the Dedekind zeta function of K, with the Laurent expansion at $s = 1$:

$$\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + c_1(s-1) + c_2(s-1)^2 + \cdots$$

Then $\gamma_K = \frac{c_0}{c_{-1}}$ is called the Euler-Kronecker constant of K. If $K = \mathbb{Q}$, then $\gamma_{\mathbb{Q}}$ is just the Euler constant $\gamma = 0.57721566\cdots$. When K is an imaginary quadratic field, the Kronecker limit formula express γ_K in terms of special values of the Dedekind η-function. It was Ihara who began a study of Euler-Kronecker constant systematically. We refer to [29] for the detail.

We can see that $\frac{\zeta_K'}{\zeta_K}(s) = -\frac{1}{s-1} + \gamma_K + (s-1)h(s)$, for some holomorphic function $h(s)$ at $s = 1$. Let $\widehat{K}$ be the Galois closure of K. Then we have $\zeta_K(s) = \zeta(s)L(s, \rho)$ for some

$n - 1$ dimensional complex representation ρ of the Galois group $Gal(\widehat{K}/\mathbb{Q})$. So we have

$$\gamma_K = \gamma + \frac{L'}{L}(1, \rho). \tag{4.1}$$

This is one of the motivations to study the logarithmic derivatives at $s = 1$. Ihara [29] found a upper bound and a lower bound for γ_K under GRH. The main term of his upper and lower bound under GRH are

$$2 \log \log \sqrt{|d_K|}, \quad -2(n - 1) \log \left(\frac{\log \sqrt{|d_K|}}{n - 1} \right).$$

In [30], page 260, the authors remarked that in the case of Dirichlet characters, the coefficient 2 can be replaced by $1 + o(1)$. In section 4.7, we prove that under the Artin conjecture, GRH and certain zero density hypothesis (Conjecture 4.19), the upper and lower bound are

$$\log \log |d_K| + O(\log \log \log |d_K|), \quad -(n - 1) \log \log |d_K| + O(\log \log \log |d_K|),$$

resp. The lower bound comes from number fields where almost all small primes split completely. This agrees with Ihara's observation ([29], page 409) that number fields with many primes having small norms have the lower bound.

When K is a quadratic field $\mathbb{Q}(\sqrt{d})$, the value $\frac{L'}{L}(1, \chi_d)$ determines $\gamma_{\mathbb{Q}(\sqrt{d})}$ where χ_d is the Dirichlet character attached to the quadratic field $\mathbb{Q}(\sqrt{d})$. Recently, Mourtada and Murty [48] showed unconditionally that there are infinitely many Dirichlet L-functions of quadratic characters whose logarithmic derivative at $s = 1$ have large values. Namely, there are infinitely many fundamental discriminants d such that $\left| -\frac{L'}{L}(1, \chi_d) \right| \gg \log \log |d|$. This implies that $\left| \gamma_{\mathbb{Q}(\sqrt{d})} \right| \gg \log \log |d|$ for infinitely many quadratic fields $\mathbb{Q}(\sqrt{d})$. We realized that the techniques we used to obtain extreme values of $L(1, \rho)$ in [6], [7], [8], can be applied to generalize their result to arbitrary Artin L-functions.

Under several assumptions (the strong Artin Conjecture, Assumptions 4.4 and 4.5), we show in section 4.2 that there are infinitely many number fields such that

$$\frac{L'}{L}(1, \rho) \geq \log \log |d_K| + O(\log \log \log |d_K|), \tag{4.2}$$

and infinitely many number fields such that

$$\frac{L'}{L}(1,\rho) \leq -(n-1)\log\log|d_K| + O(\log\log\log|d_K|). \tag{4.3}$$

4.2 Extreme values of $\frac{L'}{L}(1,\rho)$

In this section, we describe how to obtain extreme positive or negative values of $\frac{L'}{L}(1,\rho)$ in a general setting. Suppose $K_t \in \mathfrak{K}(n,G,r_1,r_2)$. Let ρ_t be the $n-1$ dimensional complex representation of the Galois group $Gal(\widehat{K}/\mathbb{Q})$ given by $\zeta_{K_t}(s) = \zeta(s)L(s,\rho_t)$. Then the conductor of ρ_t is $|d_{K_t}|$. Now we assume that ρ_t is modular, i.e., an automorphic representation of GL_{n-1}. The discriminant of $f(x,t)$ is a polynomial in t. We expect that regular Galois extension property implies that the field discriminant $|d_{K_t}|$ will increase with respect to t.

Assumption 4.4. $\log|d_{K_t}| \gg_f \log|t|$.

4.2.1 Extreme positive values of $\frac{L'}{L}(1,\rho)$

Let G be a finite group having property Gal_T and let $f(x,t) \in \mathbb{Z}[t][x]$ be an irreducible polynomial of degree n whose splitting field over $\mathbb{Q}(t)$ is a regular extension with Galois group G. Let K_t be the number field obtained by adjoining a root of $f(x,t)$ to the rational number field $\mathbb{Q}$ for a specialization $t \in \mathbb{Z}$ and let $\widehat{K_t}$ be its Galois closure. Let $L(s,\rho,t) = \sum_{l=1}^{\infty} \lambda(l,t)l^{-s}$ be the Artin L-function $\frac{\zeta_{K_t}(s)}{\zeta(s)}$.

Note that the conductor of $L(s,\rho,t)$ is $|d_{K_t}|$, and for unramified prime p, $\lambda(p,t) = N(p,t) - 1$, where $N(p,t)$ is the number of solutions of $f(x,t) \equiv 0 \pmod{p}$. Hence $-1 \leq \lambda(p,t) \leq n-1$.

The Galois group $Gal(\widehat{K_t}/\mathbb{Q}) \simeq G$ acts on the set $X = \{x_1, x_2, \cdots, x_n\}$ of roots of $f(x,t)$ transitively. Let G_0 be the set of $g \in G$ with no fixed point. Then G_0 is not empty and $\frac{|G_0|}{|G|} \geq \frac{1}{n}$ (see [62], page 430). Choose any $g_0 \in G_0$ and let $[g_0]$ be the conjugacy class

of g_0 in G. If the Frobenius element of p belongs to $[g_0]$, then $f(x,t) \equiv 0 \pmod{p}$ has no root and hence $\lambda(p,t) = -1$.

Since $f(x,t)$ gives rise to a regular extension, by Theorem 2.1, there is a constant c_f depending on f such that for any prime $p \geq c_f$, there is an integer i_p so that for any $t \equiv i_p \pmod{p}$, the Frobenius element of p belongs to $[g_0]$. For $X > 0$, let $y = \frac{\log X}{\log \log X}$ and $M = \prod_{c_f \leq p \leq y} p$. Note that $M \ll e^y = e^{\frac{\log X}{\log \log X}} \ll_\epsilon X^\epsilon$ for any $\epsilon > 0$.

Let i_M be an integer such that $i_M \equiv i_p \pmod{p}$ for all $c_f \leq p \leq y$. So if $t \equiv i_M \pmod{M}$, for all $c_f \leq p \leq y$, p belongs to $[g_0]$ and $\lambda(p,t) = -1$.

Assume that the discriminant of $f(x,t)$ is a polynomial in t of degree D. Then there is a constant C such that $|d_{K_t}| \leq Ct^D$. For $A = C^{1/D}X$, we define a set $L(A)$ of positive numbers given by

$$L(A) = \{\frac{X}{2} < t < X \mid t \equiv i_M \pmod{M}, \ Gal(\widehat{K_t}/\mathbb{Q}) \simeq G\}.$$

Under the strong Artin Conjecture, every t in $L(A)$ corresponds to an automorphic L-function of $GL(n-1)$ over $\mathbb{Q}$. But it is possible that different $t \in L(A)$ correspond to the same automorphic L-function. See Theorem 3.2.

Assumption 4.5. *There exists a finite set $T \subset \mathbb{Z}$, depending only on f, such that $L(s, \rho, t)$'s are distinct for all $t \in L(A)\backslash T$.*

Let $\widetilde{L}(A)$ be the set of automorphic L-functions coming from $L(A)$ after removing the possible repetition of the same L-functions among them. In sections 4.4 through 4.6, we consider explicit examples of families of number fields. In those cases, we may have to put more conditions in $L(A)$ in order to satisfy Assumption 4.5, or replace it by some other set. In any case, we show that $A^{1-\epsilon} \ll |\widetilde{L}(A)| \ll A$ for any fixed $\epsilon > 0$.

Let $c_0 = \frac{5(n-1)D}{2} + 1$. Or we may replace $(n-1)$ in c_0 by a smaller constant if ρ is not irreducible. Choose α with $c_0 \frac{1-\alpha}{2\alpha-1} < \frac{98}{100}$. By applying Theorem 1.10 to $\widetilde{L}(A)$ with $e = D$, $d = 1$ and $T = (\log A^D)^2$, every automorphic L-function excluding exceptional

$O(A^{98/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, \log |d_{K_t}|)^2]$. Let us denote by $\widehat{L}(A)$, the set of the automorphic L-functions with the zero-free region.

Applying Proposition 1.9 to this L-function $L(s, \rho, t)$ in $\widehat{L}(A)$ with $x = (\log CX^D)^{\frac{16}{1-\alpha}}$, we have

$$
\begin{aligned}
\frac{L'}{L}(1, \rho, t) &= -\sum_{p \leq x^2} \frac{\lambda(p, t) \log p}{p} + O_{n,x,\alpha}(1) \qquad (4.6)\\
&= \sum_{c_f \leq p \leq y} \frac{\log p}{p} - \sum_{y < p \leq x^2} \frac{\lambda(p, t) \log p}{p} + O_{n,x,\alpha}(1)\\
&= \log \log X - \sum_{y < p \leq x^2} \frac{\lambda(p, t) \log p}{p} + O(\log \log \log X).
\end{aligned}
$$

(Here we use the fact that $\sum_{p \leq y} \frac{\log p}{p} = \log y + O(1)$, and $y = \frac{\log X}{\log \log X}$.)

Now we sum the logarithmic derivative $\frac{L'}{L}(1, \rho, t)$ over $\widehat{L}(A)$, namely, consider

$$
\sum_{L(s, \rho, t) \in \widehat{L}(A)} \frac{L'}{L}(1, \rho, t).
$$

We need to deal with the sum

$$
\sum_{L(s, \rho, t) \in \widehat{L}(A)} \sum_{y < p \leq x^2} \frac{\lambda(p, t) \log p}{p} = \sum_{y < p \leq x^2} \frac{\log p}{p} \sum_{L(s, \rho, t) \in \widehat{L}(A)} \lambda(p, t).
$$

In the next section, we prove the following proposition:

Proposition 4.7. *For all $y < p \leq x^2$,*

$$
\sum_{L(s, \rho, t) \in \widehat{L}(A)} \lambda(p, t) \ll \frac{|\widehat{L}(A)|}{\sqrt{p}} + \frac{|\widehat{L}(A)|}{(\log X)^{\frac{1}{2}}}.
$$

where the implied constant is independent of p for $y < p \leq x^2$.

Proposition 4.7 implies

$$
\begin{aligned}
\sum_{L(s, \rho, t) \in \widehat{L}(A)} \sum_{y < p \leq x^2} \frac{\lambda(p, t) \log p}{p} &\ll |\widehat{L}(A)| \sum_{y < p \leq x^2} \frac{\log p}{p^{3/2}} + \frac{|\widehat{L}(A)|}{(\log X)^{\frac{1}{2}}} \sum_{y < p \leq x^2} \frac{\log p}{p}\\
&\ll \frac{|\widehat{L}(A)|}{y^{1/2}} + \frac{|\widehat{L}(A)| \log \log X}{(\log X)^{\frac{1}{2}}}.
\end{aligned}
$$

Hence we have

$$\sum_{L(s,\rho,t)\in\widehat{L}(A)} \frac{L'}{L}(1,\rho,t) = |\widehat{L}(A)|\log\log X + O(|\widehat{L}(A)|\log\log\log X).$$

Now note that $|d_{K_t}| \leq Ct^D$ and $t < X$. So if there are only finitely many L-functions with $\frac{L'}{L}(1,\rho,t) \geq \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|)$, they cannot reach the average value $\log\log X$ as X increases. Hence we proved the following under the Strong Artin conjecture and Assumptions 4.4 and 4.5,

Theorem 4.8. *There are infinitely many $L(s,\rho,t)$ in $\widehat{L}(A)$ such that*

$$\frac{L'}{L}(1,\rho,t) \geq \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|).$$

4.2.2   Extreme negative values of $\frac{L'}{L}(1,\rho)$

To generate a negative $\frac{L'}{L}(1,\rho,t)$, but whose absolute value is large, we need to manipulate $\lambda(p,t)$ so that $\lambda(p,t) = n-1$ for all primes p between c_f and $y = \frac{\log X}{\log\log X}$ in (4.6).

