1

(a) Ten different balls may be chosen from forty in $C(40,10) = \binom{40}{10}$ ways.

(b) There are $P(40;10,10,10,10) = \frac{40!}{(10!)^4}$ different ways to put 10 balls into each of 4 (different) baskets. One may see this as

$$\binom{40}{10} \cdot \binom{30}{10} \cdot \binom{20}{10} \cdot \binom{10}{10} = \frac{40!}{(10!)^2} = P(40;10,10,10,10)$$

by reasoning as follows: there are $C(40,10) = \binom{40}{10}$ ways to pick 10 balls for the first basket, then $C(30,10) = \binom{30}{10}$ ways to pick 10 balls from the remaining 30 for the second basket, then $C(20,10) = \binom{20}{10}$ ways to pick 10 balls from the 20 still left for the third basket, and finally $C(10,10) = \binom{10}{10} = 1$ way to put the final 10 balls into the fourth basket.

(c) This is really the same question as part (c): the fourth "basket" is the collection of unselected balls. Thus the answer is the same: $P(40;10,10,10,10) = 40!/(10!)^4$. Another way of thinking of this is to consider first picking 30 balls from the original 40 (there are $C(40,30)$ ways of doing this), then putting 10 balls into each of three baskets (as above, there are $P(30;10,10,10)$ ways to do this). Thus the final answer is

$$\binom{40}{30} \cdot P(30;10,10,10) = \binom{40}{30} \cdot \binom{30}{10} \cdot \binom{20}{10} \cdot \binom{10}{10} = P(40;10,10,10,10),$$

as before.

(d) This problem is different from the previous problem in that the first three piles/baskets are interchangable, whereas the "fourth pile" (or the basket of unpicked balls) is distinct. That is, as in part (c) there are $P(40;10,10,10,10)$ ways of producing three *different* piles of 10 balls each from our original 30 balls. To answer the problem, we must not distinguish between these three piles. Since there are 3! ways to permute these three piles, there are $P(40;10,10,10,10)/3!$ ways to put 10 balls into each of 3 identical piles.

2

(a) We want to pick 10 coins from 4 types; there are

$$C(10+4-1,10) = \binom{13}{10} = \binom{13}{3} = 286$$

ways to do this.

(b) This situation is the same as part (a), except that we may pick at most 4 of one type (quarters). Thus one way to do this is to add the number of ways of choosing 10 with 0 quarters, with 1 quarter, and so on, up to 4 quarters. This gives

$$\binom{10+3-1}{10} + \binom{(10-1)+3-1}{(10-1)} + \binom{(10-2)+3-1}{(10-2)} + \binom{(10-3)+3-1}{(10-3)} + \binom{(10-4)+3-1}{(10-4)}$$

or

$$\binom{12}{10} + \binom{11}{9} + \binom{10}{8} + \binom{9}{7} + \binom{8}{6} = \binom{12}{2} + \binom{11}{2} + \binom{10}{2} + \binom{9}{2} + \binom{8}{2} = 230$$

ways. Another approach is to simply take the answer from part (a) and subtract off the number of ways of choosing 4 types and at least 5 of one type (quarters). This last term is simply the number of ways of choosing 5 objects from 4 types, so the answer is

$$\binom{10+4-1}{10} - \binom{5+4-1}{5} = \binom{13}{3} - \binom{8}{3} = 230,$$

as before.

(c) There are two ways to do this problem, as in part (b). We first, however, take one penny, nickel, dime, and quarter; thus we're looking for how many ways to pick 6 (remaining) items of 4 types given that there are only 3 (more) of one type (the quarter). The first way from part (b) is now

$$\binom{6+3-1}{6} + \binom{(6-1)+3-1}{(6-1)} + \binom{(6-2)+3-1}{(6-2)} + \binom{(6-3)+3-1}{(6-3)}$$

or

$$\binom{8}{6} + \binom{7}{5} + \binom{6}{4} + \binom{5}{3} = \binom{8}{2} + \binom{7}{2} + \binom{6}{2} + \binom{5}{2} = 74$$

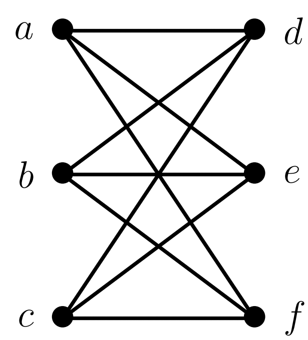
ways. The second approach (subtracting off the total using at least 5 total quarters) is

$$\binom{(10-4)+4-1}{(10-4)} - \binom{(10-3-5)+4-1}{(10-3-5)} = \binom{9}{6} - \binom{5}{2} = \binom{9}{3} - \binom{5}{2} = 74$$

