The (co)isoperimetric problem in (random) polyhedra

by

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Abstract

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We consider some aspects of the global geometry of cellular complexes. Motivated by techniques in graph theory, we develop combinatorial versions of isoperimetric and Poincaré inequalities, and use them to derive various geometric and topological estimates. This has a progression of three major topics:

1. We define isoperimetric inequalities for normed chain complexes. In the graph case, these quantities boil down to various notions of graph expansion. We also develop some randomized algorithms which provide (in expectation) solutions to these isoperimetric problems.

2. We use these isoperimetric inequalities to derive topological and geometric estimates for certain models of random simplicial complexes. These models are generalizations of the well-known models of random graphs.

3. Using these random complexes as examples, we show that there are simplicial complexes which cannot be embedded into Euclidean space while faithfully preserving the areas of minimal surfaces.

Dedication

To my parents, who have never wavered in their encouragement.

Contents

Chapter 1

Introduction

This dissertation is concerned with the geometry of cellular complexes. Cellular complexes are already a common tool in algebraic topology, combinatorics and representation theory. They provide a natural and computationally feasible model for topological and combinatorial objects.

Cellular complexes live a dual life. Ultimately, they are fundamentally a combinatorial object–a data structure. However, we most often think of them as geometric objects, and it is this geometric viewpoint which makes them so useful. With this in mind, our perspective will be to treat to complexes as abstract incidence structures, *but* to derive our language and inspiration from their geometric interpretations.

Much of graph theory has taken the same approach, especially with respect to computer science; graphs arise in abstract settings (such as low density parity-check codes) but the language we use to describe them has a distinctly geometric flavor. In fact, much of our own inspiration will come as generalizations of developments in graph theory.

The *topology* of graphs, being 1-dimensional, is most often too simple to be of interest. On the other hand, the quantitative study of the *geometry* of graphs has found application in many fields of mathematics and computer science. Our focus then will be to interpret for higher dimensional complexes some of the useful quantitative concepts in graph theory.

This task is not a simple one for two main reasons. First, any interpretation of graph-theoretic concepts for complexes must make sense in terms of algebraic topology. Second, simple concepts in graphs, such as diameter, girth or connected component, may not have obvious generalizations to higher dimensions. We will of course have to balance these two considerations as we proceed.

Our study roughly breaks down into three categories:

1. *Quantitative estimates for some topological properties*– A statement such as "X is connected" or "$\pi_1(X) = 0$" are *qualitative* statements about the topology of X. Much of this thesis concerns itself with finding natural ways of making such statements *quantitative*. In the succeeding chapters, we will discuss several applications of these quantitative estimates.

2. *Quantitative aspects of maps from complexes to Euclidean space*– We have a natural inclination when faced with an abstractly geometric object, to try to represent it or realize it in space. We do this for the purpose of visualization and comparison. Of course, most geometric objects *do not* embed into Euclidean space in a way that preserves their global geometric structure. And so we can set ourselves to the task of asking to what extent this deformation must occur when comparing an object to Euclidean space. In particular, we will explore this question in the context of simplicial complexes, while taking our intuition from graph theory.

3. *Topological properties of random complexes*– We describe a common model for constructing *random* cellular complexes. The topological quantities of these complexes are themselves random variables and we seek to understand how they are distributed. We are motivated to study these "generic" complexes and their topological properties because the inquiry gives some information about some *extremal* geomet-

ric properties of complexes and also allows us to construct examples of complexes which might otherwise be difficult to construct by explicit methods.

Each of these discussions, though distinct, will take common advantage of some core concepts which we will lay out in chapter 3. The dissertation is roughly broken into three topics, which we will outline now.

1.1 Filling problems

We will start with the topological statement "$H_k(X) \cong 0$," which means "every k-cycle is the boundary of some $(k+1)$-chain." If we are given that $H_k(X) \cong 0$, A natural quantitative specification to make is: "Every k-cycle of size x is the boundary of a chain of size no bigger than y." This quantitative question lies at the core of the ideas of this thesis. Keeping with history, we call this question the *isoperimetric problem.*

In later chapters, we will use solutions to this question to find some geometric applications. Ultimately, the usefulness of the isoperimetric problem boils down to this: If it is indeed true that cycles of size x are the boundary of a chain of size no bigger than y, then x and y put a quantitative measure on the *homological robustness* of X. Usually we measure this robustness by the ratio, $\frac{y}{x}$, or the ratio of their logarithms, $\frac{\log y}{\log x}$.

Although the isoperimetric problem is a core idea in this work, it is a classical problem. In chapter 2, we will briefly discuss some of its illustrious history. However, in our discrete context, the isoperimetric problem becomes a combinatorial optimization problem (for each cycle, find a small chain that bounds it) and we will devote considerable energy to writing algorithms which find solutions to this problem (all of chapter 6 and appendix B). These algorithms are of interest of themselves. We have several propositions and theorems in this direction throughout the dissertation. For example, chapter 6 is devoted to proving:

Theorem. *There exists a constant c_k, which depends only on k, such that for every*

k-dimensional cellular $\mathbb{Z}_2$-cycle, $z \in Z_k Q_n$, there exists a chain $y \in C_{k+1} Q_n$, such that $\partial y = z$ and

$$\|y\| \leq c_k \|z\|^{\frac{k+1}{k}}.$$

We will also show that the theorem is sharp up to a constant:

Theorem. *For each k and each n, there exists a k-cycle, $z_n^k \in Q_n$ such that*

$$\mathrm{Fill}(z_n^k) \geq \omega_k \|z\|^{\frac{k+1}{k}}.$$

The algorithms we develop for finding solutions to the combinatorial isoperimetric problem will find application in later chapters and appendices. The two most notable examples are the overlap property (Appendix A) and the cohomology of random complexes (Chapter 4).

1.2 Random complexes

Mathematicians have been studying random graphs for over sixty years. They have found application in geometry, functional analysis, physics and computer science.

From the perspective of geometry, various models of random graphs yield interesting metric structures. For this reason, random graphs have been used as prototypes for examples in all types of extremal problems (a problem which attempts to maximize or minimize some geometric quantity). Although the simplicity of graphs make them attractive candidates for examples, this same simplicity excludes some of the structures that are taken for granted in continuous geometry, such as minimal surfaces in Riemannian manifolds.

From the perspective of combinatorics and computer science, models of random graphs have proved extraordinarily useful. It seems natural to ask about random models which apply to other incidence structures (simplicial complexes and partially ordered sets are prime examples). With these two motivations in mind, in Chapter 4 we set out to define a class of models for random complexes. The models are fairly simple: start with a cellular complex, X, and simply remove k-dimensional faces from X independently with some probability.

Once we have defined our model, we can emulate some of the questions asked in the early days of random graphs. One of the first questions asked by Erdős and Rényi [21] was, "When is a random graph connected?" The analogous question for higher dimensional complexes is, "When is every cycle also a boundary?" This question was answered by Linial and Meshulam in [39]. We will address this, along with some of its quantitative reformulations, in Chapter 4.

One of the major theorems in Chapter 4 is:

Theorem. *Let $\{X_n\}$ be a sequence of finite $(k+1)$-dimensional polyhedral complexes, such that:*

$$\frac{\log |X_n^{(k)}|}{h^k(X_n)} \longrightarrow 0 \qquad \text{as } n \to \infty$$

($h^k(X_n)$ are the expansion constants to be defined in the next chapter).

Furthermore, let $\epsilon > 0$ and let $\omega = \omega(n)$ be any function such that $\lim_{n\to\infty} \omega = \infty$. Let $X_n^{(k)} \subset Y_{n,p} X_n^{(k+1)}$ be a random p-subcomplex of X_n. If

$$p \geq \frac{2\log |X_n^{(k)}| + \omega}{\epsilon^2 h^k(X_n)}$$

then

$$h^k(Y_{n,p}) \geq (1-\epsilon)p \cdot h^k(X_n) \qquad a.a.s.$$

In Appendix B, we will give several examples of families of complexes X_n which satisfy the hypotheses of the theorem.

The purpose of developing the notion of random complexes and their properties is not for their own sake. We require some of their geometric properties to provide interesting examples. As it turns out, random complexes will provide us a class of simple and effective examples in later chapters and appendices. Most notably, random complexes give an example of complexes which are difficult to embed in Euclidean space.

1.3 Mapping complexes into space

The third topic, which we will devote considerable energy to (Chapter 5 and Appendix A, as well as Appendix C to some extent), will be study of maps from simplicial complexes to Euclidean spaces.

Chapter 5 will concern itself with maps of high co-dimension, which means that the results are most interesting for maps $X^k \to \mathbb{R}^n$ for k much less than n. For such maps, we can compare the areas of minimal surfaces in $\mathbb{R}^n$ to the *combinatorially* minimal surfaces in X. This notion of comparing areas of surfaces gives us a quantitative measure of how well the embedding preserves some of the isoperimetric geometry of X. When X is a graph, this notion is exactly the metric distortion of the map.

As it turns out (just as in the case of graphs) there exist simplicial complexes for which every map to Euclidean space has a great deal of this "distortion" of areas. In fact, in Chapter 5 we will show that the Linial-Meshulam random complexes are an example. These complexes have the *simultaneous* properties that (1) divergence-free cochains have high Dirichlet energy (the "harmonics" of the complex), and (2) the set of cellular cycles is nicely distributed in terms of their filling volume (the "sparseness" of the complex). These concepts will be fully developed and discussed in Chapter 5. Our main theorem in this direction is:

Theorem. *Let $\Delta_n^{(k-1)} \subset X \subset \Delta_n^{(k)}$ be a 2-dimensional simplicial complex with complete $(k-1)$-skeleton and let $\lambda^{k-1}(X)$ denote the spectral gap of the simplicial Laplacian acting on $(k-1)$ forms. Let $\phi : X \to \mathcal{H}$ be an affine embedding of X into an infinite dimensional Hilbert space, suitably scaled so that*

$$\mathrm{Fill}_{\mathcal{H}}(\psi\tau) \geq \mathrm{Fill}_X(\tau)$$

for the boundary of every k-face, tau, in Δ, then,

$$\sum_{f \in X^{(k)}} \left(\mathrm{vol}_k(\phi f)\right)^2 \geq \frac{\lambda^{k-1}(X)}{(k+1)(n-k)} \sum_{\tau} \left(\mathrm{Fill}_X \tau\right)^2.$$

where the first sum runs over k-dimensional faces in X and the second sum runs over all boundaries of k-faces in Δ.

A corollary of this theorem is:

Theorem. *For every $\epsilon > 0$, there exists a k-dimensional simplicial complexes on n vertices and complete $(k-1)$-skeleton such that any affine embedding $\psi : X \to \mathcal{H}$ into Euclidean space must have:*

$$\max_z \frac{\mathrm{Fill}_{\mathcal{H}}(\psi z)}{\mathrm{Fill}_X(z)} \cdot \max_z \frac{\mathrm{Fill}_X(z)}{\mathrm{Fill}_{\mathcal{H}}(\psi z)} \geq C n^{\frac{k-1-k\epsilon}{k^2}}.$$

where both maximums run over simplicial $(k-1)$-cycles, z, in X.

The proof technique is adapted from the proof of a theorem of J. Bourgain: the metric distortion of a map, $G \to \mathbb{R}^n$, for an *Erdős–Rényi random graph* to Euclidean space must be at least $C\frac{\log |G|}{\log \log |G|}$.

This relationship between the harmonics of a complex and distortion of maps to Euclidean space is a central point of this work. Spectral graph theory, or the study of the eigenvalues of the Laplacian of graphs, has found application in group / representation theory, combinatorics and computer science and is a useful and pervasive tool. In some sense, by drawing geometric consequences from the eigenvalues of simplicial Laplacians, we have attempted to motivate the rudiments of a spectral theory for cellular complexes. This theory is already developing in other respects, such as a combinatorial Bochner formula [26] and a simplicial matrix tree theorem [19].

In the next chapter, we will develop our notation and concepts.

Chapter 2

History

2.1 Plateau's Problem

All of the central themes of this work will revolve around a single classical problem:

Problem. Given a closed curve in space, find an immersed surface (if possible), which has our curve as its boundary, and whose area is as small as possible. (Or failing that, do the best you can by finding a surface with small area).

This problem (without our addenda in parenthesis, and subsequent editorializing) has come to be known as *Plateau's problem.* It probably received its name first from Lebesgue [25] [36] and was named after the Belgian physicist Joseph Plateau and his experiments with soap films [47] [2]. It has also been called "The problem of least area," among other similar things. This problem has accumulated a considerable history spanning as far back as Monge [46], Lagrange [35] and, to some extent, even Euler [23].

Today, the study of the Plateau problem focuses on two specific aspects of the problem:

1. the existence of an absolute minimizing "surface," and

2. the geometric properties of the surface should it exist.

The classical history of the problem of least area is extensive in the twentieth century alone, and we must refer the reader to [25] or [2] for a more careful treatment of its developments. We will concern ourselves with a particular strain of modern incarnations of the problem and their applications in geometry.

2.2 The Federer-Fleming isoperimetric inequality

It could be argued that the birth of the modern theory of Plateau's problem occurred in the article [24]. The authors were primarily concerned with solving the existence part of Plateau's problem in broad generality. Along the way, they proved what we will call the Federer-Fleming isoperimetric inequality:

Theorem (Federer and Fleming, [24]). *For each k-dimensional Lipschitz integral cycle, z, in $\mathbb{R}^n$, then there is an integral chain, y, such that $\partial y = z$ and*

$$\|y\| \leq c_n \|z\|^{\frac{k+1}{k}}.$$

In the theorem, c_n depended on the ambient dimension. Later, Simon and Michael [43] improved the constant to depend only on k. The optimal constant was achieved by Almgren in [1]. It is achieved when z is a round k-dimensional sphere and y is a $(k+1)$-dimensional disk.

The Federer-Fleming isoperimetric inequality has acquired several applications in the succeeding years. We will discuss a few in the next section. Much of the original work in this dissertation will be derived from various combinatorial inequalities reminiscent of the Federer-Fleming isoperimetric inequality.

2.3 Some applications of the isoperimetric inequality

Our interest in Plateau's problem and various reformulations of it probably begins in its celebrated application in Gromov's proof of the systolic inequality:

Theorem (Gromov, [27]). *Let* (M^n, g) *be a closed, aspherical Riemannian manifold. Let* $\text{Systole}(M, g)$ *denote the length of the shortest non-contractible closed curve in* M*. Then*

$$\text{Systole}(M, g) \leq c(n)\text{Volume}(M, g)^{\frac{1}{n}}$$

Where $c(n)$ *is a constant which depends only on* n*.*

We encourage the reader to see [30] for a wonderful exposition on this work.

A fundamental step in the proof of the above theorem is a formulation of Plateau's problem:

Theorem ([27]). *For every* k*-cycle,* z*, in* L^∞*, there exists an* $(k+1)$*-chain,* y*, such that* $\partial y = z$*,*

$$\|y\| \leq c_k \|z\|^{\frac{k+1}{k}}$$

where c_k *is a constant which depends only on* k*.*

Another geometric application of (this modification) of Plateau's problem appears in the same paper:

Theorem ([27]). *Let* g *be a metric on the* n*-sphere,* S^n*, such that for every* k*-cycle,* z*, there exists a* $(k+1)$*-chain* y *with* $\partial y = z$ *and*

$$\|y\| \leq \omega_k \|z\|.$$

Let $m < n$ *and* $\phi : S^m \times S^{n-m} \to S^n$ *be a topologically nontrivial map, then there exists a* $\theta \in S^{n-m}$ *such that*

$$\text{vol}_m \phi(S^m \times \{\theta\}) \geq \epsilon \text{vol}_m S^m$$

where $\epsilon = \epsilon(\omega_k, \ldots, \omega_{n-1})$ *(the volume on the right hand side of the inequality is the* m*-volume of the equatorial* m*-sphere).*

We will discuss this theorem (along with its combinatorial analogue) in some detail in Appendix A.

2.4 Isoperimetric problems in combinatorics

The isoperimetric problem is a prototypical constrained optimization problem: Given a cycle, find a small surface which has the cycle as its boundary. It is a natural geometric optimization of the physical world. Maybe it is for this reason that isoperimetric-like inequalities arise in some types of combinatorial optimization.

In the discussions to follow, we will describe some "combinatorial isoperimetric inequalities" which will motivate much of the work we will find in later sections.

2.4.1 The Kruskal-Katona theorem

In [34], and independently in [33], a fundamental theorem on the combinatorics of simplicial complexes was developed. The theorem gives necessary conditions for the number of k-dimensional faces of a simplicial complex in terms of the number of $(k-1)$-dimensional faces.

Theorem (Kruskal [34] and Katona [33]). *Given a simplicial complex, X, with f_{k-1} faces of dimension $(k-1)$ and f_k faces of dimension k. Then*

$$f_k \leq c_k \cdot f_{k-1}^{\frac{k+1}{k}}$$

We have neither state this theorem in its generality nor its specificity, but instead have stated it in a way which supplies our own purposes.

The relation of this theorem to the isoperimetric inequality lies in the proof method. First we observe that every finite simplicial complex embeds simplicially into the simplex (i.e. the complex with n vertices and every possible face). As such, we can regard the complex in question as a subset of faces of the simplex. With this in mind we would like to prove the following:

For every subset, S, of k-dimensional faces of the simplex, denote by ∂S the set of $(k-1)$-dimensional faces contained in the faces of S. Then

$$|S| \leq c_k |\partial S|^{\frac{k+1}{k}}$$

Formulated this way, we see that this is some formulation of the isoperimetric inequality for the simplex (rather than $\mathbb{R}^n$).

A similar inequality was proved by Lindstrom in [37] for complexes which embed cellularly in a cube. We will discuss the geometry and combinatorics of the cube in more detail in subsequent chapters.