Since $f(x,t)$ gives rise to a regular Galois extension, by Theorem 2.1, for any prime $p \geq c_f$, there is an integer s_p so that for any $t \equiv s_p \pmod{p}$, the Frobenius element of p is the identity in G. For $X > 0$, let $M = \prod_{c_f \leq p \leq y} p$. Let s_M be an integer such that $s_M \equiv s_p \pmod{p}$ for all $c_f \leq p \leq y$. So if $t \equiv s_M \pmod{M}$, for all $c_f \leq p \leq y$, p splits completely in $\widehat{K_t}$, and $\lambda(p,t) = n-1$.

For $A = C^{1/D}X$, we define a set $L(A)$, $\widetilde{L}(A)$ and $\widehat{L}(A)$ as in the previous section. Then as in (4.6),

$$\frac{L'}{L}(1,\rho,t) = -(n-1)\log\log X - \sum_{y<p\leq x^2} \frac{\lambda(p,t)\log p}{p} + O(\log\log\log X).$$

Then by Proposition 4.7,

$$\sum_{L(s,\rho,t)\in\widehat{L}(A)} \frac{L'}{L}(1,\rho,t) = -(n-1)|\widehat{L}(A)|\log\log X + O(|\widehat{L}(A)|\log\log\log X).$$

Hence we proved the following under the Strong Artin Conjecture and Assumptions 4.4 and 4.5,

Theorem 4.9. *There are infinitely many L-functions $L(s, \rho, t)$ in $\widehat{L}(A)$ with*

$$\frac{L'}{L}(1, \rho, t) \leq -(n-1) \log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|).$$

4.3 Proof of Proposition 4.7

For a fixed prime p, consider the equation $f(x, t) \equiv 0 \pmod{p}$. Now we consider $f(x, t)$ as a algebraic curve over $\mathbb{Z}/p\mathbb{Z}$. Let A_i be the number of $t \pmod{p}$ such that $\lambda(p, t) = i$, i.e., $f(x, t) \equiv 0 \pmod{p}$ has $i + 1$ roots. Then we have

$$\sum_{i=-1}^{n-1} A_i = p + O(1),$$

where $O(1)$ is bounded by D, the degree of discriminant of $f(x, t)$.

Recall Weil's celebrated theorem on rational points of a curve over a finite field. ([58], page 75):

Theorem 4.10. *Let $f(x, y) \in \mathbb{F}_p[x, y]$ be absolutely irreducible and of total degree $d > 0$. Let N be the number of zeros of f in $\mathbb{F}_p \times \mathbb{F}_p$. Then*

$$|N - p| \leq (d-1)(d-2)\sqrt{p} + c(d),$$

for a constant $c(d)$.

Weil's theorem implies

$$\sum_{i=-1}^{n-1} (i+1) A_i = p + O(\sqrt{p}).$$

Hence we obtain

$$\sum_{i=-1}^{n-1} i A_i = O(\sqrt{p}). \tag{4.11}$$

Now we define $Q_i = \{\frac{1}{2} X < t < X \mid t \in L(A) \text{ and } t \equiv i \pmod{p}\}$ and write

$$L(A) = Q_0 \cup Q_1 \cup \cdots \cup Q_{p-1}.$$

Let R be a finite subset of $\{0, 1, 2, \cdots, p - 1\}$ for which $k \in R$ if and only if p is ramified for $t \in Q_k$. Then we prove the following in the examples in sections 4.4 through 4.6:

$$|Q_i| = c_p \frac{|L(A)|}{p} + O\left(\frac{|L(A)|}{p(\log X)^{\frac{1}{2}}}\right) \quad \text{for } \notin R \qquad (4.12)$$

where c_p is a constant close to 1, independent of i. (We can show that $\frac{1}{2} < c_p < 2$.)

Since $\displaystyle\sum_{L(s,\rho,t)\in\widehat{L}(A)} \lambda(p, t) = \sum_{L(s,\rho,t)\in L(A)} \lambda(p, t) + O(X^{98/100})$, in order to prove Proposition 4.7, it is enough to show that

$$\sum_{L(s,\rho,t)\in L(A)} \lambda(p, t) \ll \frac{|L(A)|}{\sqrt{p}} + \frac{|L(A)|}{(\log X)^{\frac{1}{2}}}.$$

When $k \in R$,

$$\left|\sum_{L(s,\rho,t)\in Q_k} \lambda(p, t)\right| \le (n - 1)\frac{|L(A)|}{p} + O(1).$$

If $k \notin R$, p is unramified for all $t \in Q_k$, and $\lambda(p, t) = j(k)$ for a unique $j(k)$. In that case,

$$\sum_{L(s,\rho,t)\in Q_k} \lambda(p, t) = j(k)c_p\frac{|L(A)|}{p} + O\left(\frac{|L(A)|}{p(\log X)^{\frac{1}{2}}}\right).$$

Hence

$$\sum_{L(s,\rho,t)\in L(A)} \lambda(p, t) = \sum_{k\in R}\sum_{L(s,\rho,t)\in Q_k} \lambda(p, t) + \sum_{k\notin R}\sum_{L(s,\rho,t)\in Q_k} \lambda(p, t).$$

Here

$$\sum_{k\in R}\sum_{L(s,\rho,t)\in Q_k} \lambda(p, t) \ll \frac{|L(A)|}{p},$$

where the implied constant is independent of p. On the other hand,

$$\sum_{k\notin R}\sum_{L(s,\rho,t)\in Q_k} \lambda(p, t) = \sum_{k\notin R} j(k)|Q_k| = c_p\frac{|L(A)|}{p}\sum_{k\notin R} j(k) + O\left(\frac{|L(A)|}{(\log X)^{\frac{1}{2}}}\right)$$

$$= c_p\frac{|L(A)|}{p}\sum_{j=-1}^{n-1} jA_j + O\left(\frac{|L(A)|}{(\log X)^{\frac{1}{2}}}\right).$$

By (4.11),

$$\sum_{L(s,\rho,t)\in L(A)} \lambda(p, t) \ll \frac{|L(A)|}{\sqrt{p}} + \frac{|L(A)|}{(\log X)^{\frac{1}{2}}}.$$

4.4 Cyclic and Dihedral extensions

Cyclic and dihedral extensions satisfy Gal_T property. Hence given G, a cyclic or dihedral group, there exists a polynomial $f(x,t) \in \mathbb{Z}[t][x]$ whose splitting field over $\mathbb{Q}(t)$ is a regular Galois extension and whose Galois group is G. We give some details for quadratic and cyclic cubic extensions.

4.4.1 Quadratic extensions

Consider $K_t = \mathbb{Q}[\sqrt{t}]$ for t square free and $t \equiv 1 \pmod 4$. Consider, for $M = 4\prod_{3 \leq p \leq y} p$,

$$L(X)_1 = \{\frac{X}{2} < t < X | \ t \text{ square-free and } t \equiv s_M \pmod M)\}$$

$$L(X)_2 = \{\frac{X}{2} < t < X | \ t \text{ square-free and } t \equiv i_M \pmod M)\}.$$

Then Assumptions 4.4 and 4.5 are clear. We verify (4.12) in the case of $L(X)_2$:

$$Q_i = \{\frac{X}{2} < t < X | t \text{ square free}, \ t \equiv i_M \pmod M), \ t \equiv i \pmod p)\}.$$

Since $p > y$, $(p, M) = 1$ and if $i \neq 0$, by [18], page 248,

$$|Q_i| = \frac{3}{\pi^2} \prod_{q|M}(1 - q^{-2})^{-1}\frac{X}{M}(1 - p^{-2})^{-1}\frac{1}{p} + O(X^{\frac{1}{2}}) = c_p\frac{|L(X)_2|}{p} + O(X^{\frac{1}{2}}),$$

where $c_p = (1 - p^{-2})^{-1}$ and $1 < c_p < 2$. Since $p \ll (\log X)^{\frac{32}{1-\alpha}}$, $X^{\frac{1}{2}} \ll \frac{|L(X)_2|}{p(\log X)^{\frac{1}{2}}}$. Here we considered real quadratic fields. However, the same argument is applicable to imaginary quadratic fields. So Theorem 4.8 and Theorem 4.9 are now stated as follows:

Theorem 4.13. *(1) There are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{t})$ (resp, imaginary quadratic fields $Q(\sqrt{t})$) with*

$$\frac{L'}{L}(1, \chi_t) \leq -\log\log|t| + O(\log\log\log|t|).$$

(2) There are infinitely many real quadratic fields $\mathbb{Q}(\sqrt{t})$ (resp, imaginary quadratic fields $Q(\sqrt{t})$) with

$$\frac{L'}{L}(1, \chi_t) \geq \log\log|t| + O(\log\log\log|t|).$$

4.4.2 Cyclic Cubic extensions

Consider

$$f(x,t) = x^3 - tx^2 - (t+3)x - 1,$$

for $t \in \mathbb{Z}^+$. Its discriminant is $g(t)^2$ with $g(t) = t^2 + 3t + 9$. Then $K_t/\mathbb{Q}$ is a C_3 Galois extension, and $L(s,\rho,t) = L(s,\chi_t)L(s,\overline{\chi_t})$, where $\chi_t, \overline{\chi_t}$ are two non-principal characters of C_3. The conductor f_{χ_t} of χ_t is $g(t)$ when $g(t)$ is square-free. Note also that $\frac{L'}{L}(1,\rho,t) = 2Re\left(\frac{L'}{L}(1,\chi_t)\right)$. Consider, for $M = 6 \prod\limits_{5 \leq p \leq y} p$,

$$L(A)_1 = \{\frac{X}{2} < t < X | \ g(t) \text{ square-free}, t \equiv s_M \ (\text{mod } M)\}$$
$$L(A)_2 = \{\frac{X}{2} < t < X | \ g(t) \text{ square-free}, t \equiv i_M \ (\text{mod } M)\}.$$

Then Assumptions 4.4 and 4.5 are clear. We verify (4.12) in the case of $L(A)_2$:

$$Q_i = \{\frac{X}{2} < t < X | \ g(t) \text{ square free}, t \equiv i_M \ (\text{mod } M), \ t \equiv i \ (\text{mod } p)\}.$$

Define R' be the set of solutions $t \ (\text{mod } p)$ for $g(t) \equiv 0 \ \text{mod } p$. Then R' has at most 2 elements. So it is enough to consider $i \notin R'$. Since $p > y$, $(p, M) = 1$ and for $i \notin R'$, by [17],

$$|Q_i| = \prod_{q \nmid M}\left(1 - \left(1 + \left(\frac{-3}{q}\right)\right)q^{-2}\right)\frac{X}{2M}\left(1 - \left(1 + \left(\frac{-3}{p}\right)\right)p^{-2}\right)^{-1}\frac{1}{p} + O(X^{\frac{2}{3}}\log X)$$

$$= c_p\frac{|L(A)_2|}{p} + O(X^{\frac{2}{3}}\log X),$$

where $c_p = (1 - (1 + (\frac{-3}{p}))p^{-2})^{-1}$ and $\frac{1}{2} < c_p < 2$. Since $p \ll (\log X)^{\frac{32}{1-\alpha}}$, $X^{\frac{2}{3}}\log X \ll \frac{|L(A)_2|}{p(\log X)^{\frac{1}{2}}}$. So Theorem 4.8 and Theorem 4.9 are now stated as follows:

Theorem 4.14. *(1) There are infinitely many $L(s,\rho,t)$ with*

$$\frac{L'}{L}(1,\rho,t) \leq -2\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|), \quad Re\left(\frac{L'}{L}(1,\chi_t)\right) \leq -\log\log f_{\chi_t} + O(\log\log\log f_{\chi_t}).$$

(2) There are infinitely many $L(s,\rho,t)$ with

$$\frac{L'}{L}(1,\rho,t) \geq \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|), \quad Re\left(\frac{L'}{L}(1,\chi_t)\right) \geq \frac{1}{2}\log\log f_{\chi_t} + O(\log\log\log f_{\chi_t}).$$

4.4.3 Dihedral and cyclic extensions

For higher degree extensions, we recall the explicit examples from Chapter 2.