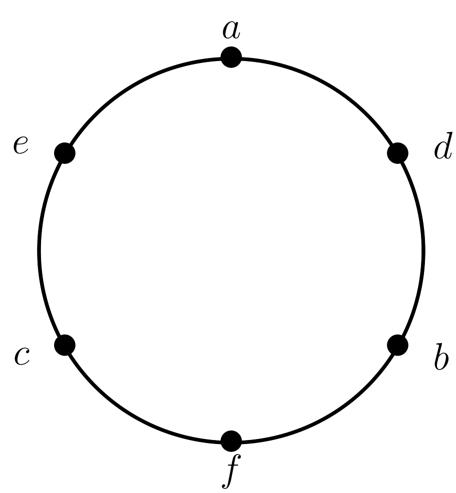
ways. (Here the $10-4$ of the first term is 10 coins minus one of each type, and the $10-3-5$ of the second term is 10 coins minus one of each non-quarter type minus at least 5 quarters.)

3 We have two ways (other than Kuratowksi's theorem) to show a connected graph is non-planar. One is from Euler's theorem: if $e \not\leq 3v-6$, then G is not planar. Unfortunately, this won't work in this case, as $v = 6$ and $e = 9$. The other is the circle-chord method, which we present here.

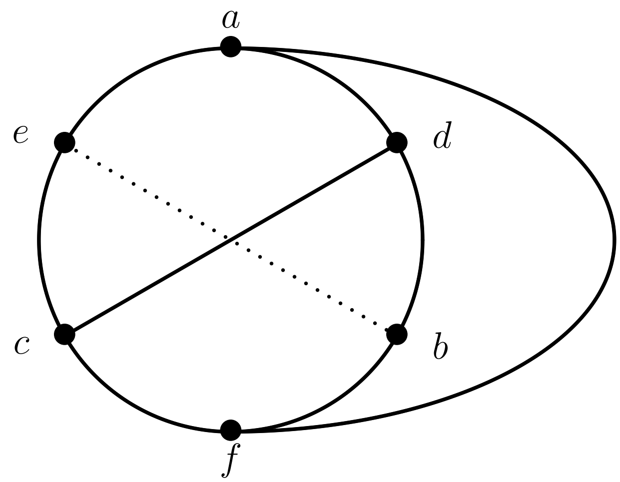
The idea is to draw the vertices of $K_{3,3}$ in a circle. First let's draw $K_{3,3}$:



Now let's draw it in a circle with only the edges involved in the circuit drawn:



We're still missing three edges: (a, f), (b, e), and (c, d). If we add two of these, one inside the circle and one (necessarily) outside, there is no way to add the third without crossing another edge:



The dotted edge (b, e) is the edge we cannot draw without crossing an existing edge. Thus $K_{3,3}$ is non-planar.

4 (a) The identity is perhaps easiest to see via a committee selection argument. Write $k = C(k, 1)$ and $n = C(n, 1)$, so the identity is

$$\binom{k}{1}\binom{n}{k} = \binom{n}{1}\binom{n-1}{k-1}$$

The left-hand side can be thought of as choosing a committee of k from n people, then choosing one of the k to be the chair of the committee. In the right-hand side, we start with choosing the chair of the committee from the original n people, then choose the remaining $k-1$ committee members from the remaining $n-1$ people. These are both choosing the same things, so the two sides are equal.

(b) We're trying to prove that

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + (-1)^k\binom{n}{k}^2 + \cdots + (-1)^n\binom{n}{n}^2 = (-1)^m\binom{2m}{m}.$$

where $n = 2m$.

Consider the polynomial $(1-x^2)^n$. By the binomial theorem, this is

$$\begin{aligned}(1-x^2)^n &= \sum_{k=0}^{n}\binom{n}{k}(-x^2)^k = \binom{n}{0} + \binom{n}{1}(-x^2) + \binom{n}{n}(-x^2)^2 + \cdots + \binom{n}{n}(-x^2)^n \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{2k} = \binom{n}{0} - \binom{n}{1}x^2 + \binom{n}{n}x^4 + \cdots + (-1)^n\binom{n}{n}x^{2n}.\end{aligned}$$