2.4.2 Connectivity of random graphs

The first model of random graphs, which we described briefly in the previous chapter, is the *Erdős–Rényi* random graph [21]. It is obtained by connecting any pairs vertices independently with some fixed probability. It is natural to think of the Erdős–Rényi random graphs as random subgraphs of the complete graph (the graph with all possible edges).

Just as easily, we can take random subgraphs of any other graph just by removing edges independently with some probability. It was exactly this thought that led Burtin to investigate random subgraphs of the cube [22]. He asked, in particular: When is a random subgraph of the cube connected? In a first attempt to answer this question one can make the observation: If H is a random subgraph of G,

$$\mathbb{P}[H \text{ is connected}] \leq \sum_{A \subset G} \mathbb{P}[\partial A \cap H \neq \emptyset]$$

where ∂A denotes the set of edges in G connecting A to its complement. We can further make the observation that:

$$\mathbb{E}|\partial A \cap H| = p \cdot |\partial A|$$

where $1 - p$ is the probability of removing any given edge from G. Furthermore, the quantity $|\partial A \cap H|$ is strongly concentrated around its expectation.

Thus, we have reduced the problem to estimating the size of ∂A for each $A \subset G$. Such a quantitative estimate is exactly a combinatorial isoperimetric inequality, and it was exactly this observation that led Burtin, and later Erdős and Spencer [22], to investigate the connectivity properties of subgraphs of the cube.

The moral of all this is that *quantitative* isoperimetric estimates can be leveraged to obtain information about *qualitative* properties of random graphs (such as connectivity). Chapter 4 will deal with higher dimensional analogues of this phenomenon.

Chapter 3

Notation and Concepts

3.1 A preliminary example

We prefer to begin with an example. Suppose we are given a finite graph, G. It consists of vertices and edges. The incidence structure can be written as:

$$0 \longrightarrow C^{-1}G \xrightarrow{d} C^0G \xrightarrow{d} C^1G \longrightarrow 0$$

where C^0G is the vector space of functions from the vertices of G to some field $\mathbb{F}$, C^1G is the vector space of functions from the edges to $\mathbb{F}$ and $C^{-1}G$ consists of all functions from the element $\{\emptyset\}$ to $\mathbb{F}$ (i.e. $C^{-1}G \cong \mathbb{F}$).

The linear map $d : C^0G \to C^1G$ is defined by $df([v,w]) = f(v) - f(w)$ ($[v,w]$ is an oriented edge of G). It is extremely useful to think about the incidence structure of a graph in this (co)chain complex format.

3.2 Complexes

The fundamental object of study we will consider will be the chain complex arising from a finite simplicial complex, cellular complex or partially ordered set. Let us define these explicitly.

Definition 3.2.1. A *simplicial complex*, X, is a collection of subsets of the set $X^{(0)}$, with the property that if $x \in X \subset \mathcal{P}(X^{(0)})$ and $y \subset x$ then $y \in X$.

The *k-dimensional faces* of X, denoted $X^{(k)}$, consists of the elements of X with order $k+1$. It is important to note that every simplicial complex contains the empty set–the lone (-1)-dimensional face.

We will always consider $X^{(0)}$ to be a finite set unless otherwise stated. Therefore we will always identify (in an arbitrary way) $X^{(0)}$ with $[n] = \{1, \dots, n\}$. Using this arbitrary ordering we will denote a k-dimensional face by $[x_0, \dots, x_k]$ where $x_j \in [n]$ and $x_0 < \cdots < x_k$.

When we fix a ring, R (usually $\mathbb{R}$ or $\mathbb{Z}_2$), we define the *chain spaces*,

$$C_k X = \Big\{ \sum_{x \in X^{(k)}} a_x x : a_x \in R \Big\}$$

i.e., just formal linear combinations of k-faces. Naturally, the chain spaces form a chain complex:

$$\cdots \xleftarrow{\partial_{k-1}} C_{k-1}X \xleftarrow{\partial_k} C_k X \xleftarrow{\partial_{k+1}} C_{k+1}X \xleftarrow{\partial_{k+2}} \cdots$$

where $\partial_k [x_0, \dots, x_k] = \sum_{i=0}^{k} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_k]$ and the notation $[x_0, \dots, \hat{x}_i, \dots, x_k]$ denotes the subset of $[x_0, \dots, x_k]$ obtained by removing x_i.

This also yields a dual *cochain* complex:

$$\cdots \xrightarrow{d_{k-2}} C^{k-1}X \xrightarrow{d_{k-1}} C^k X \xrightarrow{d_k} C^{k+1}X \xrightarrow{d_{k+1}} \cdots$$

where $C^k X = \big(C_k X\big)^\vee = \{C_k X \to R,\ R\text{-linear}\}$, the cochain space, is the dual of the chain space, and $d_k = (\partial_{k+1})^\vee$.

As a convention, we shall use Latin letters for chains (i.e. u, w, x, y, z) and Greek letters for cochains (i.e. $\alpha, \beta, \gamma, \eta$).

We shall focus most of our attention on certain subspaces of the chain space, namely:

the space of *cycles* $\quad Z_k X = \ker \partial_k \subset C_k X \quad$ (resp. *cocycles* $Z^k X = \ker d_k \subset C^k X$)

the space of *boundaries* $B_kX = \text{im}\partial_{k+1} \subset C_kX$ (resp. *coboundaries* $B^kX = \text{im}d_{k-1} \subset C^kX$)

Finally, we define the *k-th reduced homology* (respectively, *reduced cohomology*) of X as,

$$\tilde{H}_k(X;R) = Z_kX/B_kX \qquad \Big(\text{resp. } \tilde{H}^k(X;R) = Z^kX/B^kX \Big).$$

We will take the liberty of suppressing subscripts whenever unambiguous.

3.2.1 The cube as a complex

We will not always consider simplicial complexes. A significant fraction of this document is concerned with the combinatorial geometry of the n-dimensional cube thought of a *cellular complex*. Rather than define cellular complexes in abstraction, we prefer to discuss the cellular structure of the cube concretely.

Definition 3.2.2. For our purposes, the n-dimensional cube, Q_n, consists of a collection of faces defined in the following way:

1. The set of 0-faces (the underlying set), $Q_n^{(0)}$, consists of all binary strings of length n, e.g. $(0,1,1,1,0,0,1,0)$ is a 0-face in Q_8.

2. The set of k-faces, $Q_n^{(k)}$, consists of all binary strings of length n where exactly k of the coordinates are indeterminant, e.g.

$$(1,*,1,0,*,0) = \Big[(1,0,1,0,0,0),(1,0,1,0,1,0),(1,1,1,0,0,0),(1,1,1,0,1,0)\Big]$$

is a 2-dimensional face of Q_6.

3. The chain spaces,

$$C_kQ_n = \Big\{ \sum_{x \in Q_n^{(k)}} a_x x : a \in R \Big\}$$

(and the associated cochain spaces) are defined just as in the previous section. The boundary map (which then defines the coboundary map) can be defined by

$$\begin{aligned}
&\partial_k[-,\dots,-,*,-,\dots,-,\underbrace{*}_{i},-,\dots,-] \\
=\ &\textstyle\sum_{i=1}^{k}[-,\dots,-,*,-,\dots,\underbrace{1}_{i},\dots,-]-[-,\dots,-,*,-,\dots,\underbrace{0}_{i},\dots,-]
\end{aligned},$$

namely, by fixing each indeterminant coordinate separately to obtain a difference of $(k-1)$-faces.

Immediately we observe that $|Q_n^{(k)}| = \binom{n}{k}2^{n-k}$ and that the total number of faces of all dimensions (including the empty set, the lone (-1)-face) is $\sum_{k=0}^{n} |Q_n^{(k)}| = 3^n$.

3.3 Higher expansion coefficients

Our aim is to study and quantitative properties of our chain complexes (and consequences of those quantitative properties). In order to this, we must have some method of quantitative measurement. For this, we will equip our chain complexes with norms.

Unless otherwise stated we will use the following norms whenever considering various rings, R:

1. If $R = \mathbb{Z}$, we will consider the L^1-norm and L^∞-norm on chains, namely

$$\|z\|_1 = \Bigg\|\sum_{x\in X^{(k)}} a_x x\Bigg\| = \sum_x |a_x| \qquad \text{and} \qquad \|z\|_\infty = \max |a_x|.$$

   These norms will respectively induce the L^∞ and L^1 norms on cochains (naturally as dual norms).

2. If $R = \mathbb{R}$, we will consider the L^2-norm on chains (which induces the L^2-norm on cochains) defined by

$$\|z\|_2 = \sqrt{\sum_x |a_x|^2}.$$

3. In our ring of choice, $R = \mathbb{Z}_2$, chains and cochains can be canonically identified via $x \to \chi_x$ where x is a k-face and χ_x is the indicator function of that k-face. Keeping this identification in mind, shall use the norm (for both chains and cochains):

$$\|z\| = |\text{supp} z|,$$

or the number of k-faces which have a non-zero coefficient.

Also, for boundaries, $z \in B_k X$ we will frequently use the notation,

$$\text{Fill}(z) = \min \left\{ \|y\| : y \in C_{k+1}X \text{ and } \partial y = z \right\}$$

and similarly for coboundaries, $\beta \in B^k X$,

$$\text{coFill}(\beta) = \min \left\{ \|\alpha\| : \alpha \in C^{k-1}X \text{ and } d\alpha = \beta \right\}.$$

Definition 3.3.1. For a normed chain complex, the *k-th R-expansion* is defined to be

$$h_k(X; R) = \inf_{y \in C_k X} \frac{\|\partial y\|}{\inf_{w \in C_{k+1}X} \|y + \partial w\|}.$$

In most cases we will be more interested in the k-th expansion of the cochain complex,

$$h^k(X; R) = \inf_{\beta \in C^k X} \frac{\|d\beta\|}{\inf_{\alpha \in C^{k-1}X} \|\beta + d\alpha\|}.$$

This definition appeared implicitly in [39] and a modified version appeared explicitly in [28]. Since then, this definition has appeared elsewhere and has become more or less standard ([51], [17])

We now ask the reader to verify some simple facts.

Proposition 3.3.2. *1. (a)* $h_k(X; R) > 0 \Leftrightarrow \tilde{H}_k(X; R) \cong 0$

(b) $h^k(X; R) > 0 \Leftrightarrow \tilde{H}^k(X; R) \cong 0$

2. If X is a graph,

(a) $h^0(X;\mathbb{Z}_2) = c(X)$ *where* $c(X)$ *denotes the Cheeger constant of* X.

(b) $h^0(X;\mathbb{R}) = \sqrt{\lambda(X)}$ *where* $\lambda(X)$ *denotes the spectral gap of the graph Laplacian of* X.

3. *If* X *is a* $(k+1)$*-dimensional simplicial complex and* $h^k(X;\mathbb{Z}_2) = \epsilon$ *then, in order to create a* k*-dimensional reduced cohomology class, all of whose representatives have norm larger than* M*, we must remove at least* ϵM *of the* $(k+1)$*-faces from* X.

With these statements in view, we can regard $h^k(X;R)$ (and $h_k(X;R)$) as a *quantitative* measure of cohomological (and homological) robustness.

3.3.1 Polyhedral Laplacians

In this section, we will define the common Laplacian on cellular complexes. See also [31].

Definition 3.3.3. If our chain complex C_*X is equipped with an inner product, we can canonically identify $C^kX \cong C_kX$.

We will obtain this inner product by specifying a volume functional, $\nu : X^{(k)} \to \mathbb{R}_+$ satisfying the following property (which is an analogue of the triangle inequality): For every chain $z \in C^kX$ with $\partial z = 0$ (i.e. a *cycle*), and every $x \in X^{(k)}$,

$$|z(x)|\nu(x) \leq \sum_{x' \neq x} |z(x')|\nu(x').$$

Usually, we will use the uniform volume functional, $\nu \equiv 1$.

From now on we will use the more suggestive notation,

$$\int \beta \, d\nu \triangleq \sum_{x \in X^{(k)}} \beta(x)\nu(x).$$

and define the inner product to be $\langle \beta, \alpha \rangle \triangleq \int \beta \cdot \alpha \, d\nu$.

The Laplace-Beltrami operator is defined to be

$$\Delta_k = \partial_k d_{k-1} + d_k \partial_{k+1} : C^kX \to C^kX.$$

We will suppress subscripts wherever unambiguous.

Notice that

$$\ker \partial \cap \ker d = \ker \Delta \tag{3.1}$$

which follows from

$$\langle \Delta\beta, \beta \rangle = \langle \partial\beta, \partial\beta \rangle + \langle d\beta, d\beta \rangle \geq 0.$$

(this also shows that all of the eigenvalues of Δ are real and positive)

Also, the images of d and ∂ are orthogonal because,

$$\langle \partial\beta, d\alpha \rangle = \langle \partial\partial\beta, \alpha \rangle = 0.$$

Thus we have an orthogonal decomposition of $C^k X$ into Δ-invariant spaces:

$$C^k X = \mathrm{im} d \oplus \mathrm{im} \partial \oplus \ker \Delta \tag{3.2}$$

This is the discrete Hodge decomposition theorem. (3.1) and (3.2) immediately imply the following:

Lemma 3.3.4. $\mathrm{Ker}\Delta_k \cong H^k(X)$

By the Hodge decomposition theorem the positive spectrum can be partitioned into two sets: the spectrum of each Δ-invariant space associated to $d\partial$ and ∂d. Indeed, suppose ω is any eigenfunction of $d\partial$ with non-zero eigenvalue, then $\partial\omega \in C^{k-1}X$ is non-zero and

$$d\partial\omega = \lambda\omega \quad \Longrightarrow \quad \Delta_{k-1}\partial\omega = \partial d\partial\omega = \partial(\lambda\omega) = \lambda\partial\omega,$$

so $\partial\omega$ is an eigenfunction of Δ_{k-1} with eigenvalue λ. Therefore, $d\partial$ carries the same spectral information as ∂d in the next lower grading. As a result, we can restrict our attention to the spectrum of $\partial d : \ker \partial \to \ker \partial$.

We will use π_d, π_∂ to denote the orthogonal projections of $C^k X$ onto $\ker \partial$ and $\ker d$ respectively. The *spectral gap* can be characterized in terms of the Rayleigh quotient:

$$\lambda(X) \triangleq \inf_{\beta \neq 0} \frac{\|d\beta\|^2 + \|\partial\beta\|^2}{\|\beta\|^2}$$

along with,

$$\lambda_d(X) \triangleq \inf_{\beta} \frac{\|d\beta\|^2}{\inf_\alpha \|\beta + d\alpha\|^2} = \inf_{\beta} \frac{\|d\beta\|^2}{\|\pi_d\beta\|^2}$$

and

$$\lambda_\partial(X) \triangleq \inf_{\beta} \frac{\|\partial\beta\|^2}{\inf_\gamma \|\beta + \partial\gamma\|^2} = \inf_{\beta} \frac{\|\partial\beta\|^2}{\|\pi_\partial\beta\|^2}$$

Note that by lemma 3.3.4, $\lambda(X) = \min\{\lambda_+(X), \lambda_-(X)\} = 0$ if and only if $H^k(X) \neq 0$.

Furthermore, it is worth observing that

$$\|\pi_d\beta\| \leq \frac{\|\partial d\beta\|}{\lambda_d} \qquad \text{and} \qquad \|\pi_\partial\beta\| \leq \frac{\|d\partial\beta\|}{\lambda_\partial}.$$

Chapter 4

Topology of random subcomplexes

This chapter is devoted to exposing the relationship between the isoperimetric problem and random sparsifications of complexes. We will begin with the exceedingly important example, due to Linial and Meshulam [39], and then extend our view to other models. A great deal of this chapter is adapted from the paper [17].

4.1 Linial-Meshulam complexes

The random complexes of Linial and Meshulam will play a large role in this chapter and the rest of this dissertation. We will begin by constructing them and surveying some results about them. The place to start, as with much of our work, is with the 1-dimensional example.

4.1.1 Erdős-Rényi random graphs

We will begin with an important motivating example: one of the most well-studied models of random graphs [9].

Definition 4.1.1. The *Erdős–Rényi random graph* $G(n,p)$ is the probability space of all graphs on vertex set $[n] = \{1, 2, \ldots, n\}$ with each edge having probability p, independently. In other words for every graph G on vertex set $[n]$,

$$\mathbb{P}[G \in G(n,p)] = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)},$$

where $e(G)$ denote the number of edges of G.

We will say that a sequence of probability spaces $\{\Omega_n\}$ has property $\mathcal{P}$ *asymptotically almost surely* (a.a.s.) if the probability of $\mathcal{P}$ tends to 1 as n tends to infinity.

The following theorem is a seminal result in the theory of random graphs and sharp phase transtions [21].

Theorem 4.1.2 (Erdős–Rényi)**.** *Let $\omega = \omega(n)$ be any function that tends to infinity with n. If*

$$p \geq \frac{\log n + \omega}{n}$$

then $G(n,p)$ is a.a.s. connected, and if

$$p \leq \frac{\log n - \omega}{n}$$

then $G(n,p)$ is a.a.s. disconnected.

Once p is much larger than $\log n/n$, $G(n,p)$ is connected [21] and it exhibits edge expansion. The following theorem is due to Benjamini, Haber, Krivelevich, and Lubetzky [8]. Let $D(G)$ denote the maximum degree of G.

Theorem 4.1.3. *Let $0 < \epsilon < 1/2$ be fixed. Then there exists a constant $C = C(\epsilon)$ such that if $p \geq C \log n/n$, then a.a.s. $G \in G(n,p)$ has Cheeger constant bounded by*

$$(1/2 - \epsilon)D(G) \leq h(G) \leq (1/2 + \epsilon)D(G).$$

The results in [8] are more precise than what we state here, since the entire random graph process is considered, and how C depends on ϵ is made much more explicit.