$\mathfrak{K}(6, C_6, 6, 0) : f(x,t) = x^6 - 2tx^5 - 5(t+3)x^4 - 20x^3 + 5tx^2 + 2(t+3)x + 1$

$\mathfrak{K}(5, D_5, 5, 0) : f(x,t) = x^5 - tx^4 + (2t-1)x^3 - (t-2)x^2 - 2x + 1, t > 7$

$\mathfrak{K}(5, D_5, 1, 2) : f(x,t) = x^5 + tx^4 - (2t+1)x^3 + (t+2)x^2 - 2x + 1, t \geq 0$

$\mathfrak{K}(5, C_5, 5, 0) : f(x,t) = x^5 + t^2 x^4 - (2t^3 + 6t^2 + 10t + 10)x^3 + (t^4 + 5t^3 + 11t^2 + 15t + 5)x^2$
$$+ (t^3 + 4t^2 + 10t + 10)x + 1$$

$\mathfrak{K}(4, D_4, 4, 0) : f(x,t) = x^4 - tx^3 - x^2 + tx + 1$

$\mathfrak{K}(4, D_4, 2, 1) : f(x,t) = x^4 - tx^3 + 3x^2 - tx + 1$

$\mathfrak{K}(4, D_4, 0, 2) : f(x,t) = x^4 - x^3 + (t+2)x^2 - x + 1$

$\mathfrak{K}(4, C_4, 4, 0) : f(x,t) = x^4 - tx^3 - 6x^2 + tx + 1$

$\mathfrak{K}(3, D_3, 3, 0) : f(x,t) = (x-t)(x-4t)(x-9t) - t$

$\mathfrak{K}(3, D_3, 1, 1) : f(x,t) = x^3 + tx - 1$

Here in the case of C_6, we do not need to specialize t as in section 3.6.1 since we do not need to find units.

The strong Artin conjecture is valid in all of the above cases. We recall the definition of the sets $L(A)$ in each cases. We only write for the extreme positive value case.

$\mathfrak{K}(6, C_6, 6, 0) : L(A) = \{\frac{X}{2} < t < X \mid t^2 + 3t + 9 \text{ square-free, and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(5, D_5, 5, 0) : L(A) = \{\frac{X}{2} < t < X \mid 4t^3 - 28t^2 + 24t - 47 \text{ square-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(5, D_5, 1, 2) : L(A) = \{\frac{X}{2} < t < X \mid 4t^3 + 28t^2 + 24t + 47 \text{ square-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(5, C_5, 5, 0) : L(A) = \{\frac{X}{2} < t < X \mid t^4 + 5t^3 + 15t^2 + 25t + 25 \text{ cubic-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(4, D_4, 4, 0) : L(A) = \{\frac{X}{2} < t < X \mid t^2 - 4 \text{ square-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(4, D_4, 2, 1) : L(A) = \{\frac{X}{2} < t < X \mid t^2 - 4 \text{ square-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(4, D_4, 0, 2) : L(A) = \{\frac{X}{2} < t < X \mid 1 - 4t \text{ square-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(4, C_4, 4, 0) : L(A) = \{\frac{X}{2} < t < X \mid t \text{ even and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(3, D_3, 3, 0) : L(A) = \{\frac{X}{2} < t < X \mid t \text{ square-free and } t \equiv i_M \pmod{M}\}$

$\mathfrak{K}(3, D_3, 1, 1) : L(A) = \{\frac{X}{2} < t < X \mid 4t^3 + 27 \text{ square-free and } t \equiv i_M \pmod{M}\}$

Except the case of $\mathfrak{K}(3, D_3, 3, 0)$, Assumptions 4.4 and 4.5 are shown in Chapter 3. Since $f(x, t) = (x - t)(x - 4t)(x - 9t) - t$ is an Eisenstein polynomial for square-free t, $\log d_{K_t} \gg \log t$. We prove Assumption 4.5 as follows: We claim that $p \mid t$ if and only if p is totally ramified in K_t: If $p \mid t$, by [11], page 315, p is totally ramified. Conversely, if p is totally ramified and $p \nmid t$, $f(x, t) \equiv (x + a)^3 \pmod{p}$. If we compare the coefficients of $f(x, t) \pmod{p}$, we obtain contradiction. Therefore, K_t's are distinct for all $t \in L(A)$ and showed Assumption 4.5.

Now we show that (4.12) holds for these cases. For the cases of C_6, D_4 and C_4, it can be verified as in the quadratic and cyclic cubic cases. Consider the case of $D_5, (1, 2)$:

$$Q_i = \{\frac{X}{2} < t < X \mid 4t^3 + 28t^2 + 24t + 47 \text{ square-free and } t \equiv i_M \pmod{M}, \text{ and } t \equiv i \pmod{p}\}.$$

Let R' be the set of solutions $t \pmod{p}$ for $4t^3 + 28t^2 + 24t + 47 \equiv 0 \pmod{p}$. Then R' has at most 3 elements. Hence it is enough to consider $i \notin R'$. For $i \notin R'$, by Lemma 3.19,

$$|Q_i| = \prod_{q \nmid M} \left(1 - \frac{\rho(q^2)}{q^2}\right) \left(1 - \frac{\rho(p^2)}{p^2}\right)^{-1} \frac{1}{p} \frac{X}{2M} + O\left(\frac{X}{pM} \left(\log \frac{X}{pM}\right)^{-\frac{1}{2}}\right)$$

$$= c_p \frac{|L(A)|}{p} + O\left(\frac{X}{pM}\left(\log\frac{X}{pM}\right)^{-\frac{1}{2}}\right),$$

where $\rho(p^2)$ is the number of solutions for $4t^3 + 28t^2 + 24t + 47 \equiv 0 \pmod{p^2}$, and $c_p = \left(1 - \frac{\rho(p^2)}{p^2}\right)^{-1}$. Here $1 < c_p < 2$. Since $(1-\epsilon)\log X < \log\frac{X}{pM} < \log X$ for any $\epsilon > 0$, we have

$$\frac{X}{pM}\left(\log\frac{X}{pM}\right)^{-\frac{1}{2}} \ll \frac{|L(A)|}{p(\log X)^{\frac{1}{2}}}.$$

The cases of $D_3, (1,1)$ and $D_5, (5,0)$ and $D_5, (1,2)$ are similar.

Consider the case of C_5:

$$Q_i = \{\frac{X}{2} < t < X \mid t^4 + 5t^3 + 15t^2 + 25t + 25 \text{ cubic-free and } t \equiv i_M \pmod{M}, \text{ and } t \equiv i \pmod{p}\}.$$

Let R' be the set of solutions $t \pmod{p}$ for $t^4 + 5t^3 + 15t^2 + 25t + 25 \equiv 0 \pmod{p}$. Then R' has at most 4 elements. Hence it is enough to consider $i \notin R'$. For $i \notin R'$, by Lemma 3.19,

$$|Q_i| = \prod_{q \nmid M}\left(1 - \frac{\rho(p^3)}{p^3}\right)\left(1 - \frac{\rho(p^3)}{p^3}\right)^{-1}\frac{1}{p}\frac{X}{2M} + O\left(\frac{X}{pM}\left(\log\frac{X}{pM}\right)^{-\frac{3}{5}}\right)$$

$$= c_p \frac{|L(A)|}{p} + O\left(\frac{X}{pM}\left(\log\frac{X}{pM}\right)^{-\frac{3}{5}}\right),$$

where $\rho(p^3)$ is the number of solutions for $t^4 + 5t^3 + 15t^2 + 25t + 25 \equiv 0 \pmod{p^3}$, and $c_p = \left(1 - \frac{\rho(p^3)}{p^3}\right)^{-1}$. Here $1 < c_p < 2$. Clearly, we have

$$\frac{X}{pM}\left(\log\frac{X}{pM}\right)^{-\frac{3}{5}} \ll \frac{|L(A)|}{p(\log X)^{\frac{1}{2}}}.$$

Hence Theorem 4.8 and Theorem 4.9 are valid for the above examples.

4.5   Alternating Groups

4.5.1 A_4 Galois extensions

Consider from section 2.3.1 and [67]:

$$\mathfrak{K}(4, A_4, 0, 2) : f(x, t) = x^4 + 18x^2 - 4tx + t^2 + 81$$

$$\mathfrak{K}(4, A_4, 4, 0) : f(x, t) = x^4 + 18tx^3 + (81t^2 + 2)x^2 + 2t(54t^2 + 1)x + 1$$

All these polynomials generate regular Galois extensions and the strong Artin conjecture is true. In the first case, Assumption 4.4 was verified in 2.3.1. The second case is similar: Note that $disc(f(x, t)) = 16^2 t^2 (27t^2 - 4)^2 (27t^2 + 4)^2$. So if t is square free, Newton polygon argument shows that $t | d_{K_t}$. (If $p | t$, then $p\mathbb{Z}_{K_t} = \mathfrak{p}^2$ for a prime ideal $\mathfrak{p}$.) Hence $\log d_{K_t} \gg \log |t|$.

$$\mathfrak{K}(4, A_4, 0, 2) : L(A) = \{\tfrac{X}{2} < t < X : \ t(t^2 + 108) \text{ square-free and } t \equiv i_M \pmod{M}\}$$

$$\mathfrak{K}(4, A_4, 4, 0) : L(A) = \{\tfrac{X}{2} < t < X : \ t \text{ is square free and } t \equiv i_M \pmod{M}\}$$

In the case of $A_4, (0, 2)$, $t(t^2 + 108)$ is no longer irreducible so we can't apply Theorem 3.19. We need Nair's work [50] Here. He showed

Theorem 4.15 (Nair). *If*

$$f(x) = \prod_{i=1}^{m} (f_i(x))^{\alpha_i} \in \mathbb{Z}[x],$$

where each f_i is irreducible, $\alpha = \max_i \alpha_i$ and $\deg f_i(x) = g_i$, then

$$N_k(x, h) = \prod_p \left(1 - \frac{\rho(p^k)}{p^k}\right) h + O\left(\frac{h}{(\log h)^{k-1}}\right)$$

for $h = x^\theta$ where $0 < \theta < 1$ and $k \geq \max_i \{\lambda g_i \alpha_i\}, (\lambda = \sqrt{2} - 1/2)$ provided that at least one $g_i \geq 2$.

Theorem 4.15 implies that

$$|L(A)| = \prod_{p \nmid M} \left(1 - \frac{\rho(p^2)}{p^2}\right) \frac{X}{2M} + O\left(\frac{X}{M \log \frac{X}{M}}\right).$$

For both cases, (4.12) is verified easily. For the first case, Assumption 4.5 is true. However, for the second polynomial Assumption 4.5 remains to be proved. Hence Theorem 4.8 and Theorem 4.9 are valid modulo Assumption 4.5.

4.5.2 A_5 Galois extension

Consider the polynomial $f(x,t) = x^5 + 5(5t^2 - 1)x - 4(5t^2 - 1)$ from section 2.3.2.

If $K_t = \mathbb{Q}[\alpha_t]$ for $t \in \mathbb{Z}$, K_t has signature $(1,2)$. Let $\widehat{K_t}$ be the Galois closure. Then G has a subgroup H isomorphic to A_4 such that $\widehat{K_t}^H = K_t$. Let $Ind_H^G 1_H = 1 + \rho$ be the induced representation of G by the trivial representation of H where ρ is the 4-dimensional representation of A_5, so that $L(s, \rho, t) = \dfrac{\zeta_{K_t}(s)}{\zeta(s)}$.