Then the coefficient of x^n in $(1-x^2)^n$ is $(-1)^m C(2m, m)$, the right-hand side of our equation. On the other hand, the polynomial $(1-x^2)^n$ can be written as $(1-x)^n(1-x)^n$. This product expands as

$$\left[\sum_{j=0}^{n}\binom{n}{j}(-x)^j\right]\cdot\left[\sum_{k=0}^{n}\binom{n}{k}x^k\right] = \left[\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j\right]\cdot\left[\sum_{k=0}^{n}\binom{n}{k}x^k\right].$$

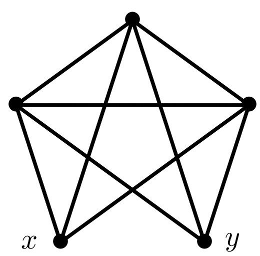
Multiplying this out, we find that the coefficient of x^n is simply

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\cdot\binom{n}{n-j}=\sum_{j=0}^{n}(-1)^j\binom{n}{j}^2,$$

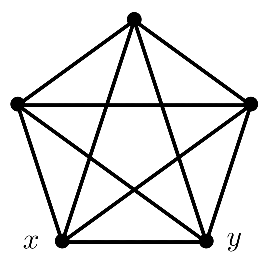
where the last equality follows from $C(n, n-j) = C(n, j)$. This is the left-hand side of the identity we wish to verify.

Since $(1-x^2)^n = (1-x)^n(1+x)^n$, these two expressions are the same, so our identity holds.

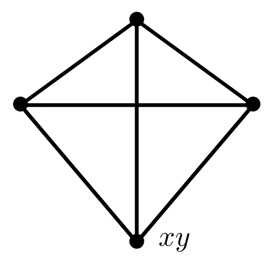
5 One can find the chromatic polynomial using methods of Chapter 2 (and it's really not very difficult). Instead, I will use the recurrence relation $P_k(G) = P_k(G^+_{xy}) + P_k(G^c_{xy})$ from Section 7.1. This works as follows: choose two vertices x and y that are not connected by an edge. Then G^+_{xy} is the graph with a new edge connecting x and y, and G^c_{xy} is the graph with x and y "coalesced" into a single vertex. (That is, we squish x and y together to form a new vertex xy; this new vertex connects to every vertex adjacent to either x or y.) These three graphs are shown here:



Original Graph G



The Graph G^+_{xy}



The Graph G^c_{xy}

Notice that $G^+_{xy} = K_5$, so $P_k(G^+_{xy}) = P_k(K_5) = k(k-1)(k-2)(k-3)(k-4)$. Moreover, $G^c_{xy} = K_4$, so $P_k(G^c_{xy}) = P_k(K_4) = k(k-1)(k-2)(k-3)$. Thus

$$\begin{aligned}P_k(G) &= P_k(G^+_{xy}) + P_k(G^c_{xy})\\ &= k(k-1)(k-2)(k-3)(k-4) + k(k-1)(k-2)(k-3)\\ &= k(k-1)(k-2)(k-3)\left[(k-4)+1\right]\\ &= k(k-1)(k-2)(k-3)^2.\end{aligned}$$

This is our final answer.

This recurrence relation is really not all that mysterious. The graph G^+_{xy} corresponds to those cases where x and y are the different colours, and the graph G^c_{xy} corresponds to those cases where x and y are the same colour.

6 (a) Each variable in our original equation can have as a value any non-negative integer. Thus each multiplicand in our generating function is of the form $1 + x + x^2 + x^3 + \cdots$. There are five variables, and therefore five terms in our generating function:

$$g(x) = \left(1 + x + x^2 + x^3 + \cdots\right)^5.$$

(b) Note that $g(x)$ from part (a) is really $1/(1-x)^5$, using the identity $1/(1-x) = 1+x+x^2+x^3+\cdots$. As done in the text, this simplifies to

$$\begin{aligned}g(x) &= \frac{1}{(1-x)^5}\\ &= 1 + \binom{1+5-1}{1}x + \binom{2+5-1}{2}x^2 + \binom{3+5-1}{3}x^3 + \cdots + + \binom{k+5-1}{k}x^k + \cdots.\end{aligned}$$

Thus the coefficient of x^7 is $C(7+5-1,7) = C(11,7) = C(11,4)$. This is the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 7.$$