4.1.2 Random complexes

Linial and Meshulam defined 2-dimensional analogues of $G(n,p)$ and proved a cohomological analogue of Theorem 4.1.2 [39], and Meshulam and Wallach extended the result to d-dimensional random complexes and arbitrary fixed finite coefficients [42].

Let Δ_n denote the $(n-1)$-dimensional simplex and $\Delta_n^{(i)}$ its i-skeleton.

Definition 4.1.4. The *Linial–Meshulam complex*

The random simplicial complex $Y_k(n,p)$ is the probability space of all simplicial complexes with complete k-skeleton and each $(k+1)$-dimensional face appearing independently with probability p. In other words for every

$$\Delta_n^{(k)} \subseteq Y \subseteq \Delta_n^{(k+1)},$$

we have

$$\mathbb{P}(Y) = p^{|Y^{(k+1)}|}(1-p)^{\binom{n}{k+2}-|Y^{(k+1)}|},$$

In particular, $Y_0(n,p)$ is equivalent to $G(n,p)$.

The main result of [39] (for $k=1$) and [42] (for $k \geq 2$) is the following.

Theorem 4.1.5 (Linial–Meshulam, Meshulam–Wallach)**.** *Let $\omega = \omega(n)$ be any function that tends to infinity with n. If $p = \frac{(k+1)\log n + \omega}{n}$ and $Y \in Y(n,p)$ then a.a.s. $H^k(Y, \mathbb{Z}_2) = 0$ and if $p = \frac{(k+1)\log n - \omega}{n}$ then a.a.s. $H^k(Y, \mathbb{Z}_2) \neq 0$.*

In the next two sections, we refine the statement that $H^k(Y) = 0$ to a more quantitative geometric statement by estimating $h^k(Y)$. At the same time we expand the statement to more general random complexes.

We now define *random p-subcomplexes*, which include all the previous examples as special cases.

Definition 4.1.6. Let $\{X_n\}$ be a family of finite regular CW complexes. Then let $X_n^{(k)} \subset Y_{n,p} \subset X_n^{(k+1)}$ be the probability space of random subcomplexs of X_n, so that each $(k+1)$-cell of X_n is included in $Y_{n,p}$ jointly independently with probability p. We refer to $Y \in Y_{n,p}$ as a *random p-subcomplex of X_n*.

4.2 Transitions for other models

We now show that under certain conditions, random p-subcomplexes of polyhedra inherit coboundary expansion from the ambient complex. In particular, this provides a threshold beyond which random subcomplexes have vanishing cohomology with high probability.

Theorem 4.2.1. *Let $\{X_n\}$ be a sequence of finite $(k+1)$-dimensional polyhedral complexes, such that:*

$$\frac{\log |X_n^{(k)}|}{h^k(X_n)} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Furthermore $\epsilon > 0$ and let $\omega = \omega(n)$ be any function such that $\lim_{n\to\infty} \omega = \infty$. Let $X_n^{(k)} \subset Y_{n,p} X_n^{(k+1)}$ be a random p-subcomplex of X_n. If

$$p \geq \frac{2\log |X_n^{(k)}| + \omega}{\epsilon^2 h^k(X_n)}$$

then

$$h^k(Y_{n,p}) \geq (1-\epsilon)p \cdot h^k(X_n) \qquad a.a.s.$$

In Appendix B, we will give several examples of families of complexes X_n which satisfy the hypotheses of the theorem.

Proof. Our argument is essentially a coarser version of the proof of Theorem 4.1.5 in [42], but our emphasis here is on the geometric expansion property h_k rather than the topological property of vanishing cohomology.

Let $\beta \in C^k X_n$ be a k-cochain in X_n (and hence also in $Y_{n,p}$ since Y contains the entire k-skeleton of X_n). We let $\|d\beta\|_Y$ denote the norm of $d\beta \in C^{k+1}Y$, as distinguished from simply $\|d\beta\|$ which denotes in norm in the full complex X_n. Since each $(k+1)$-cell is included in Y independently with probability p, by the Chernoff-Hoeffding bounds [18],

$$\mathbb{P}\big[\|d\beta\|_Y \leq (1-\epsilon)p\|d\beta\|\big] \leq e^{-\frac{\epsilon^2}{2}p\cdot\|d\beta\|}$$

Let $n = \|d\beta\|$ and $m = \|[\beta]\| = \min_\alpha\|\beta + d\alpha\|$, so that $n \geq h^k(X_n)\cdot m$. We must now check that the inequality

$$\|d\beta\|_Y \geq (1-\epsilon)p\cdot h^k(X_n)\|[\beta]\|$$

holds for every $\beta \in C^kY$. We note that it suffices to check this inequality on all β for which $\|\beta\| = \|[\beta]\|$.

Here we apply a union bound. We can upper bound the number minimizing β of m by counting *all* k-cochains of norm m.

$$\mathbb{P}\big[\exists\beta,\ ||d\beta||_Y \leq (1-\epsilon)pn\big] \leq \sum_{m\geq 1}\binom{|X_n^{(k)}|}{m}e^{-\frac{\epsilon^2 pm\cdot h^k(X_n)}{2}}$$

$$= \left[1 + e^{-\frac{\epsilon^2 p\cdot h^k(X_n)}{2}}\right]^{|X_n^{(k)}|} - 1$$

In order to show that this quantity goes to 0, we need to show that the leftmost quantity goes to 1, or simply that

$$|X_n^{(k)}|\log\big[1 + e^{-\frac{\epsilon^2 p\cdot h^k(X_n)}{2}}\big] \longrightarrow 0$$

Now,

$$\log\big[1 + e^{-\frac{\epsilon^2 p\cdot h^k(X_n)}{2}}\big] \leq e^{-\frac{\epsilon^2 p\cdot h^k(X_n)}{2}},$$

and finally, since

$$p \geq \frac{2\log|X^{(k)}| + \omega}{\epsilon^2 h^k(X_n)},$$

we have,

$$|X_n^{(k)}|e^{-\frac{\epsilon^2 p\cdot h^k(X_n)}{2}} \leq e^{-\omega} \longrightarrow 0.$$

□

4.3 A Diaconis-Stroock inequality for polyhedral complexes

In this section, we will prove a direct analogue of the Diaconis-Stroock inequality [14] [13] for polyhedral complexes. The key point is that such an inequality should be thought of as a comparison between a subcomplex and an ambient complex. In the original setting, the inequality compares an arbitrary finite reversible Markov chain to the Markov chain which mixes in a single time step.

We will recall the original (non-normalized) inequality for graphs.

Proposition 4.3.1. *[14]*

Let G be a connected graph on n vertices and $\lambda_1(G)$ be the first positive eigenvalue of the (non-normalized) graph Laplacian. For each pair $x, y \in G$, choose a path γ_{xy} connecting them and denote $\Gamma \triangleq \{\gamma_{xy}\}_{x,y\in G}$ the collection of such paths. Define

$$\xi = \xi(\Gamma) \triangleq \max_e \sum_{\gamma \ni e} L(\gamma)$$

where the maximum is taken over all edges in G and $L(\gamma)$ denotes the length of the path γ. Then

$$\lambda_1(G) \geq \frac{n}{\xi(\Gamma)}$$

This proposition is stated (both here and in [14]) in a way which does not readily indicate a higher-dimensional generalization to polyhedral complexes. For example, what is the higher dimensional analogue of a pair of vertices?

Definition 4.3.2. Let $X^{(k)} \subset Y \subset X^{(k+1)}$ be a subcomplex of the polyhedral complex X. For each $z \in \partial X^{(k+1)}$, choose $\gamma_z \in C^{k+1}Y$ such that $\partial\gamma_z = z$. The collection $\Gamma \triangleq \{\gamma_z\}_{z\in\partial X^{(k+1)}}$ will be called an L^2-*compensator*.

The *load* of Γ is defined as:

$$\xi = \xi(\Gamma) \triangleq \sup_{y \in Y^{(k+1)}} \sum_{\mathrm{supp}(\gamma_z) \ni y} \|\tau_z\|^2$$

Proposition 4.3.3. *polyhedral Diaconis-Stroock inequality ($\mathbb{R}$-coefficients)*

Let $X^{(k)} \subset Y \subset X^{(k+1)}$ as above. For any L^2-compensator, Γ, we have

$$\lambda(Y) \geq \frac{\lambda(X)}{\xi(\Gamma)}$$

Just as in the original setting, the proof of the inequality is elementary.

Proof. Let $\beta \in \Lambda^k X$

$$\begin{aligned} \|\pi_d \beta\|^2 &\leq \tfrac{1}{\lambda(X)} \|d\beta\|_X^2 \\ &= \tfrac{1}{\lambda(X)} \sum_{x \in X^{(k+1)}} \langle d\beta, x\rangle^2 \\ &= \tfrac{1}{\lambda(X)} \sum_{z \in \partial X^{(k+1)}} \langle \beta, z\rangle^2 \\ &= \tfrac{1}{\lambda(X)} \sum_{z \in \partial X^{(k+1)}} \langle d\beta, \gamma_z\rangle^2 \end{aligned}$$

Applying the Cauchy-Schwartz inequality,

$$\|\pi_d \beta\|^2 \leq \frac{1}{\lambda(X)} \sum_{z \in \partial X^{(k+1)}} \sum_{y \in Y^{(k+1)}} d\beta(y)^2 \chi_{\mathrm{supp}(\gamma_z)}(y) \|\gamma_z\|^2$$

(where $\chi_{\mathrm{supp}(\gamma_z)}$ is the indicator function of the support of γ_z)

$$\begin{aligned} &= \tfrac{1}{\lambda(X)} \sum_{y \in Y^{(k+1)}} d\beta(y)^2 \sum_{z \in \partial X^{(k+1)}} \chi_{\mathrm{supp}(\tau_z)}(y) \|\tau_z\|^2 \\ &\leq \tfrac{\xi}{\lambda(X)} \sum_{y \in Y^{(k+1)}} d\beta(y)^2 = \tfrac{\xi}{\lambda(X)} \|d\beta\|_Y^2. \end{aligned}$$

By the variational characterization we have,

$$\lambda(Y) \geq \frac{\lambda(X)}{\xi}$$

∎

We also have a version for $\mathbb{Z}_2$-coefficients.

Proposition 4.3.4. *For $X^{(k)} \subset Y \subset X^{(k+1)}$, and a $\mathbb{Z}_2$-compensator Γ we have*

$$h(Y) \geq \frac{h(X)}{\zeta(\Gamma)}$$

Proof.

$$\begin{aligned}
\min_\alpha \|\beta + d\alpha\| &\leq \tfrac{1}{h(X)}\|d\beta\| \\
&= \tfrac{1}{h(X)} \sum_{x \in X^{(k+1)}} \langle d\beta, x\rangle \\
&= \tfrac{1}{h(X)} \sum_{\Gamma} \langle d\beta, \gamma_z\rangle \\
&\leq \tfrac{1}{h(X)} \Big| \{\gamma_z \,:\, \mathrm{supp} d\beta \cap \mathrm{supp}\gamma_z \neq \emptyset\} \Big| \\
&\leq \tfrac{1}{h(X)} \sum_{y \in \mathrm{supp} d\beta} \Big| \{\gamma_z \,:\, y \in \mathrm{supp}\gamma_z\} \Big| \\
&\leq \tfrac{\|d\beta\|}{h(X)} \max_{y \in Y^{(k+1)}} \Big| \{\gamma_z \,:\, y \in \mathrm{supp}\gamma_z\} \Big| \\
&= \tfrac{\zeta(\Gamma)}{h(X)} \cdot \|d\beta\|
\end{aligned}$$

∎

Chapter 5

Filling distortion of complexes embedded in space

One of the first questions to ask when we come across a new geometric structure is to ask, "how does it compare to Euclidean space?" This is not only because we all live in Euclidean space, but also because being a subset of Euclidean space is one of the most stringent geometric conditions [20], [44], [45].

However, discrete geometric structures, by virtue of their finiteness, can only be so exotic. For example,

Theorem (J. Bourgain, [11]). *Any finite metric space, (X, d), admits an embedding, $\phi : X \hookrightarrow \mathbb{R}^{O(\log |X|)}$ with*

$$\max_{x,y\in X} \frac{\|\phi(x) - \phi(y)\|}{d(x,y)} \cdot \max_{x,y\in X} \frac{d(x,y)}{\|\phi(x) - \phi(y)\|} \leq O(\log |X|).$$

(For the convenience of the reader, this theorem is proven with attention to detail in Appendix D)

The quotient on the left side of the inequality is referred to as the *metric distortion* of ϕ. In this same article, [11], Bourgain used random graphs as examples of spaces that have the property that every embedding, ϕ, has large metric distortion.

By now, the study of metric embeddings into Euclidean space and other Banach spaces is very well developed, particularly in theoretical computer science [38] [40]. We will not attempt to add to this lively conversation, but instead we are curious about higher order phenomena.

For example, if X was a geodesic space (such as the geometric realization of a graph, or a Riemannian manifold if you prefer) we could rewrite the definition of metric distortion in a suggestive way. If $\phi : X \to \mathbb{R}^n$, the metric distortion is,

$$\delta_0(\phi) = \sup_{z \in Z_0 X} \frac{\mathrm{Fill}_{\mathcal{H}}\, \phi_* z}{\mathrm{Fill}_X\, z} \cdot \sup_{z \in Z_0 X} \frac{\mathrm{Fill}_X\, z}{\mathrm{Fill}_{\mathcal{H}}\, \phi_* z}$$

The 0-cycles here are taken with $\mathbb{Z}_2$ coefficients and the associated norm. We will use this observation to motivate and suggest a definition for higher order distortion. However, we will begin the chapter by revisiting the role of spectral/expansion properties in Euclidean embeddings.

5.1 Metric distortion of expander graphs

In this section, using spaces satisfying significantly stronger isoperimetric inequalities, we show that Bourgain's theorem is sharp.

We will use k-regular graphs, G_n, whose spectral gap of the graph Laplacian is denoted by λ (see Chapter 3, $\lambda = h^0(G_n, \mathbb{R})$).

Let us consider the graph path metric on each of these graphs. Choose any embedding, $\phi : G_n \to R^N$.

Since the metric distortion is scale and translation invariant, we can scale ϕ so that $\mathrm{Lip}(\phi^{-1}) = 1$ (and it is a metric expanding map) and we can translate it so that its center of mass is at the origin.

Then, as a vector valued function,

$$\lambda||\phi||_2^2 \leq ||d\phi||_2^2 \leq \frac{k|G_n|}{2}\mathrm{Lip}(\phi)^2$$

Now, since ϕ is an expanding map and G_n is a degree k graph, there can be no more than $O(k^R)$ vertices inside a ball of radius R. That implies that at least half of the vertices of G_n must lay outside a ball of radius $R = O(\log|G_n|)$. Thus,

$$||\phi||_2^2 = \Omega(|G_n| \log^2 |G_n|)$$

Putting these two inequalities together, we have

$$\mathrm{Lip}(\phi) = \Omega(\log|G_n|)$$

.

Therefore, expander families require the maximal metric distortion (regardless of the dimension of the Euclidean space).

5.2 Filling distortion

The rest of this chapter will outline work that can be found in the preprint [16].

We will seek to show that for every $\epsilon > 0$, there exists a 2-dimensional simplicial complexes on n vertices and complete 1-skeleton such that any affine embedding $\psi : X \to \mathcal{H}$ into Euclidean space must have:

$$\max_z \frac{\mathrm{Fill}_{\mathcal{H}}(\psi z)}{\mathrm{Fill}_X(z)} \cdot \max_z \frac{\mathrm{Fill}_X(z)}{\mathrm{Fill}_{\mathcal{H}}(\psi z)} \geq Cn^{\frac{1-2\epsilon}{4}}.$$

where both maximums run over simplicial 1-cycles, z, in X.

The reader may interpret this statement as saying that the induced map $Z_1X \to Z_1\mathcal{H}$ which maps cellular 1-cycles in X to 1-cycles in $\mathcal{H}$ has large metric distortion (with respect to the flat metric).

The section will be devoted to proving our main proposition:

Proposition 5.2.1. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a 2-dimensional simplicial complex with complete 1-skeleton. Let $\phi : X \to \mathcal{H}$ be an affine embedding of X into an infinite dimensional Hilbert space, suitably scaled so that*

$$\text{Fill}_{\mathcal{H}}(\psi\tau) \geq \text{Fill}_X(\tau)$$

for every triangle, τ, then,

$$\sum_{f \in X^{(2)}} \big(\text{Area}(\phi f)\big)^2 \geq \frac{\lambda^1(X)}{3(n-2)} \sum_{\tau} \big(\text{Fill}_X \tau\big)^2.$$

where the first sum runs over 2-dimensional faces in X and the second sum runs over all triangles in X.

Proof. We can assume that the image of the 0-skeleton, $X^{(0)}$ forms a linearly independent set in $\mathcal{H}$ (since a small perturbation of the vertices does not change the areas of triangles very much).

Choose orthonormal coordinates, $x_1, \ldots, x_n$ for span$X^{(0)} \cong \mathbb{R}^n$. We will let ϕ induce a function $\psi : X^{(1)} \to \mathbb{R}^{\binom{n}{2}}$ defined by

$$\psi_{(i<j)}(e) = \frac{1}{2} \int_{\phi(e)} x_i dx_j - x_j dx_i + \sum_{m=0}^{n} \int_{\phi(e)} y_m^{(i<j)} dx_m \quad \text{for each } e \in X^{(1)}$$

(with the $y_m^{(i<j)}$ as fixed constants to be chosen later).