Now by [36], page 498, ρ is equivalent to a twist of $\sigma \otimes \sigma^\tau$ by a character, where σ, σ^τ are the icosahedral 2-dimensional representations of $\tilde{A}_5 \simeq SL_2(\mathbb{F}_5)$. Since K_t is not totally real, σ and σ^τ are odd. Hence by [35], Corollary 10,2, σ, σ^τ are modular, i.e., they are attached to cuspidal representations π, π^τ of $GL_2/\mathbb{Q}$. By [53], the functorial product $\pi \boxtimes \pi^\tau$ is a cuspidal representation of $GL_4/\mathbb{Q}$. Hence $L(s, \rho, t)$ is a cuspidal automorphic L-function of $GL_4/\mathbb{Q}$.

Let

$$L(A) = \{\frac{X}{2} < t < X \mid 5t^2 - 1 \text{ square-free}, t \text{ even}, t \equiv i_M \pmod{M}\}.$$

Now we show Asssumption 4.5.

To verify Assumption 4.5, it is enough that K_t are not isomorphic for odd square-free $5t^2 - 1$. Suppose $5t^2 - 1$ is square free for even t. Now we prove that the primes p who ramifies totally in K_t are exactly prime divisors of $5t^2 - 1$. Since $f(x,t)$ is an Eisenstein polynomial with respect to each prime divisor of $5t^2 - 1$, p ramifies totally in K_t. Conversely, assume that a prime p ramifies totally and is not a prime divisor of $5t^2 - 1$. If $p = 2$, then $f(x,t) \equiv x(x^4 + 1) \bmod 2$. Hence p does not ramifies totally. Now assume that p is not 2 and not a prime divisor of $5t^2 - 1$. Then we should have that

$f(x,t) \equiv (x+a)^5 \bmod p$ with $a \not\equiv 0 \bmod p$ and it forces that $p = 5$ and $a = 4$. However, by Newton polygon method, $5O_{K_t} = \mathfrak{p}_1\mathfrak{p}_2^4$.

In this case, (4.12) is verified as in $D_4, (4,0)$ case. So Theorem 4.8 and Theorem 4.9 are valid.

4.6   Symmetric Groups

4.6.1 S_4 Galois extensions

Consider from section 2.4.1:

$$\mathfrak{K}(4, S_4, 4, 0) : f(x,t) = (x-t)(x-4t)(x-9t)(x-16t) - t$$

$$\mathfrak{K}(4, S_4, 2, 1) : f(x,t) = x^2(x-10t)(x-18t) + t$$

$$\mathfrak{K}(4, S_4, 0, 2) : f(x,t) = x^4 + tx^2 + tx + t$$

All these polynomials generate regular Galois extensions and the strong Artin conjecture is true. We define $L(A)$ as

$$L(A) = \{\frac{X}{2} < t < X \mid t \text{ square-free}, \ t \equiv i_M \ (\bmod \ M)\}.$$

Assumptions 4.4 and 4.5 are shown in Chapters 2 and 3.

In these cases, (4.12) is verified as in the quadratic case.

Hence Theorem 4.8 and Theorem 4.9 are valid unconditionally.

4.6.2 S_5 Galois extension

Consider from section 2.4.2:

$$f(x,t) = x^5 + tx + t$$

Since 5 does not divide $disc(f(x,t)) = t^4(256t + 3125)$, 5 is unramified in $\widehat{K_t}/\mathbb{Q}$. In addition, $f(x,t) \equiv x^5 + x + 1 \equiv (x+3)(x^2+x+1)(x^2+x+2) \ (\bmod \ 5)$, the Galois extensions $\widehat{K_t}/\mathbb{Q}$ satisfy the hypothesis of Theorem 1.4. Hence Artin L-functions $L(s, \rho, t) = \frac{\zeta_{K_t}(s)}{\zeta(s)}$ are cuspidal automorphic L-functions of $GL(4)/\mathbb{Q}$.

On the other hand, it is easy to check, for square-free t,

$$p \text{ is totally ramified in } K_t \iff p \mid t.$$

Hence K_t are not isomorphic for all square-free t and this fact implies that $L(s, \rho, t)$ are distinct. Hence we verified Assumption 4.5 for this case.

Let

$$L(A) = \left\{ \frac{X}{2} < t < X \mid t : \text{square-free and } t \equiv i_M \pmod{M} \right\}$$

Then we can see that (4.12) holds.

So Theorem 4.8 and Theorem 4.9 are valid.

4.7   Conditional result under zero density hypothesis

Until now, we obtained the average value of logarithmic derivatives of Artin L-functions in a family. In this section, we assume a zero density hypothesis and evaluate the logarithmic derivative of a single Artin L-function.

We use the same notation as in section 4.2: Let $f(x, t) \in \mathbb{Z}[t][x]$ be an irreducible polynomial of degree n whose splitting field over $\mathbb{Q}(t)$ is a regular extension with Galois group G. Let K_t, $\widehat{K_t}$ be as in section 4.2. Let $L(s, \rho, t) = \sum_{l=1}^{\infty} \lambda(l, t) l^{-s}$ be the Artin L-function $\frac{\zeta_{K_t}(s)}{\zeta(s)}$. For simplicity of notation, let $L(s, \rho) = L(s, \rho, t)$, $\lambda(p) = \lambda(p, t)$, and $N = |d_{K_t}|$.

If we assume the Artin conjecture and GRH for $L(s, \rho)$, then by [16],

$$\frac{L'}{L}(1, \rho) = - \sum_{p \leq (\log N)^{2+\epsilon}} \frac{\lambda(p) \log p}{p} + O_{n,x,\alpha}(1).$$

We show under certain zero hypothesis (Conjecture 4.19) that if $w = (\log N)(\log \log N)^2$, $x = (\log N)^{2+\epsilon}$,

$$\sum_{w < p < x} \frac{\lambda(p) \log p}{p} = O(1). \tag{4.16}$$

Proof of (4.16): By partial summation,

$$\sum_{w<p<x} \frac{\lambda(p)\log p}{p} = -\frac{1}{w}\sum_{p<w}\lambda(p)\log p + \frac{1}{x}\sum_{p<x}\lambda(p)\log p + \int_w^x \frac{\sum_{p<u}\lambda(p)\log p}{u^2}du. \quad (4.17)$$

Then $\frac{1}{w}\sum_{p<w}\lambda(p)\log p = O(1)$, and $\frac{1}{x}\sum_{p<x}\lambda(p)\log p = O(1)$ since $\sum_{p<x}\log p = O(x)$.

Let $\varrho = \beta + i\gamma$ run over the zeros of $L(s,\rho)$ in the critical strip of height up to T, with $1 \leq T \leq u$. Then by [32], page 112,

$$\psi(\rho, u) = \sum_{n\leq u}\lambda(n)\Lambda(n) = -\sum_{|\gamma|\leq T}\frac{u^\varrho - 1}{\varrho} + O(\frac{u\log u}{T}\log(u^{n-1}N)). \quad (4.18)$$

Here $\psi(\rho, u) = \sum_{p\leq u}\lambda(p)\log p + \sum_{p^k\leq u, k\geq 2}\lambda(p^k)\log p$. Since $\lambda(l) \leq d_{n-1}(l)$, where $\zeta(s)^{n-1} = \sum_{l=1}^\infty d_{n-1}(l)l^{-s}$ and $d_{n-1}(l) \leq d(l)^{n-1}$, $\lambda(p^k) \leq (k+1)^{n-1}$. Hence

$$\sum_{p^k\leq u, k\geq 2}\lambda(p^k)\log p \ll \sum_{p\leq\sqrt{u}}\log p \sum_{k<\frac{\log u}{\log p}}(k+1)^{n-1} \ll \sqrt{u}(\log u)^n.$$

So this error term contributes to $O(1)$ in the integral in (4.17). Therefore, we can use $\psi(\rho, u)$ in the integral in (4.17). We apply (4.18) with $T = (\log N)(\log\log N)^2$. The error term $O(\frac{u\log u}{T}\log(u^{n-1}N))$ gives rise to

$$\int_w^x \left((n-1)\frac{u(\log u)^2}{(\log N)(\log\log N)^2} + \frac{u\log u}{(\log\log N)^2}\right)\frac{du}{u^2}.$$

which is $O(1)$.

The sum $\sum_{|\gamma|\leq T}\frac{1}{\varrho}$ is bounded by $(\log N)\sum_{k=1}^T\frac{1}{k} \ll \log N\log T$ and it gives rise to

$$(\log N)(\log T)\int_w^x \frac{du}{u^2} \ll \frac{\log N\log T}{w} = O(1).$$

Now we assume the following zero density hypothesis for $L(s,\rho)$. (cf. [55], page 6)

Conjecture 4.19. *For $u \geq (\log(n-1)\log N)^\kappa$, and $\kappa \geq 1$*

$$\sum_{|\gamma|\leq T}\frac{u^\varrho}{\varrho} \leq u^{1-\frac{c}{(\log(n-1)\log N)^\kappa}}\, T^{\frac{d}{(\log(n-1)\log N)^\kappa}}$$

for some positive constants c, d which are independent of $L(s,\rho)$.

Remark 4.20. Conjecture 4.19 follows from GRH if u is large. However, if u is small, of size $(\log N)^a$ which is under consideration, it does not follow from GRH.

If $T = (\log N)(\log \log N)^2$,

$$T^{\frac{d}{(\log(n-1)\log N)^\kappa}} = O(1).$$

Let $b = \frac{c}{(\log((n-1)\log N)^\kappa}$, then under Conjecture 4.19,

$$\int_w^x \left(\sum_{|\gamma| \leq T} \frac{u^\varrho}{\varrho}\right) \frac{du}{u^2} \ll \int_w^x u^{-1-b}\, du \ll w^{-b} = O(1).$$

Hence under the zero density hypothesis, we proved (4.16). $\square$

Then since

$$\sum_{\frac{\log N}{\log \log N} \leq p \leq w} \frac{\log p}{p} \ll \log \log \log N,$$

we have

$$\frac{L'}{L}(1, \rho) = -\sum_{c_f \leq p < \frac{\log N}{\log \log N}} \frac{\lambda(p) \log p}{p} + O(\log \log \log N). \tag{4.21}$$

Since $-1 \leq \lambda(p) \leq n - 1$, we have

Theorem 4.22. *Under the Artin conjecture and GRH and Conjecture 4.19 for $L(s, \rho)$, the upper and lower bound for $\frac{L'}{L}(1, \rho, t)$ are*

$$\log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|), \quad -(n-1) \log \log |d_{K_t}| + O(\log \log \log |d_{K_t}|),$$

resp.

For $X > 0$, let $y = \frac{1}{100} \log X$ and define M, i_M, s_M as in section 4.2. So for all $c_f \leq p \leq y$, if $t \equiv s_M \pmod{M}$, p splits completely in $\widehat{K_t}$, and $\lambda(p, t) = n - 1$; if $t \equiv i_M$ $\pmod{M}$, $\lambda(p, t) = -1$. Assume that the discriminant of $f(x, t)$ is a polynomial in t of degree D. Then there is a constant C such that $|d_{K_t}| \leq Ct^D$. So $\log |d_{K_t}| \ll \log t$. For $A = C^{1/D} X$, we define a set $L(A)_i$ by

$$L(A)_i = \{\frac{X}{2} < t < X \mid t \equiv i_M \pmod{M}, \ Gal(\widehat{K_t}/\mathbb{Q}) \simeq G\}.$$

Similarly we define $L(A)_s$. Note that for $\frac{1}{2}X < t < X$, $\frac{\log |d_{K_t}|}{\log\log |d_{K_t}|} \leq y = \frac{1}{100}\log X$

for sufficiently large X. Hence we can control $\lambda(p)$ for $c_f \leq p < \frac{\log N}{\log\log N}$, namely, for

$t \in L(A)_s$, $\lambda(p) = n - 1$; for $t \in L(A)_i$, $\lambda(p) = -1$. Hence we have proved

Theorem 4.23. *Under the Artin conjecture and GRH and Conjecture 4.19 for $L(s, \rho, t)$,*
for all $t \in L(A)_i$, $\frac{L'}{L}(1, \rho, t) = \log\log |d_{K_t}| + O(\log\log\log |d_{K_t}|)$, and for all $t \in L(A)_s$,
$\frac{L'}{L}(1, \rho, t) = -(n-1)\log\log |d_{K_t}| + O(\log\log\log |d_{K_t}|)$.

