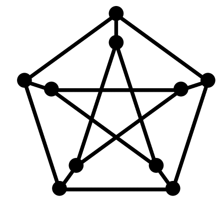
Math 344

Review Problem Solutions

Winter, 2004

1 [Graph (i)]

(a) *Is there an Euler circuit in the graph?*



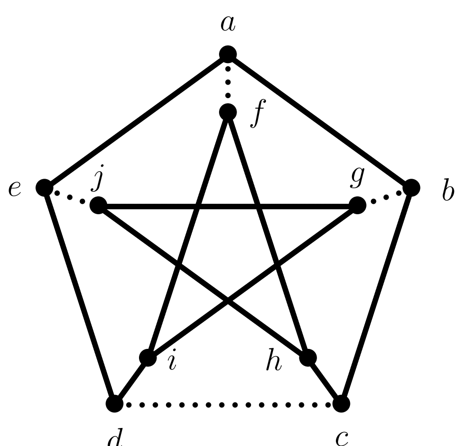
Each of the 10 vertices of this graph have degree 3, so there is no Euler circuit.

(b) *Is there an Euler path in the graph?*

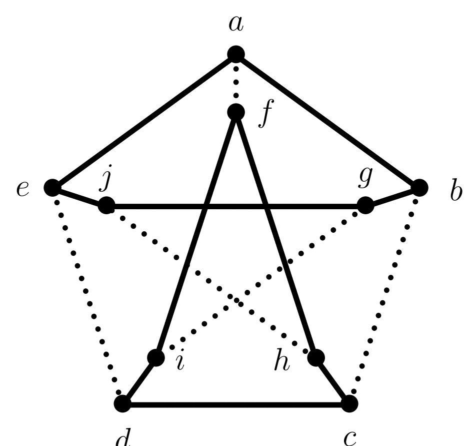
There are more than two vertices of odd degree, so there is no Euler path.

(c) *Is there a Hamilton circuit in the graph?*

There is no Hamilton circuit. The key here is to notice that a Hamilton circuit must use either two or four of the paths from the inside star to the outside pentagon. If we use only two, then one of the exterior edges must be excluded. We draw this as follows on the left, with the unused edge dotted:



Two Paths



Four Paths

Now we must include *three* edges from both h and i, which is impossible.

If, on the other hand, we include four paths from the exterior pentagon to the interior star (excluding only (a, f), say), we get the figure on the right. By including the four other paths, we must exclude the edges (d, e) and (b, c). On the other hand, since f now has degree 2, we must include (f, i) and (f, h). This forces us to exclude (g, i) and (h, j), so we end up with a disconnected graph.

(d) *Is there a Hamilton path in the graph?*

Yes. In the labelling above, $a - b - c - d - e - j - g - i - f - h$ is Hamilton path.

(e) *Is the graph bipartite?*

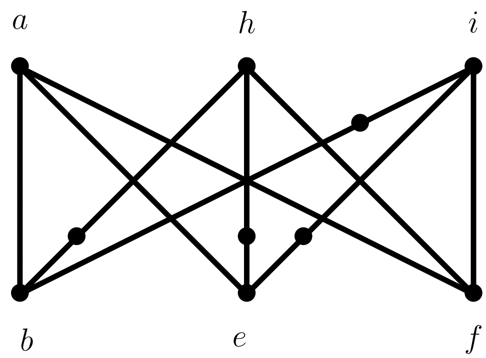
No. One simple way to see this is that $a - b - c - d - e - a$ is a cycle of odd length 5.

(f) *What is the chromatic number of the graph?*

The graph is 3-colourable (for example, colour a, h, i red, colour b, d, j yellow, and colour c, e, g green). It isn't 2-colourable, as then it would be bipartite, so it has chromatic number 3.

(g) *Is the graph planar?*

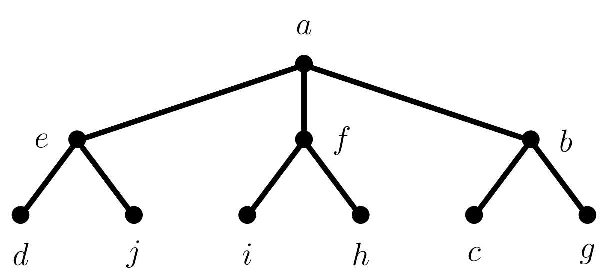
No, it's not. The simplest way to see this is to find a $K_{3,3}$ configuration in the graph. Here it is (and the four unlabelled dots are d, e, g, and j):