Now we have $d\psi : X^{(2)} \to \mathbb{R}^{\binom{n}{2}}$, and by the Stokes theorem,

$$(d\psi(f))_{(i<j)} = \int_{\phi f} dx_i \wedge dx_j.$$

Now it is easily seen that $|d\psi(f)|^2 = \big(\text{Area}(\phi f)\big)^2$. This is because the area form of ϕf can we written as

$$\omega_{\phi f} = \sum_{i<j} a_{(i<j)} dx_i \wedge dx_j \ \text{ where } \sum_{i<j} a^2_{(i<j)} = 1 \ \text{ and } \int_{\phi f} dx_i \wedge dx_j = a_{(i<j)} \text{Area}(\phi f).$$

Thus,

$$\|d\psi\|^2 = \sum_{f \in X^{(2)}} \big(\text{Area}(\phi f)\big)^2.$$

We will need to prove a small claim:

Claim. Consider the function, $\xi : X^{(1)} \to \mathbb{R}^{\binom{n}{2}}$ given by

$$(\xi(e))_{(i<j)} = \frac{1}{2}\int_{\phi(e)} x_i dx_j - y_j dx_i.$$

Then for any $\alpha : X^{(0)} \to \mathbb{R}^{\binom{n}{2}}$, we can choose $y_m^{(i<j)}$ so that $\psi = \xi + d\alpha$.

Proof. If $e = [v, w] \in X^{(1)}$,

$$\sum_m \int_{\phi(e)} y_m^{(i<j)} dx_m = \langle y^{(i<j)}, \phi(v)\rangle - \langle y^{(i<j)}, \phi(w)\rangle$$

Since $\phi(X^{(0)})$ is linearly independent, for every function $f : X^{(0)} \to \mathbb{R}$, there exists a corresponding $y \in \mathbb{R}^n$ such that

$$f(x) \equiv \langle y, \phi(x)\rangle \quad \text{for every } x \in X^{(0)}.$$

Therefore, for every function, $f : X^{(0)} \to \mathbb{R}^{\binom{n}{2}}$ we can choose $\binom{n}{2}$ such $y \in \mathbb{R}^n$ (denoted $y^{(i<j)}$) such that

$$(f_{(1<2)}(x), \ldots, f_{(n-1<n)}(x)) \equiv (\langle y^{(1<2)}, \phi(x)\rangle, \ldots, \langle y^{(n-1<n)}, \phi(x)\rangle \text{ for every } x \in X^{(0)}.$$

□

Since we can choose $(y_1, \ldots, y_n)$ so that $\partial\psi = 0$, we have the inequality:

$$\sum_{f \in X^{(2)}} \big(\text{Area}(\phi f)\big)^2 = \|d\psi\|^2 \geq \lambda^1(X)\|\psi\|^2.$$

Now we have only to prove that

$$\|\psi\|^2 \geq \frac{1}{3(n-2)} \sum_{\tau} \big(\text{Fill}_{\text{X}}(\tau)\big)^2.$$

We observe that for a triangle τ formed by the edges e_1, e_2, and e_3, we have (by the Stokes theorem again),

$$\Big[\psi(e_1) + \psi(e_2) + \psi(e_3)\Big]_{(i<j)} = \int_{\phi\tau} dx_i \wedge dx_j$$

So that,

$$\sum_{(i<j)} \Big[\psi(e_1) + \psi(e_2) + \psi(e_3)\Big]^2_{(i<j)} = \mathrm{Fill}_{\mathcal{H}}(\phi\tau) \geq \mathrm{Fill}_X(\tau)$$

and by Cauchy-Schwartz:

$$|\psi(e_1)|^2 + |\psi(e_2)|^2 + |\psi(e_3)|^2 \geq \frac{1}{3}\big(|\psi(e_1)| + |\psi(e_2)| + |\psi(e_3)|\big)^2.$$

Summing over all triangles, each edge is contained in $n-2$ triangles, we have

$$(n-2)\|\psi\|^2 \geq \frac{1}{3}\sum_{\tau} \big(\mathrm{Fill}_X(\tau)\big)^2.$$

□

In light of this proposition, it should be clear to the reader that we seek to find 2-dimensional complexes which maximize the quantity,

$$\frac{\lambda^1(X)}{|X^{(2)}|}\sum_{\tau} \big(\mathrm{Fill}_X(\tau)\big)^2.$$

As we increase the number of 2-faces, $\lambda^1(X)$ will go up, but $\frac{\sum(\mathrm{Fill}_X(\tau))^2}{|X^{(2)}|}$ will go down.

It is not clear to the author how to build exact optimizers for this quantity, so in the next section we will resort to using random complexes a la Linial and Meshulam [39].

5.2.1 Filling estimates for random complexes

Since we have given ourselves the liberty to take estimates up to a constant, we will exhibit a somewhat cavalier indifference to preserving sharp quantities. We have decided to err on the side of keeping things simple and believe the reader will forgive us for this minor transgression.

We will rely on the geometry of random complexes.

Proposition 5.2.2. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a p-random complex. There is a constant, C, so that if $p \geq \frac{C\log n}{n}$, then, with probability tending to 1 as $n \to \infty$,*

$$\lambda^1(X) \geq \frac{1}{3}pn.$$

Proof. The proof of this proposition is a simple consequence of theorem 2 in [29]:

Theorem (Gundert and Wagner, [29])**.** *Let $\hat{\lambda}^1(X)$ denote the spectral gap of normalized Laplacian on 1-forms. For all $c > 0$, there exists a constant K such that if $p \geq \frac{K \log n}{n}$ and $X = X_{n,p}$ is a random 2-complex (with complete 1-skeleton) then*

$$\hat{\lambda}^1(X;\mathbb{R}) \geq 1 - \frac{K}{\sqrt{pn}}$$

with probability greater than $1 - n^{-c}$.

In order to prove proposition 5.2.2, we simply need to prove the following claim:

Claim. With probability tending to 1 as n tends to infinity, the degree of each edge is greater than $\frac{p(n-2)}{2}$.

Proof. Our argument is a standard one.

The expected degree of each edge is $p(n-2)$. By a form of Chernoff's inequality [**?**], each edge, e, has

$$\mathbb{P}[\deg(e) < (1-\epsilon)p(n-2)] \leq e^{-\frac{\epsilon^2 p(n-2)}{2}}.$$

Taking $\epsilon = \frac{1}{2}$, and taking a union bound:

$$\sum_e \mathbb{P}\Big[\deg(e) < \frac{p(n-2)}{2}\Big] \leq e^{\frac{p(n-2)}{8}} \leq \binom{n}{2} \cdot n^{-\frac{K}{8}} \to 0 \qquad \text{by taking } K > 16.$$

□

Now applying this to the theorem of Gundert and Wagner (since the normalized Laplacian is obtained simply by dividing the differential of an edge by its degree), we have

$$\lambda(X) \geq \frac{p(n-2)}{2} - K\sqrt{pn} \geq \frac{pn}{3} \qquad \text{for large } n.$$

□

Now we are left to find an lower bound on $\sum_\tau \left(\mathrm{Fill}_X(\tau)\right)^2$.

Proposition 5.2.3. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a p-random complex with $p = n^{\epsilon-1}$, then, with probability tending to* 1 *as $n \to \infty$,*

$$\sum_{\tau} \left(\mathrm{Fill}_X(\tau)\right)^2 \geq C n^{4-2\epsilon}$$

Proof. First, shall examine a single triangle, τ, and bound the probability that $\mathrm{Fill}(\tau) < n^{\alpha}$. To achieve this, we appeal to an estimate made in [4] (later revised to [3]), but attributed as an observation of Eran Nevo, that a k-cycle in $z \in Z_k\Delta_n$ which does not contain any smaller cycles as a subset and which is supported on $f_0(z)$ vertices and $f_d(z)$ faces of dimension d must have

$$f_0 \leq \frac{f_d + (d+2)(d-1)}{d}.$$

Now, a minimal filling (i.e. does not contain a smaller filling as a subset) of τ can be obtained from a minimal cycle, z, which contains the face which τ bounds. Thus, the number of fillings of τ of size m in Δ_n can be bounded by,

$$\begin{aligned} \binom{n}{f_0-d-1}\binom{\binom{f_0}{d+1}}{m} &\leq n^{f_0-d-1}\left(\frac{ef_0^{d+1}}{m}\right)^m \\ &\leq n^{\frac{m+(d+2)(d-1)-d^2-d}{d}}(Cm^d)^m \\ &= n^{-\frac{2}{d}}(Cn^{\frac{1}{d}}m^d)^m \end{aligned}$$

Therefore, setting $d = 2$, we have,

$$\mathbb{P}[\exists y,\, \partial y = \tau,\, \|y\| < n^{\alpha}] \leq n^{-1}\sum_{m\geq 3}^{n^{\alpha}}(Cpn^{\frac{1}{2}}m^2)^m \leq n^{-1}\sum_{m\geq 3}(en^{2\alpha+\epsilon-\frac{1}{2}})^m$$

So that,

$$\mathbb{P}[\exists y,\, \partial y = \tau,\, \|y\| < n^{\alpha}] \;\leq\; Cn^{3(2\alpha+\epsilon-\frac{1}{2})-1}\frac{n^{(n^{\alpha}-2)(2\alpha+\epsilon-\frac{1}{2})}-1}{n^{2\alpha+\epsilon-\frac{1}{2}}-1}$$

Therefore, if we set $2\alpha + \epsilon - \frac{1}{2} < 0$, then we have

$$\mathcal{P}[\mathrm{Fill}_X(\tau) < n^{\alpha}] \to 0$$

and

$$\mathbb{E}[\mathrm{Fill}_X\tau] \geq cn^{\frac{1-2\epsilon}{4}} \qquad \text{for large enough } n.$$

Now we will bound the quantity

$$\mathbb{E}_Y\left[\sum_\tau \min\{\mathrm{Fill}_X(\tau), n^{\frac{1-2\epsilon}{4}}\}\right].$$

On the one hand,

$$\mathbb{E}_Y\left[\sum_\tau \min\{\mathrm{Fill}_X(\tau), n^{\frac{1-2\epsilon}{4}}\}\right] \geq \binom{n}{3} n^{\frac{1-2\epsilon}{4}} \mathbb{P}[\mathrm{Fill}_X(\tau) \geq n^{\frac{1-2\epsilon}{4}}] \geq \frac{99}{100}\binom{n}{3} n^{\frac{1-2\epsilon}{4}}.$$

On the other hand, if we let

$$H = \left\{ Y \in Y_{n,p} : \text{ at least } \frac{1}{100}\binom{n}{3} \text{ triangles have } \mathrm{Fill}_X \tau \geq \frac{n^{\frac{1-2\epsilon}{4}}}{99} \right\}$$

(Notice that H implies the proposition),

then,

$$\binom{n}{3} n^{\frac{1-2\epsilon}{4}} \mathbb{P}[H] + \left[\frac{99}{100}\binom{n}{3}\frac{n^{\frac{1-2\epsilon}{4}}}{99} + \frac{1}{100}\binom{n}{3} n^{\frac{1-2\epsilon}{4}}\right]\mathbb{P}[H^C] \geq \mathbb{E}_Y\left[\sum_\tau \min\{\mathrm{Fill}_X(\tau), n^{\frac{1-2\epsilon}{4}}\}\right]$$

So therefore, we have:

$$\mathbb{P}[H] + \frac{1}{50}(1 - \mathbb{P}[H]) \geq \frac{99}{100} \Rightarrow \mathbb{P}[H] \geq \frac{97}{98}.$$

□

Corollary 5.2.4. *Let $\Delta_n^{(1)} \subset X \subset \Delta_n^{(2)}$ be a p-random complex with $p = n^{\epsilon-1}$. Then with probability tending to 1 as $n \to \infty$, every affine embedding $\phi : X \to \mathcal{H}$ of X into an infinite dimensional Hilbert space, $\mathcal{H}$ must have,*

$$\max_{z \in Z_1 X} \frac{\mathrm{Fill}_{\mathcal{H}}(\phi_* z)}{\mathrm{Fill}_X(z)} \cdot \max_z \frac{\mathrm{Fill}_X(z)}{\mathrm{Fill}_{\mathcal{H}}(\phi_* z)} \geq C n^{\frac{1-2\epsilon}{4}}$$

Proof. Taking an affine map, $\phi : X \to \mathcal{H}$ and scaling it so that $\mathrm{Fill}_{\mathcal{H}}(\phi_*\tau) \geq \mathrm{Fill}_X(\tau)$ (again, the filling distortion is continuous with respect to small perturbations, so we may always perturb ϕ slightly and then scale it as prescribed). Then there is a 2-face of X such that:

$$\left(\mathrm{Area}(\phi f)\right)^2 \geq \frac{\lambda^1(X)}{3(n-2)|X^{(2)}|}\sum_\tau \left(\mathrm{Fill}_X \tau\right)^2 \geq c\frac{1}{n^3}\sum_\tau \left(\mathrm{Fill}_X \tau\right)^2 \geq c n^{\frac{1-2\epsilon}{2}}$$

Therefore,

$$\delta_1(\phi) \geq c n^{\frac{1-2\epsilon}{4}}.$$

□

5.2.2 Filling distortion in higher dimensions

We have chosen to state all of the theorems and propositions in this article in terms of 2-dimensional complexes. We felt that writing all arguments in their generality was cumbersome and of little use to the reader. However, we would be doing the reader a disservice if we mentioned nothing about how the theorems and propositions generalize to higher dimensions. To this end, we have devoted the current section to a brief sketch of the propositions and proofs of the preceding sections along with annotations describing what minor changes must be made in higher dimensions.

Ultimately, theorem **??** generalizes to:

Theorem. *For every large n and every ϵ, there exists a $(k+1)$-dimensional simplicial complex on n vertices and complete 1-skeleton with the property that every affine map $\phi : X \to \mathcal{H}$ has,*

$$\delta_1(\phi) \geq Cn^{\frac{k-(k+1)\epsilon}{(k+1)^2}}$$

The three tools needed for the proof are,

1. a generalization of proposition 5.2.1,
2. an estimate on the spectral gap of the Laplacian acting on the k-forms of a random complex
3. and an estimate on the average filling volume of a k-cycle of volume $k+2$ in a random complex.

To obtain the objectives (2) and (3) we need only observe that, first, the theorem of Gundart and Wagner [29] is stated for all dimensions, and second, that to get started in the proof of proposition 5.2.3 we needed only that every d-dimensional cycle, z, which does not include another d-cycle as a proper subset must have:

$$f_0(z) \leq \frac{f_d(z) + (d+2)(d-1)}{d}.$$

Objective 1 requires a bit more careful consideration. However, the most important aspect of the proof of proposition 5.2.1 is the construction of a real cochain, ψ, with $\partial\psi = 0$.

In general, we will use the (vector valued) cochain, $\psi : X^{(k)} \to \mathbb{R}^{\binom{n}{k+1}}$ defined by

$$\begin{aligned}\psi_{(i_0<\cdots<i_k)}(\sigma) &= \tfrac{1}{2}\int_{\phi\sigma}\sum_j(-1)^j x_{i_j}dx_{i_0}\wedge\cdots\wedge d\hat{x}_{i_j}\wedge\cdots\wedge dx_{i_k}\\ &\quad+\int_{\phi\sigma}\sum_{(j_1<\cdots<j_k)} y^{(i_0<\cdots<i_k)}_{(j_1<\cdots<j_k)}dx_{j_1}\wedge\cdots\wedge dx_{j_k}.\end{aligned}$$

(in light of this cumbersome formula, it may occur to the reader now why we decided to omit the general case). This cochain has the benefit of

1. $d\psi : X^{(k+1)} \to \mathbb{R}^{\binom{n}{k+1}}$ is given by

$$d\psi_{(i_0<\cdots<i_k)}(\sigma) = \int_{\phi(\sigma)} dx_{i_0}\wedge\cdots\wedge dx_{i_k}.$$

2. The constants $y^{(i_0<\cdots<i_k)}_{(j_1<\cdots<j_k)}$ can be chosen, via a general position argument just as in the proof of proposition 5.2.1 to ensure that $\partial\psi = 0$.

Chapter 6

The filling problem in the cube

This chapter is devoted to proving the following theorem,

Theorem 6.0.5. *There exists a constant c_k, which depends only on k, such that for every k-dimensional cellular $\mathbb{Z}_2$-cycle, $z \in Z_kQ_n$, there exists a chain $y \in C_{k+1}Q_n$, such that $\partial y = z$ and*

$$\|y\| \leq c_k \|z\|^{\frac{k+1}{k}}.$$

We will also show that the theorem is sharp up to a constant:

Theorem 6.0.6. *For each k and each n, there exists a k-cycle, $z_n^k \in Q_n$ such that*

$$\mathrm{Fill}(z_n^k) \geq \omega_k \|z\|^{\frac{k+1}{k}}.$$

Much of the work in this chapter can be found in the preprint [15].

6.1 A filling algorithm for the cube

We will proceed with the proof of the main theorem of this chapter:

Proof of theorem 6.0.5. This proof is philosophically influenced by the proofs of the filling inequality given by M. Gromov ([27], [30]) and S. Wenger ([53]). The proof is by induction on k and n.

6.1.1 The linear inequality

It may surprise the reader that we will need an isoperimetric inequality with the *wrong* exponent in order to achieve the *right* one. An inequality with a linear exponent will provide a stepping stone for the rest of the proof.

Lemma 6.1.1. *For each k-cycle, $z \in Z_k Q_n$, we have,*

$$\mathrm{Fill}_Q(z) \leq \frac{n-k}{2(k+1)} \|z\|$$

Notice that the constant in front depends on n. This cannot be avoided; in fact, we will show that this inequality is sharp in some cases.

Let us outline some notation that we will use throughout. There are n ways to write the cube $Q_n = Q_{n-1} \times I$, where I is a cellular interval (i.e. an edge with two vertices). One can think of this as fixing the i-th the coordinates of the cube. Since this separates two subcubes of dimension $n-1$, we will call this a *cut* and denote it by H. There are $2n$ ways to cut the cube if we consider orientation.