Let's assume the strong Artin conjecture for $L(s, \rho, t)$ instead of the Artin conjecture

and GRH. In addition we assume that Assumptions 4.4 and 4.5 are true. They by

applying Theorem 1.10 to $L(A)_s$ and $L(A)_i$, we obtain sets $\widehat{L}(A)_s$ and $\widehat{L}(A)_i$ where

every automorphic L-function excluding possible exceptional $O(A^{98/100})$ L-functions, has

a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, \log |d_{K_t}|)^2]$. Then we can prove

Theorem 4.24. *Under the strong Artin conjecture, Assumptions 4.4, 4.5 and Conjecture*
4.19 for $t \in \widehat{L}(A)_s$, $\frac{L'}{L}(1, \rho, t) = -(n-1)\log\log |d_{K_t}| + O(\log\log\log |d_{K_t}|)$. For $t \in \widehat{L}(A)_i$,
$\frac{L'}{L}(1, \rho, t) = \log\log |d_{K_t}| + O(\log\log\log |d_{K_t}|)$.

Chapter 5

A refinement of Weil's Theorem

This chapter is written based on the joint work [10] with H. Kim. We follow [10] closely.

5.1   A refinement of Weil's Theorem

First, let us recall Weil's celebrated theorem on rational points of algebraic curves over finite fields. ([58], page 75):

Theorem 5.1 (Weil). *Let $f(x, y) \in \mathbb{F}_p[x, y]$ be absolutely irreducible and of total degree $d > 0$. Let N be the number of zeros of f in $\mathbb{F}_p \times \mathbb{F}_p$. Then*

$$|N - p| \leq (d - 1)(d - 2)\sqrt{p} + c(d),$$

for a constant $c(d)$.

Let $f_t(x) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t)$, where $a_i(t) \in \mathbb{Z}[t]$. Suppose f_t is irreducible over $\mathbb{Q}(t)$. Furthermore, we assume that the splitting field E of $f_t(x)$ over $\mathbb{Q}(t)$ is regular. Let p be a prime and $N_t(p)$ be the number of solutions $f_t(x) \equiv 0 \pmod{p}$, and let $\lambda_t(p) = N_t(p) - 1$. Let $G = Gal(\widehat{K_t}/\mathbb{Q})$. Recall that

$$L(s, \rho_t) = \frac{\zeta_{K_t}(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda_t(n)n^{-s},$$

and ρ_t is the $(n-1)$-dimensional representation of G, and $\lambda_t(p)$ is the value of the character of ρ_t at the conjugacy class of p. Note that $-1 \le \lambda_t(p) \le n-1$ for unramified primes p.

Let A_i be the number of $t \pmod p$ such that $\lambda_t(p) = i$, i.e., $f_t(x) \equiv 0 \pmod p$ has $i+1$ roots. Then clearly,

$$N = \sum_{i=-1}^{n-1} (i+1)A_i + O(1).$$

When G is abelian, then $A_i = 0$ for $i \ne -1, n-1$. So $N = nA_{n-1} + O(1)$.

Theorem 5.2. *Fix a prime p. Let C_i be the union of conjugacy classes such that $\lambda_t(p) = i$. Then*

$$A_i = \frac{|C_i|}{|G|}p + O(\sqrt{p}).$$

Proof. This is essentially Chebotarev density theorem for function field, and is proved by Ree [54]. In [54], Theorem 2, it is stated only for when $f(x,t) \pmod p$ is irreducible. (He needs to assume that $f(x,t)$ gives rise to a regular Galois extension E over $\mathbb{Q}(t)$.) But it is straightforward to generalize it. We follow his exposition in [54]. Let $k = \mathbb{F}_p$, and K be the splitting field of $f(x,t)$ over $k(t)$. Since $f(x,t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$, $Gal(K/k(t)) \simeq Gal(E/\mathbb{Q}(t))$ for sufficiently large p. Let D be the set of elements $a \in k$ such that the place $\mathfrak{p}_a$ of $k(t)$ corresponding to $t-a$ does not ramify in K. Then the conjugacy class C_a of the Frobenius at $\mathfrak{p}_a$ in K is the same as factorization of $f(x,a) \pmod p$. For any conjugacy class C in G, let $N_n(C)$ be the number of elements $a \in D$ such that $C_a = C$. Then the density theorem of Weil says that there exists a constant α_n, depending only on n, such that

$$\left| N_n(C) - \frac{|C|}{|G|}p \right| < \alpha_n \sqrt{p}. \tag{5.3}$$

Hence (5.3) implies our result. $\square$

Remark 5.4. Theorem 5.2 can be thought of a refinement of Theorem 5.1. Indeed, Theorem 5.2 implies Theorem 5.1: By class equation, $\sum_{i=-1}^{n-1} |C_i| = |G|$, and so Theorem

5.2 implies

$$\sum_{i=-1}^{n-1}(i+1)A_i = \sum_{i=-1}^{n-1}(i+1)\left(\frac{|C_i|}{|G|}p + O(\sqrt{p})\right) = \frac{p}{|G|}\sum_{i=-1}^{n-1}i|C_i| + \frac{p}{|G|}\sum_{i=-1}^{n-1}|C_i| + O(\sqrt{p}).$$

Here $\sum_{i=-1}^{n-1}i|C_i| = 0$. We can prove this as follows: Note that χ_{ρ_t} is the sum of irreducible characters $\chi_1, ..., \chi_k$, and

$$\sum_{i=-1}^{n-1}i|C_i| = \sum_{g\in G}\chi_{\rho_t}(g) = \sum_{j=1}^{k}\sum_{g\in G}\chi_j(g).$$

By orthogonality of characters, $\sum_{g\in G}\chi_j(g) = 0$ for each $j = 1, ..., k$. Therefore, $\sum_{i=-1}^{n-1}(i+1)A_i = p + O(\sqrt{p})$.

This implies that $\sum_{i=-1}^{n-1}iA_i = O(\sqrt{p})$. This played a crucial role in the previous chapter to find the error term of logarithmic derivatives of Artin L-functions in a family. In the special case $f_t(x) = x^2 - g(t)$, where $g \in \mathbb{Z}[t]$ and g is square free, we recover the result of Davenport and Burgess [2] that $\sum_{t\,(mod\,p)}\left(\frac{g(t)}{p}\right) = O(\sqrt{p})$.

For the several cyclic polynomials, we determine A_i up to a constant in section 5.2.

5.2 Cyclic groups

In the case of cyclic extension, we need to determine only A_{-1} and A_{n-1}. In the case of simplest cubic fields, Duke [17] already obtained the result. The simplest cubic fields are the cubic fields parameterized by the polynomial

$$f_t(x) = x^3 - tx^2 - (t+3)x - 1, \text{ for } t \in \mathbb{Z}^+.$$

For a prime $p \geq 5$, Duke computed the number of residue classes t modulo p for which $f_t(x)$ splits completely, remains inert or ramifies respectively.

	split	inert	ramified
p ≡ 1 mod 3	$\frac{p-4}{3}$	$\frac{2p-2}{3}$	2
p ≡ 2 mod 3	$\frac{p-2}{3}$	$\frac{2p+2}{3}$	0

This can be paraphrased as

$$A_{-1} = \frac{2p-2}{3}, \text{ and } A_2 = \frac{p-4}{3} \text{ if } p \equiv 1 \bmod 3$$

and

$$A_{-1} = \frac{2p+2}{3}, \text{ and } A_2 = \frac{p-2}{3} \text{ if } p \equiv 2 \bmod 3.$$

With Duke's idea, we can extend the analogous result to simplest quartic and sextic fields. The simplest quartic fields are the fields parameterized by

$$f_t(x) = x^4 - tx^3 - 6x^2 + tx + 1$$

with the discriminant $disc(f_t(x)) = 4(t^2+16)^3$. Note that for odd primes p, $disc(f_t(x)) \equiv$ $0 \bmod p$ has a solution if and only if $p \equiv 1 \bmod 4$.

We consider the polynomial $f_t(x)$ over a finite field $\mathbb{F}_p$ of p elements. Then

$$t = \frac{x^4 - 6x^2 + 1}{x(x^2 - 1)}$$

hence x should belong to $\mathbb{F}_p \backslash \{0, \pm 1\}$.

If $p \equiv 3 \bmod 4$, $disc(f_t(x))$ has no root mod p, Hence $4 \times A_3 = p - 3$. Hence we obtain that

$$A_{-1} = \frac{3p+3}{4} \text{ and } A_3 = \frac{p-3}{4}$$

for primes p with $p \equiv 3 \bmod 4$ and $p \geq 3$.

If $p \equiv 1 \bmod 4$, $disc(f_t(x))$ has two roots $\pm 4\sqrt{-1} \bmod p$. For each t corresponding a root of $disc(f_t(x))$, $f_t(x)$ has at most 3 roots mod p, so there is a constant $0 \leq c_p \leq 6$ depending on p with $4 \times A_3 = p - 3 - c_p$. Since A_3 is of integer value, c_p is 2 or 6. For example, we can check easily that $c_5 = 2$. Hence we obtain that

$$A_{-1} = \frac{3p - 5 + c_p}{4} \text{ and } A_3 = \frac{p - 3 - c_p}{4}$$

for primes p with $p \equiv 1 \bmod 4$ and $p \geq 5$.

Next, we consider the simplest sextic fields. They are parametrized by

$$f_t(x) = x^6 - 2tx^5 - 5(t+3)x^4 - 20x^3 + 5tx^2 + 2(t+3)x + 1$$

with the discriminant $disc(f_t(x)) = 2^6 3^6 (t^2 + 3t + 9)^5$. Note that for odd prims $p \geq 5$, $disc(f_t(x)) \equiv 0 \bmod p$ has a solution if and only if $p \equiv 1 \bmod 3$.

We consider the polynomial $f_t(x)$ over a finite field $\mathbb{F}_p$ of p elements. Then

$$t = \frac{x^6 - 15x^4 - 20x^3 + 6x + 1}{x(x^2 - 1)(2x + 1)(x + 2)}$$

hence x should belong to $\mathbb{F}_p \backslash \{0, \pm 1 - 2, -1/2\}$.

If $p \equiv 2 \bmod 3$, $disc(f_t(x))$ has no root mod p, Hence $6 \times A_5 = p - 5$. Hence we obtain that

$$A_{-1} = \frac{5p + 5}{6} \text{ and } A_5 = \frac{p - 5}{6}$$

for primes p with $p \equiv 2 \bmod 3$ and $p \geq 5$.

If $p \equiv 1 \bmod 3$, $disc(f_t(x))$ has two roots $\frac{-3 \pm 3\sqrt{-3}}{2}$ mod p. For each t corresponding a root of $disc(f_t(x))$, $f_t(x)$ has at most 5 roots mod p, so there is a constant $0 \leq c_p \leq 10$ with $6 \times A_5 = p - 5 - c_p$. Since A_5 is of integer value, c_p must be 2 or 8. Especially we can check that $c_7 = 2$. Hence we obtain that

$$A_{-1} = \frac{5p - 7 + c_p}{6} \text{ and } A_3 = \frac{p - 5 - c_p}{6}$$

for primes p with $p \equiv 1 \bmod 3$ and $p \geq 7$.

Chapter 6

Simple zeros of Maass L-functions

6.1 Introduction

Many people have had an interest in zeros of various kinds of L-functions. One of the most famous open question is the Riemann Hypothesis. In 1942 Selberg [61] proved that a positive proportion of zeros lie on the critical line $re(s) = 1/2$. Levinson [44] showed that more than one third of the zeros are on the critical line. Heath-Brown [26] was the first man who showed that the Riemann zeta function has infinitely simple zeros by observing the work of Levinson. Montgomery [46], assuming the Riemann Hypothesis, showed that more than two thirds of the zeros are simple and lie on the critical line. Unconditionally, Conrey [15] showed that more than two fifths of zeros are simple and lie on the critical line.