(c) In this case, e_1 is between 2 and 5 (inclusive), so $3e_1$ is a multiple of 3 from 6 to 15. Thus this contributes $\left(x^6 + x^9 + x^{12} + x^{15}\right)$ to the generating function. Similarly, $11e_3$ contributes $\left(x^{11} + x^{22} + x^{33} + x^{44}\right)$. The $5e_2$ term contributes a series with terms whose exponents are all multiples from 0 up. Thus we get

$$g(x) = \left(x^6 + x^9 + x^{12} + x^{15}\right)\left(1 + x^5 + x^{10} + x^{15} + \cdots\right)\left(x^{11} + x^{22} + x^{33} + x^{44}\right).$$

7 This is an inclusion-exclusion problem. Let A_p be the set of integers from 1 to pqr (inclusive) that are multiples of p; define A_q and A_r similarly. Then we are interested in $N(\overline{A}_p\overline{A}_q\overline{A}_r)$. By the inclusion-exclusion theorem, this is

$$N(\overline{A}_p\overline{A}_q\overline{A}_r) = N - S_1 + S_2 - S_3,$$

where S_k is the sum of the sizes of the sets of intersection of k of the As, and $N = pqr$. To compute S_1, we notice that there are qr multiples of p between 1 and pqr (inclusive), so $N(A_p) = qr$. Thus, continuing in this vein, we see that

$$S_1 = N(A_p) + N(A_q) + N(A_r) = qr + pr + pq.$$

For S_2, we're looking at the intersections of two As. For example, A_pA_q is the set of integers between 1 and pqr that are multiples of both p and q. Since p and q are relatively prime, this means that A_pA_q is the set of integers between 1 and pqr that are multiples of pq. From this we see that $N(A_pA_q) = r$. From this we see that

$$S_2 = N(A_pA_q) + N(A_pA_r) + N(A_qA_r) = r + q + p.$$

Finally, we notice that $A_pA_qA_r$ is the set of integers from 1 to pqr that are a multiple of all of p, q, and r. There is only one such integer (namely, pqr) and so $S_3 = N(A_pA_qA_r) = 1$. Thus

$$\begin{aligned} N(\overline{A}_p\overline{A}_q\overline{A}_r) &= N - S_1 + S_2 - S_3 \\ &= pqr - (qr + pr + pr) + (p + q + r) - 1 \\ &= (p-1)(q-1)(r-1). \end{aligned}$$

You aren't really expected to come up with the factorization in the last line, but it makes things look nicer.

This is a special case of the computation of Euler's phi function. We define $\phi(n)$ to be the number of integers between 1 and n which are relatively prime to n. For example, if $n = p$ is prime, then $\phi(p) = p - 1$. More generally, if $n = p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ is a factoring of n into a product of powers of distinct primes, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right).$$

To compare, the formula we just proved is

$$\phi(pqr) = pqr\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right).$$

8 (a) This problem asks for the number of ways that five men can redistribute their five hats. This is a permutation of 5 different items, so there are 5! ways to do this.

(b) This part calls for an inclusion-exclusion argument. Let A_j be the set of redistributions of hats in which man j gets his own hat. Then the question asks for $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$, which by the inclusion-exclusion theorem is given by

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) = N - S_1 + S_2 - S_3 + S_4 - \cdots$$

(The trailing dots are there to indicate that this sum continues indefinitely, but in fact $S_k = 0$ for $k > 5$, as we will see.) Here S_k, as usual, is the sum of the number of ways that k of the A_js intersect. We simply compute these:

$$N = 5! \qquad \text{from part (a)}$$

$$\begin{aligned} S_1 &= N(A_1) + N(A_2) + N(A_3) + N(A_4) + N(A_5) \\ &= 5N(A_1) \\ &= 5\,(4!) = 5! \end{aligned}$$

$$\begin{aligned} S_2 &= \binom{5}{2} N(A_iA_j) \\ &= (5 \cdot 4/2!) \cdot (3!) = 5!/2! \end{aligned}$$

$$\begin{aligned} S_3 &= \binom{5}{3} N(A_iA_jA_k) \\ &= (5 \cdot 4 \cdot 3/3!) \cdot (2!) = 5!/3! \end{aligned}$$

$$\begin{aligned} S_4 &= \binom{5}{4} N(A_iA_jA_kA_\ell) \\ &= (5 \cdot 4 \cdot 3 \cdot 2/4!) \cdot (1!) = 5!/4! \end{aligned}$$

$$\begin{aligned} S_5 &= \binom{5}{5} N(A_iA_jA_k) \\ &= (5!/5!) \cdot (1) = 5!/5! \end{aligned}$$