Now for a chosen cut, H, the set of k-faces of Q_n fall into three types:

1. the set of faces which lie in the first Q_{n-1} (i.e. the set of k-faces whose i-th coordinate is constant 0)

2. the set of faces which lie in the second Q_{n-1} (i.e. the set of k-faces whose i-th coordinate is constant 1)

3. and the set of faces which do not lie in either cube (i.e. have part of their boundary in the first Q_{n-1} and another part of their boundary in the other Q_{n-1}).

So a cycle, z, can be written as z_+ (the set of faces in z of the first type), z_- (faces of the second type), and z_0 (faces of the third type). Since z_+ lies entirely in an $(n-1)$-dimensional subcube, we can think of each of these in the following way:

$$z_+,\, z_- \in C_k Q_{n-1} \qquad z_0 \in Z_{k-1} Q_{n-1} \qquad \text{and } \partial z_+ = z_0 = \partial z_-.$$

Now we will prove the linear inequality.

Proof. We will proceed by induction. The base case consists of filling a k-cycle, in Q_{k+1}. There are only two such cycles–the empty one and the one consisting of all k-faces ($2(k+1)$ of them). They have filling volumes 0 and 1 respectively.

Now assume that the proposition is true for all k-cycles in Q_{n-1}. Choose a cut H and construct a filling of z in the following way. Take all $(k+1)$-faces in Q_n of the third type which contain a face of z_- in their boundary; there are exactly $\|z_-\|$ such faces. Call this $(k+1)$-chain, y_0.

Now, adding z_+ to z_- obtains a k-cycle in Q_{n-1}. As a result, there must be a filling $y_+ \in C_{k+1}Q_{n-1}$ with $\partial y_+ = z_+ + z_-$ and

$$\|y_+\| \leq \frac{n-k-1}{2(k+1)}(\|z_+\| + \|z_-\|).$$

Thinking of y_+ as a collection of faces of the first type, we obtain a chain $y = y_+ + y_0$ such that $\partial y = z$.

Now we average over the oriented cut H:

$$\frac{1}{2n}\sum_H \|y\| \leq \frac{1}{2n}\sum_H \frac{n-k-1}{2(k+1)}\|z_+\| + \frac{n+k+1}{2(k+1)}\|z_-\|$$

Now, each k-face lies in exactly $n-k$ of the Q_{n-1}. So we have:

$$\frac{1}{2n}\sum \|y\| = \frac{n-k}{2n}\cdot\frac{2n}{2(k+1)}\|z\|.$$

So there must exist an oriented cut, H, for which the corresponding y satisfies the desired inequality.

□

With this tool, we will prove theorem 6.0.5 by induction.

6.1.2 Base Cases

.

For $n = k + 1$, the inequality is trivial because there is only one nontrivial k-cycle.

For $k = 1$ and n arbitrary, we observe the simple fact that if $\|z\| = 2m$ and z is connected, then z is contained entirely in some m-dimensional cube, Q_m. As a result, we can apply the linear inequality:

$$\|y\| \leq \frac{m-1}{4}\|z\| \leq \frac{1}{8}\|z\|^2.$$

We will proceed, as in the case of the linear inequality, by choosing a separation, H, of the cube Q_n into two opposite hyperfaces, Q^+_{n-1} and Q^-_{n-1}. Let us choose ϵ and δ to be some constants (to be explicitly determined later). There are three cases to consider:

6.1.3 Case 1: A small cut with a small piece

$[\exists H,\ \|z_0\| < \epsilon\|z\|^{\frac{k-1}{k}}$ **and** $\|z_+\| \leq \delta\|z_0\|^{\frac{k}{k-1}}]$

This condition states that we can find a slice, H, which intersects a relatively small part of z and either one side or the other (without loss of generality, z_+) is much smaller than a filling we would obtain from the isoperimetric inequality.

In this case, we will fill z just as we did when proving the linear inequality

$$\begin{aligned} \|y\| &\leq \|z_+\| + c_k\|z_- + z_+\|^{\frac{k+1}{k}} \\ &\leq \delta\|z_0\|^{\frac{k}{k-1}} + c_k(\|z\| - \|z_0\|)^{\frac{k+1}{k}} \end{aligned}$$

Setting $L = \frac{\delta}{c_k}$ and ensuring that $\epsilon \leq \frac{(k+1)^k}{(k+1)^k+(kL)^k}$ we can apply lemma 6.1.3 (in the next section), we obtain that

$$\|y\| \leq c_k \|z\|^{\frac{k+1}{k}}.$$

6.1.4 Case 2: A small cut with big pieces

$[\exists H,\ \|z_0\| < \epsilon \|z\|^{\frac{k-1}{k}}$ **and both** $\|z_+\|, \|z_-\| > \delta \|z_0\|^{\frac{k}{k-1}}]$

This condition states that we have found a slice in which we have cut z into two large bulbs with a relatively small cut. We will take advantage of this by filling z_0 using the isoperimetric inequality for $(k-1)$-cycles and then deal with the two resulting cycles in Q^+_{n-1} and Q^-_{n-1} separately.

Thinking of z_0 as a $(k-1)$-cycle, $z_0 \in Z_{k-1}Q_{n-1}$, we can find a filling of volume less than $c_{k-1}\|z_0\|^{\frac{k}{k-1}}$. This will leave us with cycles in Q^+_{n-1} (resp. Q^-_{n-1}) of volume $\|z_+\| + c_{k-1}\|z_0\|^{\frac{k}{k-1}}$ (resp. $\|z_-\|$...). Filling these cycles separately in each $(n-1)$-cube, we obtain:

$$\|y\| \leq c_{k-1}\|z_0\|^{\frac{k}{k-1}} + c_k\Big[(\|z_+\| + c_{k-1}\|z_0\|^{\frac{k}{k-1}}\|)^{\frac{k+1}{k}} + (\|z_-\| + c_{k-1}\|z_0\|^{\frac{k}{k-1}}\|)^{\frac{k+1}{k}}\Big]$$

Setting

$$x = \frac{\|z_+\|}{\|z_+\| + \|z_-\|} \text{ and } y = \frac{\|z_-\|}{\|z_+\| + \|z_-\|}$$

we apply lemma 6.1.2 and find that either

$$(\|z_+\| + c_{k-1}\|z_0\|^{\frac{k}{k-1}}\|)^{\frac{k+1}{k}} + (\|z_-\| + c_{k-1}\|z_0\|^{\frac{k}{k-1}}\|)^{\frac{k+1}{k}} \leq (\|z_+\| + \|z_-\|)^{\frac{k+1}{k}}$$

or

$$c_{k-1}\|z_0\|^{\frac{k}{k-1}} \geq [2^{\frac{1}{k+1}} - 1]\min\{\|z_+\|, \|z_-\|\} \geq [2^{\frac{1}{k+1}} - 1]\delta\|z_0\|^{\frac{k}{k-1}}$$

Therefore, if we can choose $\delta > \frac{c_{k-1}}{2^{\frac{1}{k+1}}-1}$ then we have,

$$\|y\| \leq c_{k-1}\|z_0\|^{\frac{k}{k-1}} + c_k(\|z_+\| + \|z_-\|)^{\frac{k+1}{k}}$$

Now setting $L = \frac{c_{k-1}}{c_k}$ and again ensuring that $\epsilon \leq \frac{(k+1)^k}{(k+1)^k+(kL)^k}$ we can apply lemma 6.1.3 to obtain that

$$\|y\| \leq c_k \|z\|^{\frac{k+1}{k}}$$

6.1.5 Case 3: Big Cuts

$[\forall H,\ \|z_0\| \geq \epsilon \|z\|^{\frac{k-1}{k}}]$

This condition states that every slice is large. As a result, we will see that the total volume of z is large enough to simply apply the linear inequality. More specifically, if we sum over all slices, each k-face in z is sliced exactly k times so:

$$\sum_H \|z_0\| = k\|z\| \geq n\epsilon \|z\|^{\frac{k-1}{k}}$$

$$\implies \|z\|^{\frac{k+1}{k}} \geq \frac{n}{k}\epsilon \|z\|$$

so that if $\epsilon \geq \frac{1}{2c_k}$ we can simply apply the linear inequality:

$$\|y\| \leq \frac{n-k}{2(k+1)}\|z\| \leq c_k \|z\|^{\frac{k+1}{k}}$$

6.1.6 The constants

Now that we have dealt with all three cases, let us wrap up our constants. In all cases, we needed that

$$\frac{1}{2c_k} \leq \epsilon \leq \frac{(k+1)^k}{(k+1)^k + (kL)^k}$$

Where L either appeared as $\frac{c_{k-1}}{c_k}$ in the second case or as $\frac{\delta}{c_k} = \frac{c_{k-1}}{c_k}[2^{\frac{1}{k+1}} - 1]^{-1}$ in the first (the value of δ was irrelevant in the first case). Since, $\frac{\delta}{c_k} \geq \frac{c_{k-1}}{c_k}$, we can take $L = \frac{\delta}{c_k}$.

Now taking, say,

$$c_k = \prod_{i=1}^{k} [2^{\frac{1}{i+1}} - 1]^{-1} = O(k!)$$

so that $L = 1$, we see that it is possible to choose ϵ and δ consistently in each case.

□

6.1.7 Two small technical inequalities

Here we prove the technical lemmas that we required in the previous sections.

Lemma 6.1.2. *Suppose* $x + y = 1$ *and* $(x+p)^{\frac{k+1}{k}} + (y+p)^{\frac{k+1}{k}} \geq 1$, *then*

$$p \geq \left(2^{\frac{1}{k+1}} - 1\right) \min\{x, y\}$$

This lemma appears in [30] as lemma 8 (without the explicit constant). For completeness, we provide it here as well.

Proof. Let $x \in [0, \frac{1}{2}]$ and consider the locus of points (x, p) satisfying:

$$(x+p)^{\frac{k+1}{k}} + (1-x+p)^{\frac{k+1}{k}} = 1.$$

We will first seek to show that $\frac{d^2p}{dx^2} \leq 0$ when $x \in [0, \frac{1}{2}]$. Implicitly differentiating:

$$\frac{dp}{dx}\left((1-x+p)^{\frac{1}{k}} + (x+p)^{\frac{1}{k}}\right) = (1-x+p)^{\frac{1}{k}} - (x+p)^{\frac{1}{k}}.$$

Noting that

$$(1+p') = \frac{2(1-x+d)^{\frac{1}{k}}}{(1-x+p)^{\frac{1}{k}} + (x+p)^{\frac{1}{k}}} \quad \text{and} \quad (p'-1) = \frac{-2(x+d)^{\frac{1}{k}}}{(1-x+p)^{\frac{1}{k}} + (x+p)^{\frac{1}{k}}}$$

and differentiating again:

$$p''\left((1-x+p)^{\frac{1}{k}} + (x+p)^{\frac{1}{k}}\right)^2 +$$

$$p'\left[(1-x+p)^{\frac{1}{k}}(x+p)^{\frac{1-k}{k}} - (x+p)^{\frac{1}{k}}(1-x+p)^{\frac{1-k}{k}}\right] =$$

$$-\Big[(1-x+p)^{\frac{1}{k}}(x+p)^{\frac{1-k}{k}} + (x+p)^{\frac{1}{k}}(1-x+p)^{\frac{1-k}{k}}\Big]$$

And since on the interval $x \in [0, \frac{1}{2}]$ we have

$$(1-x+p)^{\frac{1}{k}}(x+p)^{\frac{1-k}{k}} \geq (x+p)^{\frac{1}{k}}(1-x+p)^{\frac{1-k}{k}}$$

we see that $p'' \leq 0$ on $x \in [0, \frac{1}{2}]$.

Therefore $p \geq 2\Big[p(\frac{1}{2}) - p(0)\Big]x$. Since $p(0) = 0$ and $p(\frac{1}{2}) = \frac{1}{2}[2^{\frac{1}{k+1}} - 1]$, the lemma follows.

□

Lemma 6.1.3. *Let* $e = \left(\frac{k+1}{k}\right)^k$. *If* $0 \leq x \leq \left(\frac{eS}{e+L^k}\right)^{\frac{k-1}{k}}$ *then*

$$(S-x)^{\frac{k+1}{k}} + Lx^{\frac{k}{k-1}} \leq S^{\frac{k+1}{k}}$$

Proof.

$$\begin{aligned}
eS &\geq ex^{\frac{k}{k-1}} + L^k x^{\frac{k}{k-1}} \\
\Longrightarrow e(S-x) &\geq e(x^{\frac{k}{k-1}} - x) + L^k x^{\frac{k}{k-1}} \geq L^k x^{\frac{k}{k-1}} \\
\Longrightarrow \tfrac{k+1}{k}(S-x)^{\frac{1}{k}} &\geq Lx^{\frac{1}{k-1}} \\
\text{So since } \ S^{\frac{k+1}{k}} - (S-x)^{\frac{k+1}{k}} &\geq \Big[\tfrac{d}{dw}|_{S-x} w^{\frac{k+1}{k}}\Big]x = \tfrac{k+1}{k}(S-x)^{\frac{1}{k}}x \\
&\geq Lx^{\frac{k}{k-1}} \\
\Longrightarrow S^{\frac{k+1}{k}} &\geq (S-x)^{\frac{k+1}{k}} + Lx^{\frac{k}{k-1}}
\end{aligned}$$

□

6.2 A class of isoperimetric-minimizing cubical cycles

In this section, we will construct a family of cycles $z_n^k \in Z^k Q_n$, which will show that the exponent in theorem 6.0.5 cannot be improved.

Definition 6.2.1. Recall, that a k-dimensional face in Q_n is defined by allowing some k coordinates to vary while fixing the other $n-k$ coordinates as either 0 or 1.

We will define $z_n^k \in Z^k Q_n$ as the set of faces satisfying the following:

1. allow *any* k coordinates (say $i_1, \ldots, i_k$) to vary.

2. for all coordinates, i, such that $i_j < i < i_{j+1}$, either $i = 0$ or $i = 1$.

3. if $i_j < i < i_{j+1}$ and $i_{j+1} < i' < i_{j+2}$, then $i = 0$ implies $i' = 1$ and $i = 1$ implies $i' = 0$.

This codification is probably not very clear at first glance, so let me give an example.

Example 1. The chain z_{10}^3 consists of all three faces of the form:

$$(0, 0, *, 1, *, 0, *, 1, 1, 1) \quad \text{or say} \quad (1, *, *, 1, 1, 1, *, 0, 0, 0)$$

This is simply to say that we specify 3 (or k) coordinates to be allowed to vary and then in-between these coordinates, we force the entries to be constant in blocks, *alternating only once we reach a varying coordinate.*

Lemma 6.2.2. *The chain z_n^k satisfies the following:*

1. *z_n^k is a cycle.*

2. $\|z_n^k\| = \|z_{n-1}^{k-1}\| + \|z_{n-1}^k\| = 2\binom{n}{k}$.

3. $\text{Fill}(z_n^k) = \text{Fill}(z_{n-1}^{k-1}) + \text{Fill}(z_{n-1}^k) = \binom{n}{k+1}$.

It is important to note that in particular,

$$\text{Fill}(z_n^k) = \frac{n-k}{2(k+1)} \|z_n^k\|$$

and so therefore, this class of cycles make the linear inequality (lemma **??**) sharp.

Proof. The second statement is clear by definition. However, we would like to point out a geometric way of seeing it.

Let us specify a splitting, H, of the cube, just as we have done above, by deleting the first entry of the binary string. Then

$$(z_n^k)_+ + (z_n^k)_- = z_{n-1}^k \quad \text{and} \quad (z_n^k)_0 = z_{n-1}^{k-1}.$$

Now suppose we have a filling, y, of z_n^k. Then y_0 is a filling of z_{n-1}^{k-1} and $y_+ + y_-$ is a filling of z_{n-1}^k. Therefore,

$$\|y\| = \|y_+\| + \|y_0\| + \|y_-\| \geq \text{Fill}(z_{n-1}^{k-1}) + \text{Fill}(z_{n-1}^k).$$

Now, $\text{Fill}(z_{k+1}^k) = 1 = \binom{k+1}{k+1}$, while $\text{Fill}(z_n^0) = n = \binom{n}{1}$. Therefore, by induction,

$$\text{Fill}(z_n^k) = \binom{n}{k+1}.$$

Let us briefly remark on why z_n^k is a cycle. A $(k-1)$-face lies in the boundary of a face of z_n^k if it consists of alternating blocks of 1's and 0's in between varying coordinates *except* in one spot, where the block alternates unprompted. In that case, this $(k-1)$-face lies in exactly two faces of z_n^k: the one obtained by varying the 1 (at the unprompted alternation), and the one obtained by varying the 0. To give an example, the 2-face given by:

$$(1, 1, *, 0, 0, 1, 1, *, 0, 0)$$

belongs to both

$$(1, 1, *, 0, *, 1, 1, *, 0, 0) \quad \text{and} \quad (1, 1, *, 0, 0, *, 1, *, 0, 0) \quad \text{in } z_{10}^3$$

□

Corollary 6.2.3.

$$\text{Fill}(z_n^k) \geq \omega_k \|z_n^k\|^{\frac{k+1}{k}}$$

where $\omega_k \geq \frac{\sqrt[k]{k!}}{2^{\frac{k+1}{k}}(k+1)}(1-\epsilon)$ *for any* ϵ *and large enough* n.

In particular, this shows that the exponent in theorem 6.0.5 is optimal (though possibly the constant can be improved; $c_k = O(k!)$ while $\omega_k = \Omega(1)$).