In 1983 Hafner [23] showed that L-functions attached to Hecke cusp form for $SL(2, \mathbb{Z})$ has a positive proportion of zeros with odd multiplicity on the critical line. Later he [24] obtained the same conclusion for the L-functions attached to even Maass forms.

Except the case of the Riemann zeta function, people did not know about simple zeros of L-functions. In 1986, Conrey, Ghosh and Gonek [13] showed that the zeta function of a quadratic field has infinitely many simple zeros. In 1988, Conrey and Ghosh [14]

got a breakthrough about simple zeros of modular L-functions. They showed that the L-function attached to Ramanujan tau-function has infinitely many simple zeros. In fact, they showed that one simple zero of a Heck cusp L-function implies infinitely many simple zeros if it has no non-trivial real zero.

The purpose of this article is to extend Conrey and Ghosh [14] to Maass L-functions. First, we recall a Maass form and its L-function briefly.

Let $\Gamma = PSL(2, \mathbb{Z})$ act on the upper half-plane $\mathfrak{H} = \{z : Imz > 0\}$ by linear fractional transformations. The Maass cusp forms which are also eigenfunctions of the Hecke operators is functions f in $L^2(\Gamma \backslash \mathfrak{H})$ satisyfing

$$(1)\ \Delta f = (\tfrac{1}{4} + r^2)f, \Delta = -y^2(\partial_x^2 + \partial_y^2),$$

$$(2)\ f(\gamma z) = f(z) \text{ for all } \gamma \in \Gamma,$$

$$(3)\ \int_0^1 f(z)dx = 0,$$

$$(4)\ T_n f = \lambda_n f \text{ for } n \geq 1$$

where T_n is the nth Hecke operator for Γ.

Then we have the Fourier-Whittaker expansion of $f(z)$,

$$f(z) = \sum_{n \neq 0}^{\infty} a_n y^{1/2} K_{ir}(2\pi n y)e^{2\pi i n x}.$$

Since f is an eigenfunction of the T_n's, it follows that if we normalize $a_1 = 1$, then $a_n = \lambda_n$,which is real, and

$$a_n a_m = \sum_{d|(n,m)} a_{nm/d^2}.$$

The L-function attached to f is defined by

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

and it satisfies the functional equation

$$\Lambda_f(s) := \pi^{-s}\Gamma\left(\frac{s + \epsilon + ir}{2}\right)\Gamma\left(\frac{s + \epsilon - ir}{2}\right)L_f(s) = (-1)^\epsilon \Lambda_f(1 - s)$$

where $\epsilon = 0$ if f is even and $\epsilon = -1$ if f is odd.

Strömbergsson [68] found simple zeros of three different L-functions attached to even Maass forms on $SL(2,\mathbb{Z})$. He considered three even Hecke eigenforms corresponding to $r = 13.779751351891, 17.738563381058$ and 125.313840177018. He showed that there are $156, 157$ and 170 nontrivial zeros under height $t = 200$ in the three respective cases. The zeros are all found to be simple and to lie on the critical line $Re(s) = 1/2$. We will show that at least these three Maass L-functions have infinitely many simple zeros.

Throughout this article, we assume that our Maass form is even except in section 6.5. This is because a minor modification of the proof is required when the form is odd. Another reason is that Strömbergsson [68] found simple zeros of even Maass L-function. In section 5, we explain about the difference when the form is odd in detail.

We imitate Conrey and Ghosh [14] and Booker [3]. In case of Conrey and Ghosh [14], the gamma factor of a Hecke modular L-function consists of one gamma function. On the other hand, the gamma factor of a Maass L-function is the product of 2 gamma functions, that is $H(s) = \pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s+ir}{2}\right)$.

Via Stirling's formula, we can see

$$\pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)(s-1/2)$$

$$= \sqrt{8\pi}(2\pi)^{-s}\Gamma\left(s+1/2\right) + b(2\pi)^{-s}\Gamma\left(s-1/2\right) + (2\pi)^{-s}\Gamma\left(s-1/2\right)E_{(1,r)}(s)$$

where b is some constant constants and $E_{(1,r)}(s)$ is holomorphic and $O(1/s)$ in $Re(s) > 1$.

First we consider the difference of two integrals,

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(1-s)\frac{L'}{L}(s)H(s)L(s)(s-1/2)e^{i(\pi/2-\delta)s}ds$$

$$-\frac{1}{2\pi i}\int_{(-\epsilon)}\frac{L'}{L}(1-s)\frac{L'}{L}(s)H(s)L(s)(s-1/2)e^{i(\pi/2-\delta)s}ds$$

And this equals, by the functional equation ,

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(1-s)\frac{L'}{L}(s)H(s)L(s)(s-1/2)f(s,\delta)ds \qquad (6.1)$$

where $f(s, \delta) = e^{i(\pi/2-\delta)s} + e^{i(\pi/2-\delta)(1-s)}$. The integral is ,via Cauchy's theorem, expressed as the following sum over zeros together with $O(1)$ term.

$$- \sum_{0 < re(\rho) < 1} L'(\rho) H(\rho)(\rho - 1/2) e^{i(\pi/2-\delta)\rho} + O(1)$$

where $O(1)$ is contributed by trivial zeros $1 \pm ir$ of $L(1-s)$ and simple poles $\pm ir$ of $H(s)$.

We show that, in Section 6.2, this integral is very large if there is a simple zero of $L(s)$. This implies the infinity of the number of simple zeros of a Maass L-function.

6.2 Estimates of integrals

Define $X(s)$ be $\frac{H(1-s)}{H(s)}$. Then we have an asymmetric functional equation $L(s) = X(s)L(1-s)$. If we replace $H(s)$ by the above linear combination of Gamma functions, the integral (6.1) equals

$$- \int_{(1+\epsilon)} \frac{L'}{L}(s) L'(s)(2\pi)^{-s} G_1(s) f(s, \delta) ds + \int_{(1+\epsilon)} \frac{X'}{X}(s) L'(s)(2\pi)^{-s} G_1(s) f(s, \delta) ds \quad (6.2)$$

where $G_1(s) = \left(\sqrt{8\pi} \Gamma(s + 1/2) + b\Gamma(s - 1/2) + \Gamma(s - 1/2) E_{(1,r)}(s) \right)$.

We divide (6.2) into 4 integrals. We estimate the value of each integral. For this purpose, we need the following Lemma.

Lemma 6.3. *Suppose that*

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where the sum is absolutely convergent for $\sigma > \sigma_0 > 0$ and $0 < \delta < \pi/2$.

Then for $l > \sigma_0$ and $l + c > 0$,

$$\frac{1}{2\pi i} \int_{(l)} F(s)(2\pi)^{-s} \Gamma(s + c) e^{i(\pi/2-\delta)\pm s} ds$$

$$= \frac{1}{2\pi i} \int_{(l)} F(s)(2\pi)^{-s} \Gamma(s + c) (2\sin(\delta/2))^{-s-c} \left(e^{\pm i\delta/2} \right)^{-s+c} (\mp i)^c ds.$$

Proof. The proof is almost identical with the proof of Lemma 1 in Conrey and Ghosh [14] or Lemma 2 in Booker [3]. For the sake of completeness, we re-write the proof. First recall that $z^c e^{-z} = \int_l \Gamma(z + c) z^{-s}$ for $re(z) > 0$ and $l + c > 0$.

The integral equals

$$= \sum_{n=1}^{\infty} f(n) \frac{1}{2\pi i} \int_{(l)} \Gamma(s+c) \left(\mp 2\pi i n e^{\pm i\delta}\right)^{-s} ds$$

$$= \sum_{n=1}^{\infty} f(n) (\mp 2\pi i n e^{\pm i\delta})^c e^{\pm 2\pi n i (e^{\pm i\delta}-1)}$$

$$= \sum_{n=1}^{\infty} f(n) (\mp 2\pi i n e^{\pm i\delta})^c e^{-2\pi n (2\sin(\delta/2)) e^{\pm i\delta/2}}$$

$$= (\mp i)^c (2\sin(\delta/2))^{-c} (e^{\pm i\delta/2})^c \sum_{n=1}^{\infty} f(n) (2\pi n (2\sin(\delta/2)) e^{\pm i\delta/2})^c e^{-2\pi n (2\sin(\delta/2)) e^{\pm i\delta/2}}$$

$$= (\mp i)^c (2\sin(\delta/2))^{-c} (e^{\pm i\delta/2})^c$$
$$\times \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{(l)} f(n) n^{-s} \Gamma(s+c) (2\pi)^{-s} (e^{\pm i\delta/2})^{-s} (2\sin(\delta/2))^{-s} ds$$

which equals the right hand side. $\square$

Let's consider the following integral.

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s) L'(s) (2\pi)^{-s} \Gamma\left(s+1/2\right) f(s,\delta) ds.$$

By Lemma 6.3, this integral is equal to

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s) L'(s) (2\pi)^{-s} \Gamma\left(s+1/2\right) (2\sin(\delta/2))^{-(s+1/2)} g(s,\delta) ds$$

where $g(s,\delta) = \left[\left(e^{i\delta/2}\right)^{-s+1/2} (-i)^{1/2} + e^{i(\pi/2-\delta)} \left(e^{-i\delta/2}\right)^{-s+1/2} (i)^{1/2} \right]$.

Let's assume that $L(s)$ has a simple zero $\rho = \beta + it$ for $0 < \beta < 1$. From the functional equation $H(s)L(s) = H(1-s)L(1-s)$, we may assume that $\beta \geq 1/2$. Since zeros of $g(s,\delta)$ are $\frac{3}{2} - \frac{2\pi(n+1)}{\delta}$ for $n \in \mathbb{Z}$, the integrand has a simple pole at $s = \beta + it$.

Lemma 6.4 (Booker [3]). *Let $\psi(s)$ be meromorphic in the complex plane, and holomorphic for $\sigma > \sigma_0$ and of rapid decay in vertical strips in a right hand plane. If $\psi(s)$ has a pole at $s = \beta + it$, then for $l > \sigma_0$*

$$\frac{1}{2\pi i} \int_{(l)} \psi(s) x^{-s} ds = \Omega_\epsilon \left(x^{-(\beta-\epsilon)}\right) \text{ as } x \to 0$$

for all $\epsilon > 0$.

Hence by Lemma 6.4,

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}\Gamma\left(s+1/2\right)f(s,\delta)ds = \Omega_\epsilon\left(\delta^{-(\beta+1/2-\epsilon)}\right)$$

as $\delta \to 0$.

Let's consider the integral

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}\left(b\Gamma\left(s-1/2\right)+\Gamma\left(s-1/2\right)E_{(1,r)}(s)\right)f(s,\delta)ds$$

The value of this integral is dominated by the value of the integral

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}\Gamma\left(s-1/2\right)e^{i(\pi/2-\delta)(-s)}ds$$

and this integral is equal to, by Lemma 6.3,

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(s)L'(s)(2\pi)^{-s}\Gamma\left(s-1/2\right)\left(2\sin(\delta/2)\right)^{-(s-1/2)}\left(e^{-i\delta/2}\right)^{-s-1/2}(i)^{-1/2}ds$$

Then, the value of this integral is $O\left(\left(\frac{1}{\delta}\right)^{1/2+\epsilon}\right)$.

Nextly, let's consider the integral

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}\left(b\Gamma\left(s-1/2\right)+\Gamma\left(s-1/2\right)E_{(1,r)}(s)\right)f(s,\delta)ds$$

which is dominated by

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma\left(s-1/2\right)e^{i(\pi/2-\delta)(-s)}ds$$

.

If shift the contour of the integral to $1/2 + \epsilon$, then

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma\left(s-1/2\right)e^{i(\pi/2-\delta)(-s)}ds$$
$$= \frac{1}{2\pi i} \int_{(1/2+\epsilon)} \frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma\left(s-1/2\right)e^{i(\pi/2-\delta)(-s)}ds + O(1).$$

Meurman [45] showed that $L(1/2+it) \ll_\epsilon t^{1/3+\epsilon}$. Hence the integrand is $O\left(|t|^{5/6+\epsilon-1}e^{-\delta|t|}\right)$ and this integral is $O\left((1/\delta)^{5/6+\epsilon}\right)$.