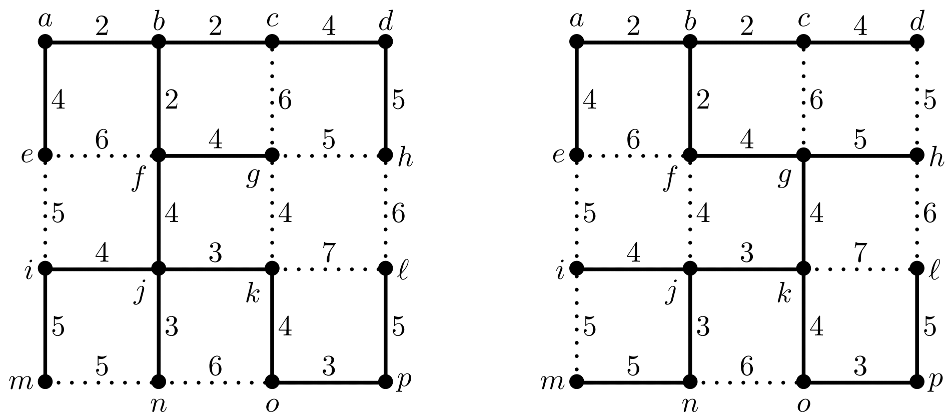
Here the only fact that we've used is that if k men get their own hat, then there are $(5 - k)!$ ways for the other men to pick hats. Summing, we get

$$\begin{aligned} N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) &= N - S_1 + S_2 - S_3 + S_4 - \cdots \\ &= 5! - 5! + 5!/2! - 5!/3! + 5!/4! - 5!/5! \\ &= 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right). \end{aligned}$$

This is our final answer.

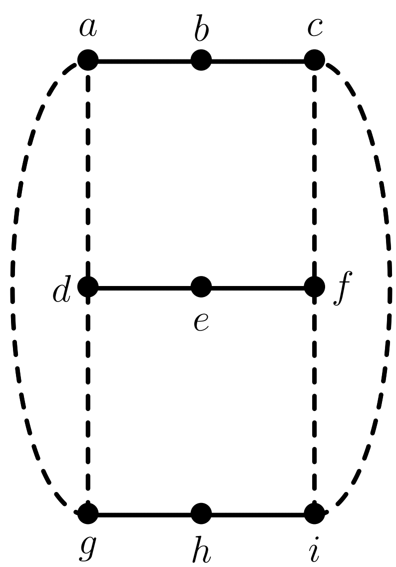
An interesting aside: the formula continues in the same way if we have n men and hats, not just 5. The series in the parentheses above will then converge to $e^{-1} = 1/e$ as n approaches infinity. Another way of saying this is: a permutation of n objects has no fixed points (that is, objects remaining in the same place) with probability roughly $1/e$.

9 We know two algorithms for finding a minimal spanning tree: Prim's algorithm (extend the existing tree by the minimal cost edge attached to the tree) and Kruskal's algorithm (add the minimal cost edge, whether or not it's connected to the existing tree). (In neither algorithm is one allowed to form a circuit). These two algorithms can produce the same, or different, minimal spanning trees, depending on choices made. Two possible minimal spanning trees are shown here:

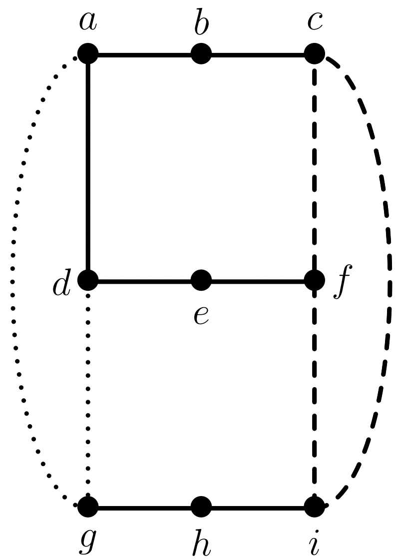


As usual with these sorts of problems, variations are possible, and some choices have been made. For example, in the graph on the left, using Kruskal's algorithm, we have chosen edges (f, j) and (f, g), which forces the exclusion of edge (g, k). All three of these edges are cost 4, however, so we could have chosen any two of them.