Chapter 7

Appendix A: Gromov's point selection theorem

The lineage of *point selection theorems* traces back to Radon's theorem for configurations of points in $\mathbb{R}^m$, which says that any configuration of $m+2$ points can always be separated into two subsets whose respective convex hulls share a common point.

This theorem was generalized to more than two subsets by Tverberg [50]. Quantitative bounds on the size of the subsets in Tverberg's theorem led to the first selection lemmas considered by Bárány, Boros, and Füredi in [10] and [6]. The lemma can be stated as follows:

Theorem. *The First Selection Lemma [10] [6]*

Suppose the m-dimensional simplex is affinely submerged in $\mathbb{R}^n$, then there exists a point $p \in \mathbb{R}^n$ so that the image of $c_n\binom{m+1}{n+1}$ faces of dimension n contain p.

The content of the theorem is that the constant c_n depends only on n. The optimal value of the constant c_n is only known in dimension 2 (in this case it is $c_2 = \frac{2}{9}$). However, it is known to be quite small in high dimensions ($c_n \leq e^{-\Theta(n)}$, [12]).

In the common proofs of the first selection lemma, the rectilinearity of the submersions is essential; the proofs rely heavily on techniques from convex geometry. Nonetheless, the question remains whether it is possible to relax the rectlinearity of these maps and consider instead smooth or continuous maps.

An extension to the continuous setting was achieved by Bajmóczy and Bárány. In [5], they extended Radon's theorem to continuous maps:

Theorem. *Topological Radon Theorem [5]*

Let P be a convex polytope in $\mathbb{R}^{n+1}$ with non-empty interior, and consider a continuous map $\phi : P \to \mathbb{R}^n$. Then there exist two faces of P, $x, y \subset P$, whose closures are disjoint such that $\phi(x) \cap \phi(y) \neq \emptyset$.

Later the Tverberg theorem was given a similar treatment by Bárány, Shlosman, and Szucs:

Theorem. *Topological Tverberg Theorem [7]*

Let p be a prime and let $\phi : \Delta_m \to \mathbb{R}^n$ be a continuous map from the m-dimensional simplex to $\mathbb{R}^n$ with $m = (n+1)(p-1)$. Then there exists p faces, $x_1, \ldots, x_p \subset \Delta_m$ with disjoint closures such that

$$\phi(x_1) \cap \cdots \cap \phi(x_p) \neq \emptyset.$$

These developments indicate that there might be an extension of the first selection lemma to the continuous setting. Namely,

Question. Topological Selection Lemma

Suppose the m-dimensional simplex is continuously mapped into $\mathbb{R}^n$. Does there exists a point $p \in \mathbb{R}^n$ so that the image of $c_n\binom{m+1}{n+1}$ faces of dimension n contain p?

Just as with the proofs of the topological Radon and Tverberg theorems, the verification of this question would require completely different techniques than those employed in the traditional point selection theorems. While the traditional theorems relies on tools from convex geometry, a *topological* point selection theorem would be required to explicitly use some of the topological properties of m-dimesional simplex.

Luckily, we have use of an important analogy:

Theorem. *The Waist of the Sphere [27]*

Suppose we are given a continuous map $\phi : S^m \to \mathbb{R}^n$ from the unit m-sphere to $\mathbb{R}^n$. Then there exists a point $p \in \mathbb{R}^n$ such that

$$\mathrm{vol}_{n-k}[\phi^{-1}(p)] \geq \mathrm{vol}_{n-k}(S^{n-k})$$

where $\mathrm{vol}_{n-k}(S^{n-k})$ is the Riemannian volume of the unit $(n-k)$-sphere.

(Other, more robust, techniques for estimating waists of maps were developed later in [48]; see also [41])

There are several distinct methods for proving of the waist of the sphere theorem. One such method that will lend itself directly to the combinatorial setting in question can be found in [27] (actually, the waist theorem proved in [27] is a variant of the one stated above).

Upon inspection, we find that the dynamo which drives the proof in [27] is the repeated use of a certain spherical isoperimetric inequality. Indeed, the isoperimetric inequalities in codimension m $(0 \leq k \leq m)$ are the only quanitative estimates necessary for the proof. In particular, *the specifically round metric on the sphere can be dispensed with in favor of another metric, provided we are given suitable isoperimetric inequalities!*

Modelling the proof of this theorem, we can answer our question affirmatively. In this appendix, we will provide a clear proof of the following theorem of Gromov:

Theorem. *Let $\phi : \Delta_m \to \mathbb{R}^n$ be a generic C^1 map from the m-dimensional simplex to $\mathbb{R}^n$. Then there exists $p \in \mathbb{R}^n$ which lies in the image of at least $c_n\binom{m+1}{n+1}$ faces of dimension n in Δ_m.*

Here, the proof will provide $c_n \geq \frac{1}{(n+1)!} - o(1)$ (we think of n fixed, while m tends to infinity). Another treatment of the proof of this theorem can be found in [52] or [32], but they are significantly different in flavor and focus.

Just as in the case of the specificity of the round sphere, the specific combinatorial or topological properties of the simplex can be dispensed with in favor of vastly more general complexes. The salient properties in the proof are exactly the expansion constants described in Chapter 3. These inequalities will play a central role in the proof.

7.1 The Point Selection Theorem

The proof of Gromov's point selection theorem relates to the rest of this dissertation because it gives geometric information about maps to Euclidean space in terms of expansion inequalities. However, it will be easier to deal with *normalized* expansion constants.

Definition 7.1.1. For a polyhedral complex, X, we will define the constants ω_k as:

$$\omega_k = \max_{\beta \in Z^{k+1}X} \frac{\min_{d\alpha=\beta} \|\alpha\|}{\|\beta\|} \cdot \frac{|X^{(k+1)}|}{|X^{(k)}|}.$$

In other words, for each cocycle, $\beta \in Z^{k+1}X$, there exists a primitive, $\alpha \in C^kX$ such that $d\alpha = \beta$ and

$$\frac{\|\alpha\|}{|X^{(k)}|} \leq \omega_k \frac{\|\beta\|}{|X^{(k+1)}|}.$$

Or said in yet another way: $\omega_k = \frac{|X^{(k+1)}|}{h^k \cdot |X^{(k)}|}$.

We will also need a notion of *degree proportion*:

$$\mu_k = \mu_k(X) \triangleq \frac{1}{|X^{(k+1)}|} \cdot \max_{x \in X^{(k)}} \left| \left\{ y \in X^{(k+1)} \ : \ x \subset y \right\} \right|.$$

For example, if X is the m-dimensional simplex, Δ_m, then $\mu_k(X) = \frac{m-k}{\binom{n+1}{k+2}}$. In all of the families, X_m, of complexes we will consider, $\mu_k(X_m)$ will tend to 0 as m tends to infinity.

With this notation, we will state Gromov's theorem in more specificity.

Theorem 7.1.2. *[28]*

Let X be a polyhedral complex as above and let $\phi : \tilde{X} \to \mathbb{R}^n$ be a map from the geometric realization $\tilde{X}$ of the complex X, which is C^1 on each face and is "generic" in the sense that the images of faces are pairwise transverse (and faces are transverse with themselves). Then there exists $p \in \mathbb{R}^n$ such that

$$\left|\phi^{-1}(p) \cap X^{(n)}\right| \geq C_n \left|X^{(n)}\right|$$

where

$$C_n = C_n(\omega_1, \ldots, \omega_{n-1}, \mu_0, \ldots, \mu_{n-1}).$$

Remark 7.1.3. The reader should note that the proportion C depends only on the co-isoperimetric sequence and the degree proportions. The second term should be viewed as a small error term because the degree proportions will usually be quite small.

We should also remark on the meaning of $\left|\phi^{-1}(p) \cap X^{(n)}\right|$. Due to our choice to use $\mathbb{Z}$ coefficients, this quantity represents the number of times the pre-image of p intersects the n-skeleton. It may be, however, that many of these intersections all lie in a single n-cell. If, instead we choose to use $\mathbb{Z}_2$ coefficients, the reader will note that our arguments proceed without any difficulty and, in that case, the quantity $\left|\phi^{-1}(p) \cap X^{(n)}\right|$ represents the number of n-cells whose image contains p.

Proof. The proof will rely on a very fine triangulation Γ of $\mathbb{R}^n$ (which will depends on ϕ). We will require that each k-cell $\eta \in \Gamma^{(k)}$ only intersects the image of a few $(n-k)$-cells of X. If fact, we can choose a trangulation fine enough so that each $(n-k)$-face of Γ intersects less than μ_{k-1} faces of dimension k in the image of ϕ (here we are using that the images of faces of ϕ are in general position). We will omit the details on the construction of this triangulation.

Let us begin by assuming that every $p \in \mathbb{R}^n$ lies in the image of less than $\epsilon|X^{(n)}|$ faces of X.

Now, ϕ induces a map

$$\Phi : C_k(\Gamma) \to C^{n-k}(X), \quad \eta \mapsto \Phi_\eta$$

where $\Phi_\eta(x)$ is the (oriented) intersection number of η with $\phi(x)$.

Therefore, we have the following diagram of normed $\mathbb{Z}$-modules:

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\partial} & C_{k-1}\Gamma & \xleftarrow{\partial} & C_k\Gamma & \xleftarrow{\partial} & C_{k+1}\Gamma & \xleftarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} & & \\
\cdots & \xleftarrow[d]{} & C^{n-k+1}X & \xleftarrow[d]{} & C^{n-k}X & \xleftarrow[d]{} & C^{n-k-1}X & \xleftarrow[d]{} & \cdots
\end{array}$$

We should check that this diagram commutes. Consider some $\eta \in \Gamma^{(k)}$ and all the $(n-k)$-cells in $x \in X^{(n-k)}$ such that $\Phi_\eta(x) \neq 0$.

We are in search of a point $p \in \Gamma^{(0)}$ contained in the image of a large number of n-simplicies of X. Another way to say this is that we would like to show that the map $\Phi : C_0\Gamma \to C^n X$ has large operator norm. The essence of this argument is show that if the operator norm is too small, we will be able to construct a chain homotopy, ψ, from Φ to the 0 map.

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\partial} & C_{k-1}\Gamma & \xleftarrow{\partial} & C_k\Gamma & \xleftarrow{\partial} & C_{k+1}\Gamma & \xleftarrow{\partial} & \cdots \\
 & \searrow^{\psi} & \downarrow{\scriptstyle\Phi} & \searrow^{\psi} & \downarrow{\scriptstyle\Phi} & \searrow^{\psi} & \downarrow{\scriptstyle\Phi} & \searrow^{\psi} & \\
\cdots & \xleftarrow[d]{} & C^{n-k+1}X & \xleftarrow[d]{} & C^{n-k}X & \xleftarrow[d]{} & C^{n-k-1}X & \xleftarrow[d]{} & \cdots
\end{array}$$

$$\Phi = d\psi + \psi\partial$$

Obviously, this will not result in a contradiction unless we have, in addition, some sort of topological restriction for the map Φ.

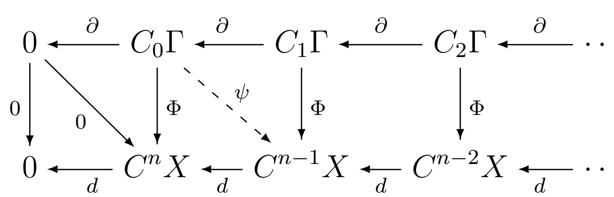
At first, it is not obvious what sort of topological restriction we might use. However, we notice that under the map

$$\Phi : C_n\Gamma \to C^0X$$

The sum of enough n-faces of Γ is sent to the constant function $1 \in C^0X$. This is because each vertex of X lands in one and only one k-face of Γ. The constant function is a representative of a non-trivial cohomology class in $H^0(X)$. This will be our topological restriction and we will attempt to exploit it.

Let us then begin with the construction of the chain homotopy. It will be constructed inductively:

1. Starting at the bottom of the diagram,



In order to define $\psi(p)$ for each $p \in \Gamma^{(0)}$, let γ be a curve starting at p and traveling (transversely to $\phi(X)$) to infinity. This curve γ determines a cochain in $C^{n-1}X$ (again by oriented intersection), whose coboundary is exactly $\Phi(p)$. Thus, $\Phi(p)$ is a coboundary, and, by assumption there exists a primitive α, such that $d\alpha = \Phi(p)$ and

$$||\alpha|| \leq \omega_n ||\Phi(p)|| \leq \omega_n \epsilon$$

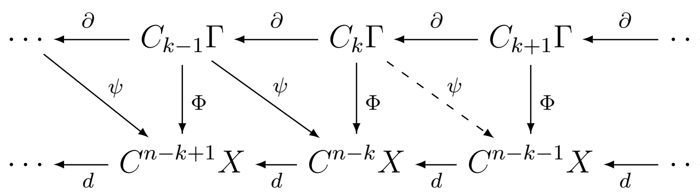
Define $\psi(p) \triangleq \alpha$ and extend ψ linearly to all of $C_0\Gamma$. Thus, we have, by definition,

$$\Phi = d\psi = d\psi + 0\partial$$

2. Suppose we have defined ψ up to $\psi : C_{k-1}\Gamma \to C^{n-k}X$, such that for each $(k-1)$-cell, τ, in Γ ,

$$||\psi(\tau)|| \leq C(\omega_{n-k+1}, \dots, \omega_{n-1}, \mu_{n-k}, \dots, \mu_{n-1})$$

We would like to define $\psi : C_k\Gamma \to C^{n-k-1}X$,



For each k-cell, η, in Γ, consider $(\Phi - \psi\partial)\eta \in C^{n-k}X$. This cochain is a cocycle; indeed,

$$d(\Phi - \psi\partial) = \Phi\partial - d\psi\partial = \Phi\partial - (\Phi - \psi\partial)\partial = 0$$

Since X is acyclic, this cocycle is therefore also a coboundary. By assumption there exists a primitive α, such that $d\alpha = (\Phi - \psi\partial)\eta$ and

$$||\alpha|| \leq \omega_{n-k}||\Phi\eta - \psi\partial\eta||$$

$$\leq \omega_{n-k}[||\Phi\eta|| + ||\psi\partial\eta||]$$

Define $\psi(\eta) \triangleq \alpha$ and extend ψ linearly to all of $C_k\Gamma$. Thus we have,

$$\Phi = d\psi + \psi\partial$$

Now, by construction, η intersects at most μ_{n-k-1}, $(n-k)$-cells in $\phi(X)$, thus $||\Phi\eta|| \leq \frac{\mu_{n-k-1}}{|X^{(n-k)}|}$. Now, $\partial\eta$ consists of $(k+1)$ many cells, each of dimension $(k-1)$, thus

$$||\psi\partial\eta|| \leq (k+1)C(\omega_{n-k+1}, \dots, \omega_{n-1}, \mu_{n-k}, \dots, \mu_{n-1})$$

Now we see that,

$$||\psi(\eta)|| = ||\alpha|| \leq \omega_{n-k}\left[\frac{\mu_{n-k-1}}{\left|X^{(n-k)}\right|} + (k+1)C(\omega_{n-k+1},\dots,\omega_{n-1},\mu_{n-k},\dots,\mu_{n-1})\right]$$

$$= C(\omega_{n-k},\omega_{n-k+1},\dots,\omega_{n-1},\mu_{n-k-1},\mu_{n-k},\dots,\mu_{n-1}).$$

3. Now let us proceed to the top of the chain complex,

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{\partial} & C_{n-2}\Gamma & \xleftarrow{\partial} & C_{n-1}\Gamma & \xleftarrow{\partial} & C_n\Gamma & \xleftarrow{0} & 0 \\
 & \searrow^{\psi} & \downarrow{\Phi} & \searrow^{\psi} & \downarrow{\Phi} & \searrow^{\psi} & \downarrow{\Phi} & & \\
\cdots & \xleftarrow[d]{} & C^2X & \xleftarrow[d]{} & C^1X & \xleftarrow[d]{} & C^0X & \xleftarrow[d]{} & 0
\end{array}$$

Once again, for each n-cell, η in Γ, consider the 0-cochain given by $(\Phi - \psi\partial)\eta$. As noted before, this cochain is indeed a *cocycle*. However, the only cocycles in C^0X are the functions on vertices which are locally constant.

Since every vertex of X is mapped into some n-cell of Γ, we have that

$$\sum_{\eta\in\Gamma^{(k)}} \Phi(\eta) \equiv 1$$

(where 1 denotes the function which is constant with value 1 on the vertices of X).

And since

$$\sum_{\eta\in\Gamma^{(k)}} \psi\partial\eta = \psi\left[\sum \partial\eta\right] = 0$$

(because every $(k-1)$-cell of Γ is contained in exactly two k-cells, contributing a positive orientation to one and a negative orientation to the other). Therefore,

$$\sum_{\eta}(\Phi - \psi\partial)\eta \equiv 1.$$

Recalling that each $(\Phi - \psi\partial)\eta$ is individually a constant function, we know that at least one of them has norm greater than 1, i.e.

$$\begin{aligned} 1 &\leq ||\Phi\eta - \psi\partial\eta|| \leq ||\Phi\eta|| + ||\psi\partial\eta|| \\ &\leq \left|X^{(0)}\right|^{-1} + (n+1)C(\omega_0, \ldots, \omega_{n-1}, \mu_1, \ldots, \mu_{n-1}) \end{aligned}$$

□

Chapter 8

Appendix B: Randomized algorithms for the linear filling inequality

The purpose of this appendix is to solve the linear filling problem for a number of important examples. A linear filling inequality is an inequality of the form,

For each $(k=1)$*-cocycle,* β*, in* X,

$$\text{coFill}_X(\beta) \leq \frac{\|\beta\|}{h^k(X)}.$$

In many applications of the applications we have seen, a bound on the constant h^k suffices. The prototypical example of of a linear filling inequality comes from Federer and Fleming:

Theorem (Federer and Fleming, [24])**.** *Each* k*-cycle,* z*, in the unit* n*-sphere* S^n *has,*

$$\text{Fill}(z) \leq \frac{\text{vol}_{k+1} S^{k+1}}{2\text{vol}_k S^k} \text{vol}_k(z)$$

The proof technique for this linear inequality involves a randomized algorithm which, in expectation, produces the desired filling. The algorithms which we will develop in this appendix are all combinatorial adaptations of this original technique.