Lastly, the remaining integral is

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma\left(s+1/2\right)f(s,\delta)ds \qquad (\diamondsuit)$$

The estimate of the integral ($\diamondsuit$) is the most difficult part. In section 6.3, we show

Lemma 6.5.

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{X'}{X}(s)L'(s)(2\pi)^{-s}\Gamma\left(s+1/2\right)f(s,\delta)ds \ll \left(\frac{1}{\delta}\right)^{5/6+\epsilon}$$

for any $\epsilon > 0$.

Our estimates of the 4 integrals show that

$$\frac{1}{2\pi i}\int_{(1+\epsilon)}\frac{L'}{L}(1-s)\frac{L'}{L}(s)H(s)L(s)(s-1/2)f(s,\delta)ds = \Omega_\epsilon\left(\frac{1}{\delta}\right)^{\beta+1/2-\epsilon}$$

as $\delta \to 0$.

6.3   An estimate of the integral ($\diamondsuit$)

Recall that $H(s) = \pi^{-s}\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)$. By Stirling's formula, we have

$$H(s) = (2\pi)^{-s}\Gamma\left(s-\frac{1}{2}\right)E_{(2,r)}(s)$$

where $E_{(2,r)}(s)$ is holomorphic in the complex plane except when $s = \pm ir - 2n$ for $n = 0, 1, 2, \cdots$ and $E_{(2,r)}(s) = \sqrt{8\pi} + O(1/s)$.

Then
$$X(s) = \frac{H(1-s)}{H(s)} = (2\pi)^{2s-1}\frac{\Gamma(1/2-s)E_{(2,r)}(1-s)}{\Gamma(s-1/2)E_{(2,r)}(s)}$$

$$= (2\pi)^{2s-1}\frac{\Gamma(3/2-s)(s-1/2)E_{(2,r)}(1-s)}{\Gamma(s+1/2)(1/2-s)E_{(2,r)}(s)} = -(2\pi)^{2s-1}\frac{\Gamma(3/2-s)E_{(2,r)}(1-s)}{\Gamma(s+1/2)E_{(2,r)}(s)}.$$

Hence we have,

$$\frac{X'(s)}{X(s)} = 2\log 2\pi - \frac{\Gamma'}{\Gamma}(3/2-s) - \frac{\Gamma'}{\Gamma}(s+1/2) - \frac{E'_{(2,r)}}{E_{(2,r)}}(1-s) - \frac{E'_{(2,r)}}{E_{(2,r)}}(s).$$

Since $\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} + O(1/z^2)$, it is easy to see that $\frac{E'_{(2,r)}}{E_{(2,r)}}(s) = O(1/s)$. Then, the contribution of $\frac{E'_{(2,r)}}{E_{(2,r)}}(1-s) + \frac{E'_{(2,r)}}{E_{(2,r)}}(s)$ to the integral ($\Diamond$) is $O\left((1/\delta)^{5/6+\epsilon}\right)$ when we move the contour of integral from $c = 1 + \epsilon$ to $c = 1/2$.

We plug in the remaining part of $\frac{X'(s)}{X(s)}$ into the integral ($\Diamond$), we have

$$\int_{(1+\epsilon)} L'(s)G_2(s)f(s,\delta)ds$$

where $G_2(s) = 2\log 2\pi \Gamma(s + 1/2) - \frac{\Gamma'}{\Gamma}(3/2 - s)\Gamma(s + 1/2) - \Gamma'(s + 1/2)$.

By Lemma 6.3,

$$\int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma(s + 1/2)e^{i(\pi/2 - 1/2)s}ds$$
$$= \int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-s-1/2}\left(e^{i\delta/2}\right)^{-s+1/2}(-i)^{1/2}ds$$
$$= \int_{(\epsilon)} L'(s)(2\pi)^{-s}\Gamma(s + 1/2)(2\sin(\delta/2))^{-s-1/2}\left(e^{i\delta/2}\right)^{-s+1/2}(-i)^{1/2}ds$$
$$= O\left((1/\delta)^{1/2+\epsilon}\right)$$

Secondly,

$$\int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\Gamma'(s + 1/2)e^{i(\pi/2 - 1/2)s}ds$$
$$= \int_{(1+\epsilon)} \sum_{n=1}^{\infty} \frac{b(n)}{n^s}\log(-2\pi i n e^{i\delta})(2\pi)^{-s}\Gamma(s + 1/2)\left(2\sin(\delta/2)\right)^{-s-1/2}\left(e^{i\delta/2}\right)^{-s}ds$$
$$= \int_{(1+\epsilon)} \sum_{n=1}^{\infty} \frac{b(n)(\log(n) + \log(-2\pi i e^{i\delta}))}{n^s}$$
$$\times (2\pi)^{-s}\Gamma(s + 1/2)\left(2\sin(\delta/2)\right)^{-s-1/2}\left(e^{i\delta/2}\right)^{-s}ds = O\left((1/\delta)^{1/2+\epsilon}\right)$$

where $L'(s) = -\sum_{n=1}^{\infty} a_n \log(n)n^{-s} = \sum_{n=1}^{\infty} b(n)n^{-s}$.

The remaining integral to estimate is

$$\int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\frac{\Gamma'}{\Gamma}(3/2 - s)\Gamma(s + 1/2)e^{i(\pi/2 - \delta)s}ds.$$

For this purpose, we needs analogous lemmas of Lemma 2 and Lemma 3 in [14].

Lemma 6.6. *Assume that* $|arg(z)| < \pi/2$, *then* $\frac{1}{2\pi i}\int_{(1+\epsilon)} \frac{\Gamma'}{\Gamma}(3/2 - s)\Gamma(s + 1/2)z^{-s} =$
$z^{1/2}e^{-z}\left(\frac{\Gamma'}{\Gamma}(2) - \int_0^1 \frac{e^{tz}-1}{t}(1 - t)dt\right).$

Proof. $\Gamma(s+1/2)$ has a simple pole at $s = -1/2 - n$ for $n = 0, 1, 2, \cdots$ with residue $\frac{(-1)^n}{n!}$.

Hence the integral equals

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{1/2+n}}{n!} \frac{\Gamma'}{\Gamma}(2+n)$$

Since $s\Gamma(s) = \Gamma(s+1)$, we have

$$\frac{\Gamma'}{\Gamma}(2+n) \quad = \frac{\Gamma'}{\Gamma}(2) + \sum_{l=0}^{n-1} \frac{1}{l+2}$$

$$= \frac{\Gamma'}{\Gamma}(2) + \int_0^1 t \frac{(1-t^n)}{(1-t)} dt.$$

Then the integral is equal to

$$= \frac{\Gamma'}{\Gamma}(2) z^{1/2} e^{-z} + z^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!} \int_0^1 t \frac{(1-t^n)}{(1-t)} dt$$

$$= \frac{\Gamma'}{\Gamma}(2) z^{1/2} e^{-z} + z^{1/2} \int_0^1 \frac{t}{1-t} (e^{-z} - e^{-zt}) dt$$

and when we change t by $1 - t$, we obtain the right hand side.

$\square$

Lemma 6.7. *Suppose $v \geq 1/2$. Define*

$$\omega(v,\delta) = \frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{\Gamma'}{\Gamma}(3/2 - s)\Gamma(s+1/2)(2\pi)^{-s} v^{-s} e^{i(\pi/2 - \delta)s} ds.$$

Then

$$\omega(v,\delta) \ll v^{1/2} \left(e^{-x} \log \frac{1}{\delta} + min\{1, \frac{1}{x^2}\} \right)$$

where $x = 2\pi v \sin \delta$. Moreover, we have

$$\omega(v + 1/2, \delta) + \omega(v, \delta) \ll e^{-x} \left(\delta \log \frac{1}{\delta} + \frac{1}{v^{1/2}} \log \frac{1}{\delta} \right) + \delta min\{1, \frac{1}{x^2}\} + \frac{1}{v^{1/2}} min\{1, \frac{1}{x}\}.$$

Proof. Since the process of the proof is identical with Lemma 3 in [14], we skip some detail and refer to [14].

By Lemma 6.6, $\omega(v, \delta) = z^{1/2} e^{-z} \left(\frac{\Gamma'}{\Gamma}(2) - \int_0^1 \frac{e^{tz}-1}{t}(1 - t)dt \right) = e^{-z} G(z)$ where $z = -2\pi v i e^{i\delta} = 2\pi v \sin \delta - 2\pi i \cos \delta = x - iy$.

Since $\int_0^1 \frac{e^{tz}-1}{t}(1 - t)dt \ll \log \frac{|z|}{x} + e^x min\{1, x^{-2}\}$, the first statement follows.

Next, $\omega(v + 1/2, \delta) + \omega(v, \delta) = e^{-(z+\eta)}(G(z + \eta) - G(z)) + G(z)(e^{-z} + e^{-(z+\eta)})$ where $\eta = -\pi i e^{i\delta}$. We have

$$e^{-(z+\eta)} + e^{-z} \ll e^{-x} |e^{-eta} + 1| = e^{-x} |1 - e^{\pi i(e^{2i\delta}-1)}| \ll \delta e^{-x}.$$

Also we have

$$G(z+\eta) - G(z) = \sqrt{z+\eta}\left(-\int_0^1 \frac{e^{t(z+\eta)} - e^{tz}}{t}(1-t)dt\right)$$

$$+(\sqrt{z+\eta} - \sqrt{z})\left(\frac{\Gamma'}{\Gamma}(2) - \int_0^1 \frac{(e^{tz}-1)(1-t)}{t}dt\right)$$

$$\ll |z|^{1/2}\frac{1}{|z|}e^x\min\{1, x^{-1}\} + \frac{1}{|z|^{1/2}}\left(\log\frac{1}{\delta} + e^x\min\{1, x^{-2}\}\right).$$

Then the second statement follows.

$\square$

Now we are ready to estimate the integral

$$F(\delta) = \int_{(1+\epsilon)} L'(s)(2\pi)^{-s}\frac{\Gamma'}{\Gamma}(3/2-s)\Gamma(s+1/2)e^{i(\pi/2-\delta)s}ds = -\sum_{n=1}^{\infty} a_n\log n\,\omega(n,\delta).$$

By Proposition 1.1 in [21] and partial summation, we have

Lemma 6.8. *Let* $S_m = \sum_{n=1}^m a_n\log n$. *Then* $S_m \ll m^{1/3+\epsilon}$.

We notice that by Lemma 6.7,

$$\omega(n+1,\delta) - \omega(n,\delta) = (\omega(n+1,\delta) + \omega(n+1/2,\delta)) - (\omega(n+1/2,\delta) + \omega(n,\delta))$$

$$\ll \left(e^{-(n\delta)}\left(\delta\log\frac{1}{\delta} + \frac{1}{n^{1/2}}\log\frac{1}{\delta}\right) + \delta\min\{1, \frac{1}{(n\delta)^2}\} + \frac{1}{n^{1/2}}\min\{1, \frac{1}{n\delta}\}\right)$$

Since by Lemma 6.7 and Lemma 6.8, $S_n\omega(n+1,\delta) \to 0$ as $n \to \infty$ and the series

$$\sum_{n=1}^{\infty} S_n(\omega(n+1,\delta) - \omega(n,\delta))$$

converges absolutely. Hence we rewrite $F(\delta)$ as

$$F(\delta) = -\sum_{n=1}^{\infty} S_n(\omega(n+1,\delta) - \omega(n,\delta)).$$

Finally, we have, by Lemma 6.8,

$$F(\delta) \ll \sum_{n=1}^{\infty} n^{1/3+\epsilon}|\omega(n+1,\delta) - \omega(n,\delta)|$$

$$\ll \sum_{n=1}^{\infty} n^{1/3+\epsilon}\left(e^{-n\delta}\left(\delta\log\frac{1}{\delta} + \frac{1}{n^{1/2}}\log\frac{1}{\delta}\right) + \delta\min\{1, \frac{1}{(n\delta)^2}\} + \frac{1}{n^{1/2}}\min\{1, \frac{1}{n\delta}\}\right)$$

$$\ll \delta\left(\log\frac{1}{\delta}\right)\left(\frac{1}{\delta}\right)^{4/3+\epsilon} + \left(\log\frac{1}{\delta}\right)\left(\frac{1}{\delta}\right)^{5/6+\epsilon} + \left(\frac{1}{\delta}\right)^{1/3+\epsilon} + \left(\frac{1}{\delta}\right)^{5/6+\epsilon} \ll \left(\frac{1}{\delta}\right)^{5/6+2\epsilon}.$$

Hence Lemma 6.5 follows.