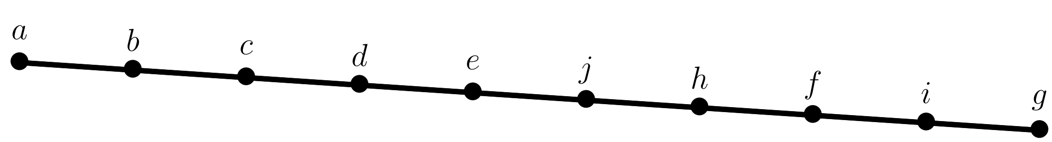
(h) *Find a spanning tree for the graph using the breadth-first method.*

(i) *Find a spanning tree for the graph using the depth-first method.*

Using the two methods, we produce very different trees. (Your answers may vary; it depends which vertex you choose to be root.) Breadth-first:

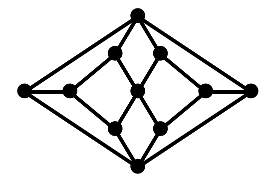


Depth-first:



1 [Graph (ii)]

(a) *Is there an Euler circuit in the graph?*



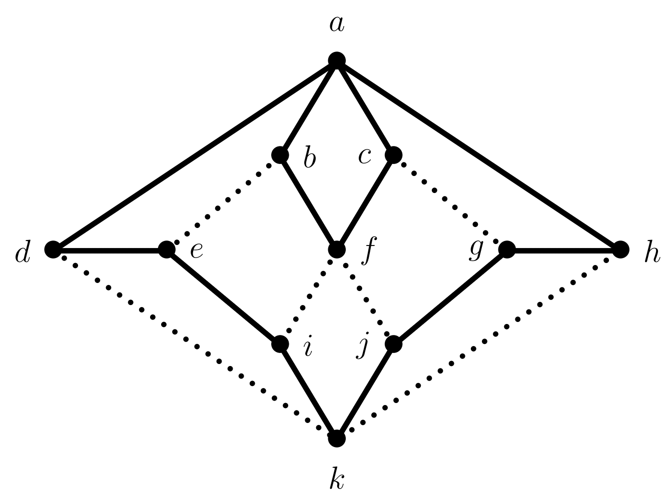
There are 10 vertices of degree 3, so there is no Euler circuit.

(b) *Is there an Euler path in the graph?*

No. There is more than two vertices of odd degree, so there is no Euler path.

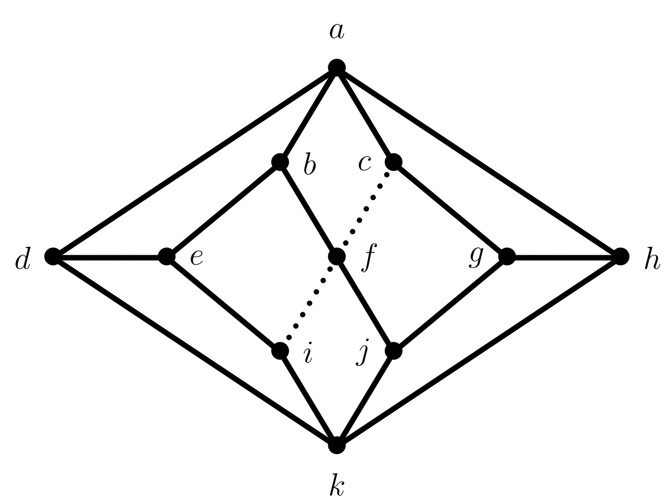
(c) *Is there a Hamilton circuit in the graph?*

This is a more difficult question. Let's consider two cases, each centred around the vertex f. First, consider the case when two *adjacent* edges (such as (b, f) and (c, f)) are included in the circuit, and second, when two *opposite* edges (such as (b, f) and (f, j)) are included. In the first case the picture is this (again, we've drawn dotted lines for edges excluded from the circuit):



We thus must exclude the other two edges at f: (f, i) and (f, j). This forces us to include the two remaining edges at i and j, and we proceed along to obtain the picture above. This is *not* a Hamilton circuit, as there are four edges incident to a.

In the other case, we include opposite edges at f, such as (b, f) and (f, j):



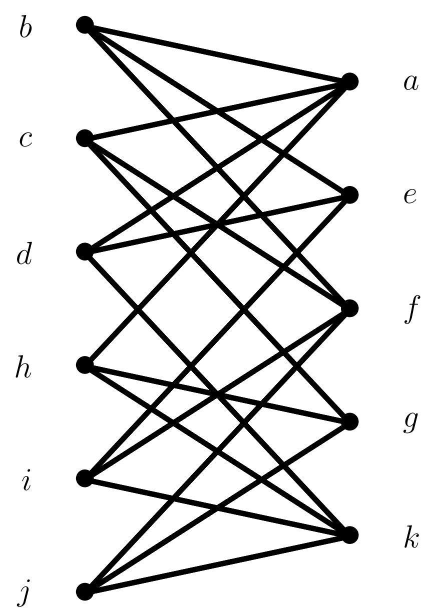
We now consider two more cases, based on whether or not the edge (g, j) is part of the circuit. Both cases produce problems for the circuit at the vertex d, so there is no Hamilton circuit.