8.1 The simplex

Let Δ be the simplex on $n+1$ ordered vertices and $\beta \in d(C^k\Delta)$ be a $k+1$ dimensional coboundary in $C^*\Delta$, then there exists a primitive $\alpha \in C^k\Delta$, $d\alpha = \beta$, such that

$$||\alpha|| \leq \frac{n-k}{n+1}||\beta||$$

Proof. Proofs of the linear inequality for the simplex appear in [28] and [**?**].

For each $v \in \Delta^{(0)}$, let $\alpha_v \in C^k\Delta$ be defined by

$$\alpha_v[v_0,\dots,v_k] \triangleq \begin{cases} (-1)^l\beta[v,v_0,\dots,v_k], & \text{if } v \notin \{v_0,\dots,v_k\} \\ 0, & \text{otherwise} \end{cases}$$

where $l = \Big|\{v_i \in \{v_0,\dots,v_k\} \,:\, v_i < v\}\Big|$ according to the ordering of $\Delta^{(0)}$.

$$\begin{aligned} d\alpha_v[v_0,\dots,v_{k+1}] &= \textstyle\sum_i (-1)^i \alpha_v[v_0,\dots,\hat{v}_i,\dots,v_{k+1}] \\ &= \begin{cases} \beta[v_0,\dots,v_{k+1}], & \text{if } v = v_i \\ \sum\limits_{i=-1} (-1)^{i+l}\beta[v,v_0,\dots,\hat{v}_i,\dots,v_{k+1}] & \text{otherwise} \end{cases} \end{aligned}$$

(note that in the above sum, l depends on i).

Now, since $\beta \in \mathrm{im}d$ we have,

$$d\beta[v,v_0,\dots,v_{k+1}] = -\beta[v_0,\dots,v_{k+1}] + \sum_i (-1)^{i+l}\beta[v,v_0,\dots,\hat{v}_i,\dots,v_{k+1}] = 0$$

$$\Rightarrow \sum_i (-1)^{i+l}\beta[v,v_0,\dots,\hat{v}_i,\dots,v_{k+1}] = \beta[v_0,\dots,v_{k+1}]$$

which shows that $d\alpha_v = \beta$. We shall take the average norm over all of these α_v's..

$$\begin{aligned} \tfrac{1}{n+1}\textstyle\sum_v ||\alpha_v|| &= \tbinom{n+1}{k+1}^{-1}\tfrac{1}{n+1}\sum_v \sum_{v\notin\{v_0,\dots,v_k\}} \Big|\beta[v,v_0,\dots,v_k]\Big| \\ &= \tbinom{n+1}{k+1}^{-1}\tfrac{k+2}{n+1}\sum_{[v_0,\dots,v_{k+1}]} \Big|\beta[v_0,\dots,v_{k+1}]\Big| = \tfrac{k+2}{n+1}\tbinom{n+1}{k+2}\tbinom{n+1}{k+1}^{-1}||\beta|| \end{aligned}$$

The second equality follows because in choosing the $(k+1)$-face $[v_0, \ldots, v_{k+1}]$ by method of first choosing a k-subface and then choosing the remaining vertex, we choose the $(k+1)$-face exactly $k+2$ times.

Now since, α is an average of the α_v, there must be some v so that

$$||\alpha_v|| \leq \frac{k+2}{n+1}\binom{n+1}{k+2}\binom{n+1}{k+1}^{-1}||\beta||_1 = \frac{n-k}{n+1}||\beta||$$

□

8.2 The cube

Proposition 8.2.1. *Let Q_n denote the cellular n-cube.*

$$h^k(Q_n) = 1$$

Proof. Just as in the proof technique from Chapter 6, we will use a decomposition of the cube into three pieces by "cutting" it along one coordinate (see section 6.1.1). In this way, we will write β as,

$$\beta_+,\, \beta_- \in Z^k Q_{n-1} \qquad \beta_0 \in C^{k-1}Q_{n-1} \qquad \text{and } d\beta_0 = \beta_+ - \beta_-.$$

We will construct a primitive inductively. Certainly the proposition holds true for Q_1, so suppose it is true for Q_{n-1}.

There is a cochain, $\alpha_+ \in C^{k-1}Q_{n-1}$ such that $d\alpha_+ = \beta_+$ and $\|\alpha_+\| \leq \|\beta_+\|$. As a result, $\alpha_+ - \beta_0$ is a primitive for α_-.

Therefore, we will consider the primitive, $\alpha \in C^{k-1}Q_n$, which when restricted to one of the $(n-1)$-faces is α_+ and is $\alpha_+ - \beta_0$ on the opposing face (and is 0 on the $(k-1)$-faces which do not lie in either of the $(n-1)$-faces. So we have

$$\|\alpha\| \leq 2\|\alpha_+\| + \|\beta_0\|.$$

Now we will average over cuts (just as in section 6.1.1), keeping in mind that each k-face lies in exactly $n-k$ faces of dimension $n-1$, and similarly, each k-face occurs in a β_0 exactly $2k$ times (the cut is oriented).

$$\frac{1}{2n}\sum_{H}\|\alpha\| \leq \frac{1}{2n}\Big(2(n-k)\|\beta\| + 2k\|\beta\|\Big) = \|\beta\|.$$

So therefore, there must exist one such α with $d\alpha = \beta$ and $\|\alpha\| \leq \|\beta\|$. □

8.3 The cross-polytope

Proposition 8.3.1. *Let $\widehat{Q}_n$ denote the n-dimensional cross-polytope. Then*

$$h^k(\widehat{Q}_n) \geq \frac{2(n-k-1)}{k+2}.$$

Proof. The cross-polytope can be written as the n-fold simplicial join of two points, i.e.

$$\widehat{Q}_n = \overbrace{\{x,y\} * \cdots * \{x,y\}}^{n}.$$

Obviously, if we remove any such pair, we will obtain a cross-polytope of one lower dimension. Let us fix such a pair $\{x,y\}$. Now suppose that we have proved the proposition for all cross-polytopes of dimension less than n (There is little to check in the base case $n = k+1$, because there is only *one* nontrivial cocycle and it has only one cofilling), and, suppose that $\beta \in Z^{k+1}\widehat{Q}_n$ is a cocycle.

1. define $\alpha_0 \in C^k\widehat{Q}_n$ by

$$\begin{aligned} \alpha_0[v_0, v_1, \ldots, v_k] &= \beta[x, v_0, \ldots, v_k] \\ \alpha_0[x, v_1, \ldots, v_k] &= 0 \\ \alpha_0[y, v_1, \ldots, v_k] &= 0 \end{aligned}$$

2. define $\beta_- \in C^{k+1}\widehat{Q}_n$ by

$$\begin{aligned}\beta_-[v_0, v_1, \dots, v_{k+1}] &= 0 \\ \beta_-[x, v_1, \dots, v_{k+1}] &= 0 \\ \beta_-[y, v_1, \dots, v_{k+1}] &= \beta[x, v_1, \dots, v_{k+1}]\end{aligned}$$

3. define $\beta_+ \in C^{k+1}\widehat{Q}_n$ by:

$$\begin{aligned}\beta_+[v_0, v_1, \dots, v_{k+1}] &= 0 \\ \beta_+[x, v_1, \dots, v_{k+1}] &= 0 \\ \beta_+[y, v_1, \dots, v_{k+1}] &= \beta[y, v_1, \dots, v_{k+1}]\end{aligned}$$

4. Notice that since $d\beta = 0$, we have $d\beta_+ + d\beta_- = 0$.

5. By the induction hypothesis (the chain $\beta_+ + \beta_-$ can be thought of as the suspension by y of a cocycle in $Z^k\widehat{Q}_{n-1}$) there exists a cochain $\alpha_+ \in C^k\widehat{Q}_n$ such that $d\alpha_+ = \beta_+ + \beta_-$ and

$$\|\beta_+ + \beta_-\| \geq \frac{2(n-k-1)}{k+1}\|\alpha_+\|$$

6. Let $\alpha = \alpha_+ + \alpha_0$. Then $d\alpha = \beta$ and

$$\|\alpha\| = \|\alpha_+\| + \|\alpha_0\| \leq \frac{k+1}{2(n-k-1)}\Big(\|\beta_-\| + \|\beta_+\|\Big) + \|\beta_-\|$$

We constructed this naive filling by the fixing the pair $\{x, y\}$. If we chose such a pair randomly and uniformly among the n pairs and take the expected norm, we obtain:

$$\mathbb{E}\|\alpha\| \leq \frac{1}{n}\left[\frac{k+1}{2(n-k-1)}\sum\|\beta_+\| + \frac{2n-k-1}{2(n-k-1)}\sum\|\beta_-\|\right]$$

However, each face in the support of, say, β_+, has $k+2$ vertices, and therefore is over-counted with multiplicity $k+2$, i.e.

$$\sum\|\beta_+\| = (k+2)\|\beta\| = \sum\|\beta_-\|.$$

Therefore we have:

$$\mathbb{E}\|\alpha\| \leq \frac{k+2}{n-k-1}\|\beta\|$$

Therefore, there exists an $\alpha \in C^k\widehat{Q}_n$ such that $d\alpha = \beta$ and

$$\|\beta\| \geq \frac{n-k-1}{k+2}\|\alpha\|.$$

□

8.4 The complete $(k+2)$-partite complex

Let $\Lambda_{n,k}$ be the complete $(k+2)$-partite $(k+1)$-complex with $(n,\ldots,n)$ vertices. In other words,

$$\Lambda_{n,k} := \overbrace{[n] * \cdots * [n]}^{k+2}$$

where $*$ denotes join and $[n]$ denotes a set of of n vertices.

Proposition 8.4.1. *We have*

$$h^k(\Lambda_{n,k}) \geq \frac{n}{2^{k+2}-1}.$$

For example, when $k=0$, $\Lambda_{n,0}$ is the complete bipartite graph, $K_{n,n}$. The Cheeger constant of the graph $K_{n,n}$ is given by $h(K_{n,n}) \geq \frac{n}{2}$.

Proof. The proof is by induction on k.

We will suppose that for all $\beta \in C^{k-1}\Lambda_{n,k-1}$,

$$\frac{\|d\beta\|}{\|[\beta]\|} \geq c_{k-1}.$$

There are two types of k-faces in $\Lambda = \Lambda_{n,k}$. We say that a k-face, $[v_0,\ldots,v_k]$, is in type one, T_1, if it contains a vertex from the first factor $[n]$ of the simplicial join; there are $(k+1)n^{k+1}$ many faces of type 1. We say that $[v_1, \ldots, v_k]$ is of type 2, T_2, if all its vertices come from the second through $(k+1)$-th factors of the join; there are n^{k+1} many faces of type 2.

For example, in a $(k+1)$-dimensional face $[v_0, \ldots, v_{k+1}]$, the subface $[\hat{v}_0, v_1, \ldots v_{k+1}]$ is of type 2, whereas the subfaces $[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}]$ are all of type 1.

For each $m \in [n]$, in the first factor of the join, we have the contraction $\iota_m : C^\ell \Lambda_{n,k} \to C^{\ell-1}\Lambda_{n,k-1}$ given by:

$$(\iota_m \beta)[v_0, \ldots, v_\ell] = \beta[m, v_0, \ldots, v_\ell].$$

Let $\beta \in C^k \Lambda_{n,k}$. We will begin by finding a naive filling of the coboundary $d\beta$.

1. Notice that for $\ell, m \in [n]$,

$$\iota_\ell d\beta - \iota_m d\beta = d(\iota_\ell \beta - \iota_m \beta)$$

   (It is not true, however, that $\iota_\ell d\beta = d\iota_\ell \beta$).

2. By the induction hypothesis, let us find $\alpha_{\ell,m} \in C^{k-1}\Lambda_{n,k-1}$ such that $d\alpha_{\ell,m} = \iota_\ell d\beta - \iota_m d\beta$, and

$$\|\iota_\ell d\beta - \iota_m d\beta\| \geq c_{k-1} \|\alpha_{\ell,m}\|.$$

3. Now define $\alpha_m \in C^k \Lambda_{n,k}$ by

$$\alpha_m[v_0, \ldots, v_k] := \begin{cases} \alpha_{v_0,m}[\hat{v}_0, \ldots, v_k], & \text{if } [v_0, \ldots, v_k] \in T_1 \\ \iota_m d\beta[v_0, \ldots, v_k], & \text{if } [v_0, \ldots, v_k] \in T_2 \end{cases}$$

4. By construction, $d\alpha_m = d\beta$ and

$$\|\alpha_m\| = \|\iota_m d\beta\| + \sum_\ell \|\alpha_{\ell,m}\|$$

5. Now we average the norms of the α_m:

$$\frac{1}{n}\sum_m \|\alpha_m\| = \frac{1}{n}\Big(\|d\beta\| + \sum_{\ell\neq m}\|\alpha_{\ell,m}\|\Big)$$

$$\leq \frac{\|d\beta\|}{n} + \frac{1}{n\cdot c_{k-1}}\sum_{\ell\neq m}\Big\|d\beta[\ell,\cdot] - d\beta[m,\cdot]\Big\|$$

$$\leq \frac{\|d\beta\|}{n} + \frac{1}{n\cdot c_{k-1}}\sum_m\Big(\|d\beta\| - \|d\beta[m,\cdot]\|\Big)$$

$$+\frac{n-1}{n\cdot c_{k-1}}\sum_m \|d\beta[m,\cdot]\|$$

$$= \frac{1}{n}\Big(1 + \frac{2n-2}{c_{k-1}}\Big)||d\beta||$$

Therefore we have that $c_k \geq \frac{nc_{k-1}}{c_{k-1}+2n}$

Finally, noting that $c_0 \geq \frac{n}{2}$, we can check that $c_k \geq \frac{n}{2^{k+2}-1}$:

$$c_k = n\left(\frac{c_{k-1}}{c_{k-1}+2n-2}\right) \geq \frac{c_{k-1}}{c_{k-1}+2n} = \frac{n}{2^{k+2}-1}$$

That is,

$$h^k(\Lambda) \geq \frac{n}{2^{k+2}-1}$$

$\square$

8.5 The L^1-expansion of graphs

Suppose G is a finite, connected, symmetric graph, satisfying the isoperimetric inequality:

$$\inf\left\{\frac{|\partial A|}{|A|} \,:\, A\subset G^{(0)},\, |A| \leq \frac{|G^{(0)}|}{2}\right\} \geq c$$

Then, for every coboundary with $\mathbb{Z}$-coefficients, $\beta \in d(C^0G) \subset C^1G$, there is a primitive with $\mathbb{Z}$-coefficients, $\alpha \in C^0G$ with $d\alpha = \beta$ and

$$c\|\alpha\|_1 \leq \|\beta\|_1$$

where $\| - \|_1$ is understood to be the L^1 norm on $\mathbb{Z}$-cochains.

Proof. Suppose we have a coboundary $df \in C^1G$. Since G is connected ker$d$ consists of the constant integer-valued functions. Therefore, f is determined up to a global additive constant. Let us choose this constant so that the median of f is 0. Let us define

$$A_n = \{v \in G^{(0)} \,:\, f(v) \geq n\}$$

(notice that these sets are nested).

Since the median of f is 0, we have that

$$|A_0| > \frac{|G^{(0)}|}{2} \text{ but } |A_1| \leq \frac{|G^{(0)}|}{2}.$$

The reader will further notice that $||df||_1 = |G^{(1)}|^{-1} \sum_{-\infty}^{\infty} |\partial A_n|$. Indeed, if a given edge has energy k, then one of its vertices is in A_n, for some n, and the other is in A_{n+k}. Therefore, the edge occurs in $\partial A_{n+k}, \partial A_{n+k-1}, \ldots, \partial A_{k+1}$, i.e. it is counted k times in the above sum. We split up this sum as

$$\sum_{-\infty}^{\infty} |\partial A_n| = \sum_{1}^{\infty} |\partial A_n| + \sum_{-\infty}^{0} |\partial(G \setminus A_n)|$$

since $\partial A = \partial(G \setminus A)$ as sets. Now $|A_n| \leq \frac{|G^{(0)}|}{2}$ for $n \in \{1, 2, \ldots,\}$ and $|G \setminus A_n| \leq \frac{|G^{(0)}|}{2}$ for $n \in \{0, -1, -2, \ldots\}$, so by the isoperimetric inequality we have:

$$||df|| \geq \frac{c}{|G^{(1)}|} \Big[\sum_{1}^{\infty} |A_n| + \sum_{-\infty}^{0} |G \setminus A_n| \Big]$$

Now, we can write

$$\begin{aligned}
|G^{(0)}| \cdot ||f|| &= \textstyle\sum_{-\infty}^{\infty} n|A_n \setminus A_{n+1}| = \sum_{-\infty}^{\infty} n(|A_n| - |A_{n+1}|) \\
&= \textstyle\sum_{1}^{\infty} n(|A_n| - |A_{n+1}|) - \sum_{-\infty}^{0} n(|G \setminus A_{n+1}| - |G \setminus A_n|) \\
&= (|A_1| - |A_2|) + 2(|A_2| - |A_3|) + 3(|A_3| - |A_4|) + \dots \\
&+ 0(|G \setminus A_1| - |G \setminus A_0|) + (|G \setminus A_0| - |G \setminus A_{-1}|) + 2(|G \setminus A_{-1}| - |G \setminus A_{-2}|) + \dots \\
&= \textstyle\sum_{1}^{\infty} |A_n| + \sum_{-\infty}^{0} |G \setminus A_n| \\
&\leq \tfrac{|G^{(1)}|}{c} ||df||
\end{aligned}$$

□

8.6 Suspensions of expanders

Let G be a finite, connected, symmetric graph with isoperimetric constant c. Let $X = G * \{\infty_1, \dots, \infty_k\}$ denote the simplicial join of G with k points. That is,

$$\begin{aligned}
X^{(0)} &= G^{(0)} \cup \{\infty_1, \dots, \infty_k\} \\
X^{(1)} &= G^{(1)} \cup \Big\{[v, \infty_i] \,:\, v \in G^{(0)} \text{ and } 1 \leq i \leq k\Big\} \\
X^{(2)} &= \Big\{[v, w, \infty_i] \,:\, [v, w] \in G^{(1)} \text{ and } 1 \leq i \leq k\Big\}.
\end{aligned}$$

Then

$$h^0(X) \leq \frac{|G^{(1)}| + k|G^{(0)}|}{(k+c)|G^{(0)}|} \qquad \text{and} \qquad h^1(X) \leq \frac{|G^{(1)}|}{k|G^{(0)}| + |G^{(1)}|}\Big[\frac{2k-2}{c} + 1\Big]$$

Proof. Take any $A \subset X^{(0)}$ such that $|A| \leq \frac{|G^{(0)}|+k}{2} = \frac{1}{2}|X|$. Let $j = |\{\infty_i \,:\, \infty_i \in A\}|$.