10 A Hamilton circuit is a single circuit passing through all the vertices of the graph. We begin by noticing that, if a Hamilton circuit exists in the given graph, it must path through vertex b, and therefore through edges (a, b) and (b, c). (This is Tucker's Rule I.) Similarly, edges (d, e), (e, f), (g, h), and (h, i) must be included. The graph therefore must include the following solid edges (edges not yet included are shown dashed):



Now consider vertex d. One of the two edges (a, d) and (d, g) must be included (Tucker's Rule 3); by the symmetry of the graph it doesn't matter which one. Let's choose (a, d). Then by Tucker's Rule 1, no other edges at a or d may be included; this eliminates edges (a, g) and (d, g). We now have a problem, as there is only the one single edge at g (excluded edges are shown dotted):

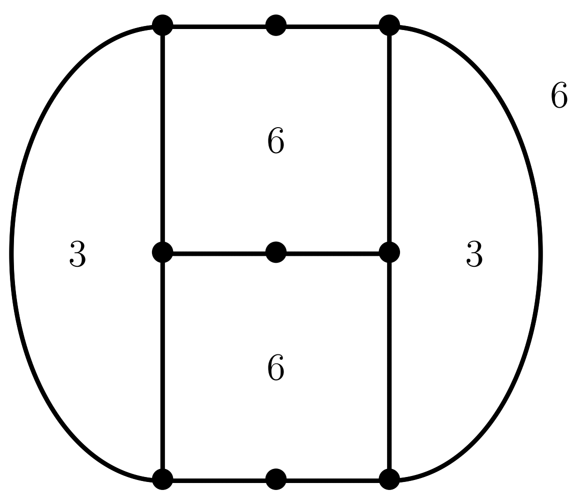


Thus there is no Hamilton circuit for this graph.

Another approach uses Grinberg's theorem (see page 64 of Tucker). Suppose that a planar graph G has a Hamilton circuit. Let r_i be the number of regions inside this circuit with i bounding edges, and r'_i be the number of regions outside the circuit with i bounding edges. Grinberg's theorem then says that

$$\sum_i (i-2)(r_i - r'_i) = 0. \tag{$*$}$$

Our first step is to figure out what kind of regions we have. From this picture below, it is easy to see that we have two regions with 3 bounding edges and three regions with 6:



That is, we know that

$$r_3 + r'_3 = 2$$
$$r_6 + r'_6 = 3.$$

On the other hand, equation $(*)$ becomes

$$(r_3 - r'_3) + 4(r_6 - r'_6) = 0.$$

It is now fairly easy to argue that this cannot happen. (Notice that $r_6 - r'_6 \neq 0$ since $r_6 + r'_6 = 3$; also notice that $r_3 - r'_3$ is one of $\{-2, 0, +2\}$.)

11 (a) This question requires that we place the letters $\{V, S, B, L, T, Y\}$ in the following boxes:

$$\square \ \mathrm{I} \ \square \ \mathrm{I} \ \square \ \mathrm{I} \ \square \ \mathrm{I} \ \square$$

so that the middle three boxes are not empty. First we place 3 of the 6 letters in these middle 5 boxes, then put the other 3 letters in the 5 boxes in

$$\binom{(6-3)+5-1}{(6-3)} = \binom{7}{3} \text{ ways.}$$

This is the number of ways to put 6 letters into these 5 boxes so that the middle three boxes are non-empty. But the letters are not identical – they are all different! – so they may be permuted in 6! ways. Thus the final answer is $C(7,3) \cdot 6!$.

(b) The six vowels here are two A's, one E, two I's, and one O. These six letters must be placed into the six boxes in:

$$\square \ \mathrm{M} \ \square \ \mathrm{N} \ \square \ \mathrm{N} \ \square \ \mathrm{T} \ \square \ \mathrm{X} \ \square$$

That is, each box must have precisely one vowel in it. Now there are w

$$\binom{6}{1} \cdot \binom{5}{1} \cdot \binom{4}{2} \cdot \binom{2}{2}$$

ways to make the selection. First choose where the E goes: there are $C(6,1)$ ways to choose a box for the E. Now choose the box for the O from the remaining 5 boxes; this gives $C(5,1)$ choices. Now choose *two* boxes for the A's; there are $C(4,2)$ ways to do this. Now the remaining 2 boxes must hold the two I's; that is, there is $C(2,2) = 1$ way to make this choice. Notice that the answer given above is

$$\frac{6!}{5! \cdot 1!} \cdot \frac{5!}{4! \cdot 1!} \cdot \frac{4!}{2! \cdot 2!} \cdot \left(\frac{2!}{2! \cdot 0!}\right) = \frac{6!}{1!1!2!2!} = P(6; 1, 1, 2, 2).$$

Thus our answer is $P(6; 1, 1, 2, 2)$. Why is this? Because we're really simply selecting 6 objects of four types: two each of two types, and one each of two types.