6.4   Main Theorem and its Corollary

Now we state the main Theorem and its Corollary.

Theorem 6.9. *Let $L(s)$ is the L-function attached to a Maass Hecke eigenform on $SL(2, \mathbb{Z})$ and we assume that $L(s)$ has a simple zero $\rho = \beta + it$ for $0 < \beta < 1$. Then, there is an arbitrary large T so that $L(s)$ has $\gg_\epsilon T^{1/6-\epsilon}$ simple zeros in the region $0 < t < T$.*

Corollary 6.10. *There are at least three L-functions attached to even Maass Hecke eigenforms for $SL(2, \mathbb{Z})$ which have infinitely many simple zeros.*

Meurman [45] showed that $L(1/2 + it) \ll_\epsilon t^{1/3+\epsilon}$. By Pharagmen-Lindelöf argument, we have

$$|L(\sigma + it)| \ll_\epsilon |t|^{-2/3\sigma+2/3+\epsilon}$$

for $\frac{1}{2} \leq \sigma \leq 1$.

Now let $\beta_0 = sup\{\beta | \rho = \beta + it$ is a simple zero of $L(s)\}$.

Let's return to our original sum

$$\sum_{0<re(\rho)<1} L'(\rho)H(\rho)(\rho-1/2)e^{i(\pi/2-\delta)\rho} + O(1)$$

For any $\epsilon > 0$, we have $H(s)L'(s) \ll_\epsilon e^{-\frac{\pi}{2}|t|}|t|^{\frac{1}{3}\sigma-\frac{1}{3}+\epsilon}$. Let $\frac{1}{\delta} = T$ and for sufficiently large T, the sum is

$$\ll_\epsilon \sum_{|t|<T, \rho=\beta+it \, simple} T^{\frac{1}{3}\beta_0+\frac{2}{3}+\epsilon}$$

By the calculation in Section 6.2, there is an arbitrary large T for which this sum is $\gg_\epsilon T^{\beta_0+\frac{1}{2}-\epsilon}$. Hence we finish the proof.

6.5   In case of an odd Maass form

When a Maass form is odd, we need some modifications in our arguments.

We consider the integral

$$\frac{1}{2\pi i} \int_{(1+\epsilon)} \frac{L'}{L}(1-s)L'(s)H(s)(s-1/2)^2(e^{i(\pi/2-\delta)s} - e^{i(\pi/2-\delta)(1-s)})ds$$

where $H(s) = \pi^{-s}\Gamma\left(\frac{s-1+ir}{2}\right)\Gamma\left(\frac{s-1-ir}{2}\right)$. Via Stirling's formula, we also see

$$\pi^{-s}\Gamma\left(\frac{s-1+ir}{2}\right)\Gamma\left(\frac{s-1-ir}{2}\right)(s-1/2)^2$$

$$= \sqrt{8\pi}(2\pi)^{-s}\Gamma(s+1/2)+b_1(2\pi)^{-s}\Gamma(s-1/2)+b_2(2\pi)^{-s}\Gamma(s-3/2)+(2\pi)^{-s}\Gamma(s-3/2)E_{(1,r)}(s)$$

where b_1, b_2 are some constants and $E_{(1,r)}(s)$ is holomorphic and $O(1/s)$ in $Re(s) > 1$.
Define $X(s) = \frac{H(1-s)}{H(s)}$. Then $X(s)$ equals

$$(2\pi)^{2s-1}\frac{\Gamma(-1/2-s)E_{(2,r)}(1-s)}{\Gamma(s-3/2)E_{(2,r)}(s)} = (2\pi)^{2s-1}\frac{\Gamma(3/2-s)(s-3/2)E_{(2,r)}(1-s)}{\Gamma(s+1/2)(s+1/2)E_{(2,r)}(s)}.$$

Then, we have

$$\frac{X'}{X}(s) = 2\log 2\pi - \frac{\Gamma'}{\Gamma}(3/2-s) - \frac{\Gamma'}{\Gamma}(1/2+s)$$

$$+\frac{1}{s-3/2} - \frac{1}{s-1/2} - \frac{E'_{(2,r)}}{E_{(2,r)}}(1-s) - \frac{E'_{(2,r)}}{E_{(2,r)}}(s).$$

Except these differences, all calculations work similarly as in the case of an even Maass form.

Chapter 7

Appendix

7.1 Proof of Proposition 3.16

This proposition is a linear sieve problem and it is well-known, at least, to those familiar with sieve method. However, for the sake of completeness, we provide the proof of Proposition 3.16. We refer to Halberstam and Richert [25] for unexplained notations.

Let $\mathcal{A}$ be a finite sequence of integers. The for a square-free integer d, we define

$$\mathcal{A}_d := \{a \mid a \in \mathcal{A}, \ a \equiv 0 \bmod d\}.$$

Let X be a convenient approximation to $\mid \mathcal{A} \mid$. For example, we put $X = \mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$ for $\mid \mathcal{A} \mid = \pi(x)$. For each prime p, we choose $\omega_0(p)$ so that $\frac{\omega_0(p)}{p}X$ approximates to $\mid \mathcal{A}_p \mid$.

and write the remainders as

$$r_p := \mid \mathcal{A}_p \mid - \frac{\omega_0(p)}{p}X \text{ for all } p.$$

With these choices of X and $\omega_0(p)$, we define, for each square-free integer d,

$$\omega_0(1) := 1, \ \omega_0(d) = \prod_{p \mid d} \omega_0(p).$$

so that $\omega_0(d)$ is a multiplicative function. Then we introduce for each square-free integer d,

$$r_d := \mid \mathcal{A}_d \mid - \frac{\omega_0(d)}{d} X.$$

Let $\mathfrak{P} = \{p\}$ be a set of primes and $\overline{\mathfrak{P}}$ denotes the complement of $\mathfrak{P}$ with respect to the set of all primes. After our choice of $\mathfrak{P}$, we define

$$\omega(p) = \begin{cases} \omega_0(p), & p \in \mathfrak{P} \\ 0, & p \in \overline{\mathfrak{P}}. \end{cases}$$

and it is extended to the set of all square-free integers by

$$\omega(1) := 1, \ \omega(d) = \prod_{p \mid d} \omega(p).$$

Correspondingly we introduce for a square-free integer d,

$$R_d := \mid \mathcal{A}_d \mid - \frac{\omega(d)}{d} X.$$

With the new function ω we form the product

$$W(z) := \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right).$$

Theorems in sieve method are formulated subject to certain basic conditions and they are expressed as symbols. We need to introduce some conditions and their symbols necessary for our purpose. Here A_0, A_1, A_2 and L are some constants bigger than or equal to 1 and k is a positive constant. Especially, if $k = 1$, then the corresponding sieve problem is called a linear sieve.

(Ω_0) $\qquad\qquad \omega(p) \leq A_0$ for all p.

(Ω_1) $\qquad\qquad 0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1}$ for all p.

(R) $\qquad\qquad |R_d| \leq \omega(d)$ if $\mu(d) \neq 0$, $(d, \overline{\mathfrak{P}}) = 1$.

$(\Omega_2(k))$ $\qquad\qquad \sum_{w \leq p < z} \frac{\omega(p) \log p}{p} \leq k \log \frac{z}{w} + A_2$ if $2 \leq w \leq z$.

$$(\Omega_2(k, L)) \qquad -L \leq \sum_{w \leq p < z} \frac{\omega(p) \log p}{p} - k \log \frac{z}{w} \leq A_2 \text{ if } 2 \leq w \leq z.$$

Then, our main interest is evaluating the sifting function

$$S(\mathcal{A}; \mathfrak{P}, z) :=| \{a \mid a \in \mathcal{A}, \ (a, P(z)) = 1\} | \ .$$

We quote a lower bound theorem of a linear sieve necessary to prove Proposition 3.16.

Theorem 7.1 (Halberstam and Richert). $(\Omega_1),(\Omega_2(1, L))$: *For $\xi \geq z$, we have*

$$S(\mathcal{A}; \mathfrak{P}, z) \geq XW(z) \left\{ f\left(\frac{\log \xi^2}{\log z} \right) - B \frac{L}{(\log \xi)^{1/14}} \right\} - \sum_{d < \xi^2, d|P(z)} 3^{\nu(d)} \mid R_d \mid$$

where $f(u)$ is a positive function for $u > 2$ and monotonically increasing toward 1, $\nu(d)$ is the number of distinct prime divisors of d and B is a positive constant.

For given positive integer l of the size of x^u with $0 < u \leq 0.3$, let define a finite set $\mathcal{A}$ to be

$$\mathcal{A} = \{0 < n < x \mid n \equiv b \bmod l\}$$

for any co-prime integers b to l. Put $\mathfrak{P} = \{p \mid p : \text{prime and } p \nmid l\}$ and $z = x^{1/3+\epsilon}$ for some $\epsilon > 0$. If $n \in \mathcal{A}$ and $(n, P(z)) = 1$, then the number of prime divisors of n is at most 2. Hence we have

$$S(\mathcal{A}; \mathfrak{P}, z) + O(\sqrt{x}) \leq \mid \{0 < n < x \mid n : \text{square-free}, \ n \equiv b \bmod l, \nu(n) \leq 2\} \mid \ .$$

It is easy to check that $\omega(p) = 1$ if $p \in \mathfrak{P}$. Hence our sieve problem satisfies Ω_1 condition with $A_1 = 2$.

$$(\Omega_1) \qquad 0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1} \text{ for all } p.$$

Since, for $2 \leq w \leq z$,

$$\sum_{w \leq p < z} \frac{\omega(p) \log p}{p} = \sum_{w \leq p < z} \frac{\log p}{p} - \sum_{w \leq p < z, p|l} \frac{\log p}{p} = \log \frac{z}{w} + O_l(1),$$

our sieve problem satisfies $(\Omega_2(1, L))$. So we can apply Theorem 7.1 to our problem. We choose X to be $\frac{x}{l} = x^{1-u}$ and we have $W(z) = \frac{\prod_{p < z}\left(1 - \frac{1}{p}\right)l}{\phi(l)}$. Now we put $\xi^2 = \frac{x^{1-u-\delta}}{\log^4 z}$.

Then, for large x and very small ϵ and δ,

$$\frac{\log \xi^2}{\log z} = \frac{(1 - u - \delta) \log x - 4(\log \log x + \log(1/3 + \epsilon))}{(1/3 + \epsilon) \log x} > 2.$$

Since, for $x \geq 2$ we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right)$$

with Euler constant γ, we obtain the main term of Proposition 3.16.

Now we should show that the error term is smaller than the main term. We have that $\mid R_d \mid \leq \omega(d) \leq 1$ for all square-free d and especially $R_d = 0$ if d is not co-prime to l. Then,

$$\sum_{d < \xi^2, d \mid P(z)} 3^{v(d)} \mid R_d \mid \leq \sum_{d < \xi^2, d \mid P(z)} 3^{v(d)} \omega(d) \leq \xi^2 \sum_{d \mid P(z)} 3^{v(d)} \frac{\omega(d)}{d}$$

$$\leq \xi^2 \prod_{p \mid P(z)} \left(1 + \frac{\omega(p)}{p}\right)^3 \leq \frac{\xi^2}{W^3(z)} \leq \xi^2 W(z) \log^4 z$$

where in the last inequality we used the fact that $\frac{1}{W(z)} = O(\log z)$ under (Ω_1) and $(\Omega_2(1))$ conditions. By choice of ξ^2, the error term is ignorable compared with the main term. Since there are at most $x^{1/2}$ squares in $\mathcal{A}$, we finish the proof.

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