(d) *Is there a Hamilton path in the graph?*

Yes: $c - a - d - k - h - g - j - f - i - e - b$ is one.

(e) *Is the graph bipartite?*

Yes:



(f) *What is the chromatic number of the graph?*

It is a bipartite graph, so the chromatic number is 2.

(g) *Is the graph planar?*

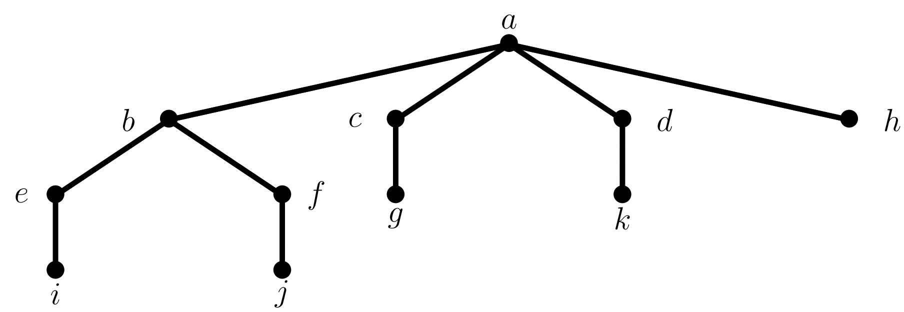
Of course. The original picture is a planar representation.

(h) *Find a spanning tree for the graph using the breadth-first method.*

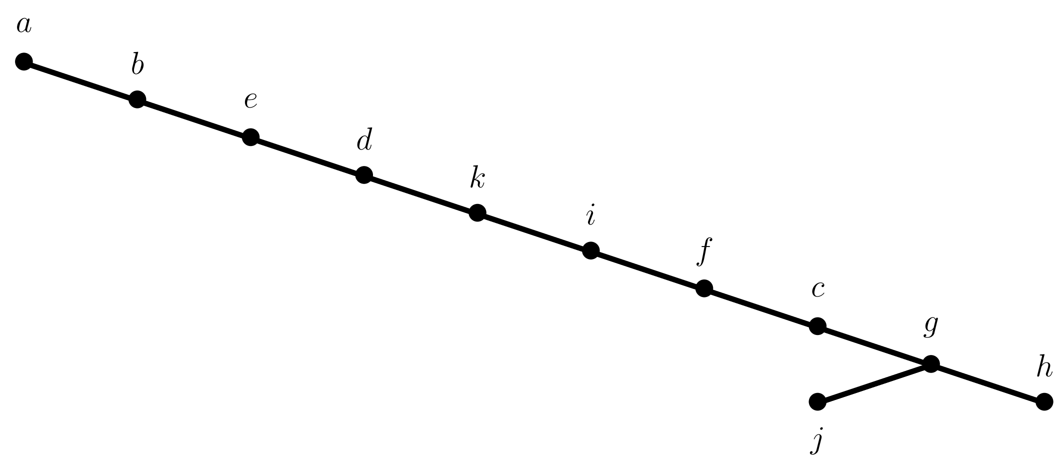
(i) *Find a spanning tree for the graph using the depth-first method.*

Using the two methods, we produce very different trees. (Your answers may vary; it depends which vertex you choose to be root.) (See the pictures on the next page.)

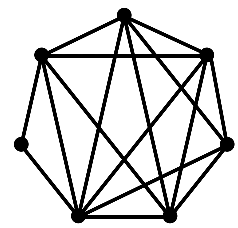
Breadth-first:



Depth-first:



1 [Graph (iii)]



(a) *Is there an Euler circuit in the graph?*

There are five vertices of odd degree, so there is no Euler circuit.

(b) *Is there an Euler path in the graph?*

There are five vertices of odd degree, so there is no Euler path.

(c) *Is there a Hamilton circuit in the graph?*

Yes, there is an obvious Hamilton circuit: just proceed around the outside of the graph in a circuit.

There are five vertices of odd degree, so there is no Euler path.

(d) *Is there a Hamilton circuit in the graph?*

Yes, there is an obvious Hamilton circuit: just proceed around the outside of the graph in a circuit.