Then,

$$
\begin{aligned}
|\partial A| &\geq j|G \setminus A| + (k-j)|A \cap G| + c\min\{|G \setminus A|,\, |A \cap G|\} \\
&= j(|G| - |A| + j) + (k-j)(|A| - j) + c\min\{|G| - |A| + j,\, |A| - j\} \\
&\geq j|G| - j(|G| + k) + k|A| - kj + 2j^2 + c\min\{|G| - |A| + j,\, |A| - j\} \\
&\qquad \text{since} \quad 2|A| \leq |G| + k \\
&\geq k|A| + 2j^2 - 2kj + c\min\{|G| - |A| + j,\, |A| - j\}
\end{aligned}
$$

Now, letting $j = \frac{k}{2}$ and $|A| = \frac{|G|+k}{2}$ we obtain:

$$\min \frac{|\partial A|}{|A|} \geq k + c\frac{|G|}{|G|+k} - \frac{k^2}{|G|+k}$$

So that $\omega_1 \leq \frac{|G^{(1)}|+k|G^{(0)}|}{(k+c)|G^{(0)}|}$

Now suppose $df \in d(\Lambda^1(\Sigma^k G)) \subset \Lambda^2(\Sigma^k G)$, we seek to find a minimal filling. We can vary f by an element in $\ker d \subset \Lambda^1(\Sigma^k G)$.

Now, $\ker d \cong \Lambda^0(G) \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ with exactly $k-1$ copies of $\mathbb{Z}$. This is because we can choose any function we like on the vertical edges of one of the cones $G * \infty_1$ This will determine the value on the horizontal edges, and thus determine the values on the horizontal edges of the other cones up to a choice of integer (hence, the $k-1$ copies of $\mathbb{Z}$).

So suppose we have chosen a function $g \in \Lambda^0(G)$. Then we can vary f to get:

$$h[x, \infty_1] = f[x, \infty_1] + g[x] \text{ on the vertical edges of the first cone.}$$

$$h[x, y] = f[x, y] + g[x] - g[y] \text{ on the horizontal edges}$$

and once we have specified an integer i_j for each of the remaining cones,

$$h[x,\infty_j] = f[x,\infty_j] + g[x] + i_j.$$

The reader should verify that $dh = df$, thus h is a filling for df.

In our choice of h we must be judicious in order to get a good upper bound on

$$(k|G^{(0)}| + |G^{(1)}|) \cdot ||h|| = \sum_{x\in G^{(0)}} \sum_{j} |f[x,\infty_j] + g[x] + i_j| + \sum_{[x,y]\in G^{(1)}} |f[x,y] + g[x] - g[y]|$$

in terms of

$$(k|G^{(1)}|) \cdot ||df|| = \sum_{j} \sum_{[x,y]} |f[x,y] + f[y,\infty_j] - f[x,\infty_j]|.$$

Notice that for any choice of g and any j, $f[x,\infty_j] - g[x]$ determines a function in $\Lambda^0(G)$, and so there exists an integer i_j so that

$$\sum_{x} |f[x,\infty_j] - g[x] + i_j| \leq \frac{|G^{(1)}|}{c|G^{(0)}|} \sum_{[x,y]} |f[y,\infty_j] - f[x,\infty_j] - g[y] + g[x]|$$

For simplicity, let us denote,

$$(k|G^{(1)}| \cdot ||df_j|| = \sum_{[x,y]} |f[x,y] + f[y,\infty_j] - f[x,\infty_j]|$$

which is df restricted to the jth cone. That is, $||df|| = \sum_j ||df_j||$.

Now returning to our choice of g, we shall *not* be judicious at all, but rather choose g randomly. Namely, let

$$g(x) = -f[x,\infty_r] \text{ for a uniformly chosen } r \in \{1,\ldots,k\}$$

. Then we have,

$$
\begin{aligned}
(k|G^{(0)}| + |G^{(1)}|) \cdot \mathbb{E}||h|| &= \tfrac{1}{k}\sum_r \sum_{x\in G^{(0)}} \sum_j |f[x,\infty_j] - f[x,\infty_r] + i_j| \\
&+ \tfrac{1}{k}\sum_r \sum_{[x,y]\in G^{(1)}} |f[x,y] - f[x,\infty_r] + f[y,\infty_r]| \\
&\leq \tfrac{k|G^{(1)}|}{kc}\left[\sum_r \sum_{j\neq r} ||df_j|| + ||df_r||\right] + |G^{(1)}|\cdot ||df|| \\
&= \tfrac{|G^{(1)}|}{c}\left[\sum_r ||df|| + (k-2)||df_r||\right] + |G^{(1)}|\cdot ||df|| \\
&= |G^{(1)}| \cdot \tfrac{2k-2}{c} \cdot ||df|| \\
&\Rightarrow \omega_2 \leq \tfrac{|G^{(1)}|}{k|G^{(0)}|+|G^{(1)}|}\left[\tfrac{2k-2}{c} + 1\right]
\end{aligned}
$$

□

Corollary 8.6.1. *For simplicity, let us assume that $\{G_n\}$ is sequence of d-regular graphs with Cheeger constants $c_n \geq c > 0$ and $|G_n| \to \infty$. Let G_n^k denote the simplicial join of G_n with k points.*

Then our filling constants for this 2-dimensional complex are

$$\omega_1 \leq \frac{d+2k}{2(k+c)} \quad \textit{and,} \quad \omega_2 \leq \frac{d}{2k+d}\cdot\left[\frac{2k-2}{c}+1\right]$$

Gromov's theorem now says that any generic map from G_n^k into $\mathbb{R}^2$ must have a fibre of size:

$$\left[\frac{c(k+c)}{3(2k-2+c)}\right]\cdot\frac{d|G_n|}{2} = \Omega(|G_n|)$$

Chapter 9

Appendix C - Bourgain's embedding theorem

Theorem 9.0.2. *Let (X, d) be a finite metric space. Then there exists an embedding*

$$\phi : X \to \mathbb{R}^n$$

such that $\mathrm{Lip}(\phi) \cdot \mathrm{Lip}(\phi^{-1}) \leq C \log |X|$ *and* $n = K \log^2 |X|$*, where* ϕ^{-1} *is the inverse of* ϕ *(defined only on the image of* ϕ*).*

Proof. We will start by considering a generalization of the Frechet-Kuratowski embedding, i.e. choose some set of subsets of X, $\Lambda \subset 2^X$, and map

$$I : X \to \ell^\infty(\Lambda), \qquad x \mapsto (A \in \Lambda \mapsto d(x, A))$$

For now, Λ will remain an unknown. Let $\Lambda_k = \{A \in \Lambda : |A| = k\}$.

The reader should notice that

$$|d(x, A) - d(y, A)| \leq d(x, y) \implies \mathrm{Lip}(I) \leq 1$$

Extending the map into $\ell^1(\Lambda)$,

$$J : \ell^\infty(\Lambda) \to \ell^1(\Lambda)$$

where for the Dirac mass, δ_A,

$$J\delta_A = \left[|A||\Lambda_{|A|}|\right]^{-1}\delta_A,$$

i.e. J is a diagonal matrix whose diagonal entry associated to $A \in \Lambda$ is $\left[|A||\Lambda|A||\right]^{-1}$

The norm of J is maximized on the vector of all 1's, and so,

$$||J|| = \sum_{A\in\Lambda}\left[|A||\Lambda_{|A|}|\right]^{-1} \leq \sum_{s=1}^{|X|}\frac{1}{s} = \log|X|.$$

Remark 9.0.3. The choice of the coefficient $\left[|A||\Lambda_{|A|}|\right]^{-1}$ may not be immediately obvious to the reader. Indeed, it is quite a clever choice. The philosophy behind it is simple, however. For larger sets, A, the functions $d(-, A)$ do not separate points as well as smaller sets (in fact, in what is to come, the reader will notice that we will need to separate whole neighborhoods as well). Therefore, we weight the larger sets with the weight $\frac{1}{|A|}$. This choice of specific value of the weight will be illumined below.

Now we will factor J through $\ell^2(\Lambda)$,

$$\begin{array}{ccccc} X & \xrightarrow{I} & \ell^\infty(\Lambda) & \xrightarrow{J} & \ell^1(\Lambda) \\ & & {\scriptstyle J_0}\downarrow & & \uparrow{\scriptstyle J_1} \\ & & \ell^2(\Lambda) & = & \ell^2(\Lambda) \end{array}$$

Where, as matrices, J_0 and J_1 are simply the square root of J, that is, diagonal matrices with diagonal entries $\left[|A|\binom{|X|}{|A|}\right]^{-\frac{1}{2}}$ associated to each $A \in \Lambda$.

Immediately, we have

$$||J_0|| = \left[\sum_{A\in\Lambda}\left[|A||\Lambda_{|A|}|\right]^{-1}\right]^{\frac{1}{2}} \leq \sqrt{\log|X|}$$

Now choose a unit vector $v \in \ell^2(\Lambda)$ which realizes the norm of J_1. Using the Cauchy-Schwartz inequality,

$$||J_1|| = ||J_1 v|| = \sum_{\Lambda}\left[|A||\Lambda_{|A|}|\right]^{-\frac{1}{2}} v(A) \leq \left[\sum_{\Lambda}\left[|A||\Lambda_{|A|}|\right]^{-1}\right]^{\frac{1}{2}}\left[\sum_{\Lambda} v(A)^2\right]^{\frac{1}{2}} \leq \sqrt{\log|X|}$$

At this point we have constructed an embedding $\phi = J_0 \circ I$ into a Euclidean space of dimension $|\Lambda|$ such that $\text{Lip}(\phi) \leq \sqrt{\log |X|}$.

Furthermore, $\text{Lip}(\phi) \leq ||J_1|| \cdot \text{Lip}[(J \cdot I)^{-1}]$, yielding

$$\text{Lip}(\phi) \cdot \text{Lip}(\phi^{-1}) \leq \text{Lip}[(J \cdot I)^{-1}] \log |X|$$

And so we seek to show that $\text{Lip}[(J \cdot I)^{-1}]$ is bounded above by a universal constant, that is,

$$C \sum_{\Lambda} \Big[|A||\Lambda_{|A|}| \Big]^{-1} |d(x, A) - d(y, A)| \geq d(x, y)$$

for every $x, y \in X$ and some universal constant C.

When is it that $|d(x, A) - d(y, A)| \geq \delta$? This occurs exactly when there is a radius r, such that

$$A \cap B_r(y) \neq \emptyset \text{ and } A \cap \breve{B}_{r+\delta}(x) = \emptyset \tag{9.1}$$

or visa-versa for x and y ($\breve{B}_R(x)$ is the open ball around x).

Let us choose some numbers $A_1, A_2, \ldots, A_T$ (like Λ, they will have to remain a mystery for the moment), and define

$$r_t = \inf\{r : |B_r(x)| \geq A_t \text{ and } |B_r(y)| \geq A_t\}$$

with the largest, $r_T = \frac{d(x,y)}{2}$.

Without any loss of generality, we can assume $|B_{r_t}(y)| \geq |B_{r_t}(x)|$. This implies that $|\breve{B}_{r_t}(x)| < A_t$.

We consider the two events,

$$E_1 = [A \cap \breve{B}_{r_t}(x) = \emptyset]$$

and,

$$E_2 = [A \cap B_{r_{t-1}}(y) \neq \emptyset]$$

among sets A uniformly chosen among all subsets of X of a *fixed* size.

Now, since $B_{r_{t-1}}(y) \cap \breve{B}_{r_t}(x) = \emptyset$, $\mathbb{P}[E_2|E_1] \geq \mathbb{P}[E_2]$, and so

$$\mathbb{P}[E_1 \wedge E_2] = \mathbb{P}[E_2|E_1] \cdot \mathbb{P}[E_1] \geq \mathbb{P}[E_2] \cdot \mathbb{P}[E_1]$$

Now,

$$\mathbb{P}[E_1] \geq \binom{|X|}{|A|}^{-1} \binom{|X| - A_t}{|A|} \geq \Big(1 - \frac{A_t}{|X| - |A|}\Big)^{|A|}$$

Choosing $|A| = \frac{|X|}{A_t}$, we get $\mathbb{P}[E_1] \geq e^{-2}$.

Similarly,

$$\mathbb{P}[E_2] \geq 1 - \binom{|X|}{|A|}^{-1} \binom{|X| - A_{t-1}}{|A|} \geq 1 - \Big(\frac{|X| - A_{t-1}}{|X|}\Big)^{|A|}$$

At this point, we need to specify some relationship between A_{t-1} and A_t *which does not depend on* t. So choosing, say, $\frac{A_t}{A_{t-1}} = e$ and using again that $|A| = \frac{|X|}{A_t}$,

$$\mathbb{P}[E_2] \geq \frac{\sqrt{e} - 1.1}{\sqrt{e}}$$

So, for our choices, $\mathbb{P}[E_1 \wedge E_2] \geq p$ for some universal p. Now, we have chosen $|\Lambda_k|$ sets of size k, each of which, has probability at least p of satisfying $E_1 \wedge E_2$. Now suppose, we want to choose enough sets for Λ_k so that there is a very high probability (say $1 - \frac{1}{|X|}$) that at least a fixed proportion (say $a|\Lambda_k|$) satisfy $E_1 \wedge E_2$. We refer to the following central limit-type theorem:

Theorem 9.0.4. *Cramer's Inequality (see [49])*

Let $\{X_n\}_1^\infty$ *be a sequence of* $\mathbb{P}$*-independent random variables with common distribution* μ*, assume that the associated moment generating function* $M_\mu(\xi)$ *satisfies,*

$$M_\mu(\xi) \equiv \int_{\mathbb{R}} e^{\xi x}\, d\mu(x) < \infty \text{ for each } \xi.$$

If m *is the mean of* μ *and* $a \in (-\infty, m]$*, then*

$$\mathbb{P}(\bar{S}_n \leq a) \leq e^{-nI_\mu(a)}$$

Where $\bar{S}_n$ is the average of the first n random variables, and

$$I_\mu(a) \equiv \sup\{\xi a - \log(M_\mu(\xi)) \,:\, \xi \in \mathbb{R}\}$$

is the Legendre transform of $\log M_\mu$.

Now, in our context, the random variables are $\{0,1\}$ valued and are given by choosing a subset of X uniformly from the all the sets of size k. It is valued 1 if it satisfies $E_1 \wedge E_2$ and 0 otherwise.

Then $M_\mu(\xi) \leq 1 - p + pe^\xi$ for $\xi \leq 0$, and for $a < p$ we have $I_\mu(a) = O(1)$.

Therefore, by choosing n sets independently, the probability that at least $a \cdot n$ of them satisfy $E_1 \wedge E_2$ is greater than $1 - e^{-nO(1)}$. Now if we would like our sets to achieve our goal asymptotically almost surely, then we can set

$$e^{-nO(1)} = \frac{1}{|X|}$$

which tells us that $|\Lambda_k| = O(\log|X|)$

Now, turning back to our inequality–notice that $\frac{A_t}{A_{t-1}} = e$ implies $A_t = e^t$ and $|X| = e^t|A|$. Writing $|A| = 2^w$, we have $w = \log|X| - t$. Writing this out,

$$\sum_\Lambda \Big[|A||\Lambda_{|A|}|\Big]^{-1} |d(x,A) - d(y,A)| = \int_1^{|X|} \frac{1}{s} \sum_{\Lambda_s} [|d(x,A) - d(y,A)|]\, ds,$$

and choosing $w = \log s$

$$= \int_0^{\log|X|} |\Lambda_{e^w}|^{-1} \sum_{|A|=e^w} [|d(x,A) - d(y,A)|]\, dw = \int_0^{\log|X|} |\Lambda_{\frac{|X|}{A_t}}|^{-1} \sum_{|A|=\frac{|X|}{A_t}} [|d(x,A) - d(y,A)|]$$

$$\geq \sum_{t=0}^{T} c(r_t - r_{t-1}) = cr_T = \frac{c}{2} d(x,y)$$

with probability greater than $1 - \frac{1}{|X|}$.

Notice further that there are $\log|X|$ sets Λ_k, and each one contains $O(\log|X|)$ elements. So, asymptotically almost surely we have embedded X into a Euclidean space of dimension $O(\log^2|X|)$ with metric distortion less than $O(\log|X|)$.

□

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