(e) *Is there a Hamilton path in the graph?*

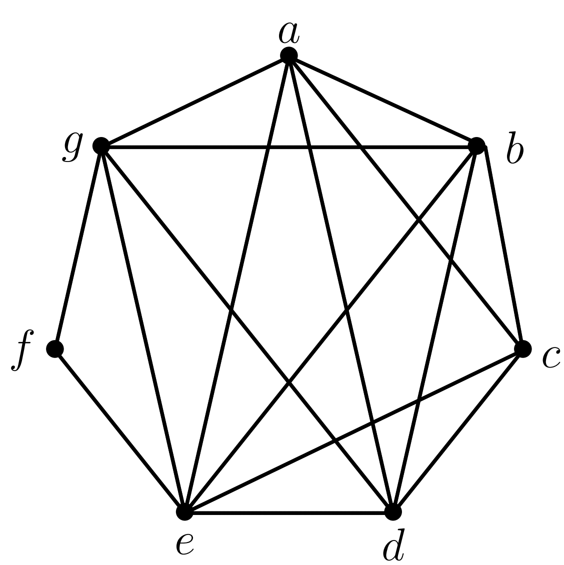
Yes. Simply use the Hamilton circuit, above.

(f) *Is the graph bipartite?*

No. There are, for example, several simple circuits of length 3. Since a bipartite graph only has circuits of even length, this is enough to show that the graph is not bipartite.

(g) *What is the chromatic number of the graph?*

The chromatic number is 5. One simple way to see this is to notice that the vertices labelled a through e (with their connecting edges) form a K_5 subgraph of the original graph.



Thus we need at *least* 5 colours. One can then colour f and g with the same colours fairly simply: put g the same colour as c, and f the same colour as d.

(h) *Is the graph planar?*

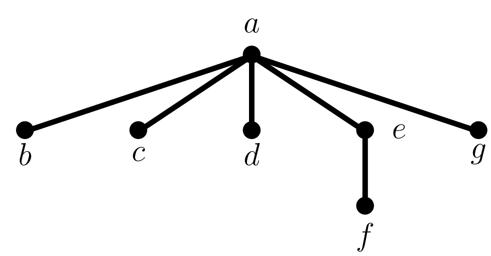
No. One way to do this is to notice the K_5 subgraph (as mentioned in the previous part) and apply Kuratowski's theorem. Another way to see this is to notice that the graph has $v = 7$ vertices and $e = 16$ edges. If it was a planar graph, then $e \leq 3v - 6$. Alas, $16 \not\leq 3(7) - 6 = 15$, so the graph is not planar.

(i) *Find a spanning tree for the graph using the breadth-first method.*

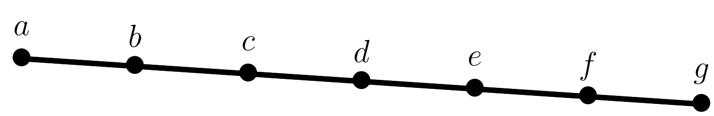
(j) *Find a spanning tree for the graph using the depth-first method.*

As above, the two methods will produce very different trees. (Your answers also may vary; it depends which vertex you choose to be root.)

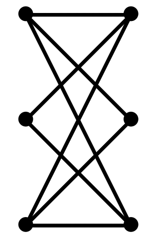
Breadth-first:



Depth-first:



1 [Graph (iv)]



(a) *Is there an Euler circuit in the graph?*

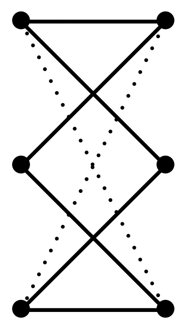
No, four of the vertices are of odd degree (degree 3).

(b) *Is there an Euler path in the graph?*

No, again, more than two of the vertices are of odd degree.

(c) *Is there a Hamilton circuit in the graph?*

Yes, along the solid paths below:



(d) *Is there a Hamilton path in the graph?*

Yes. See the Hamilton circuit.

(e) *Is the graph bipartite?*

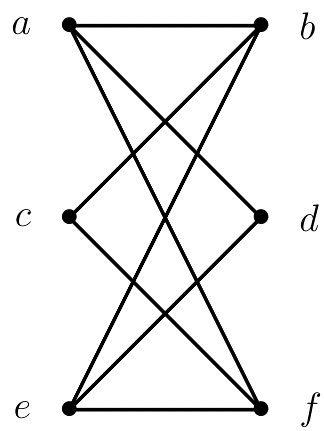
Yes, obviously.

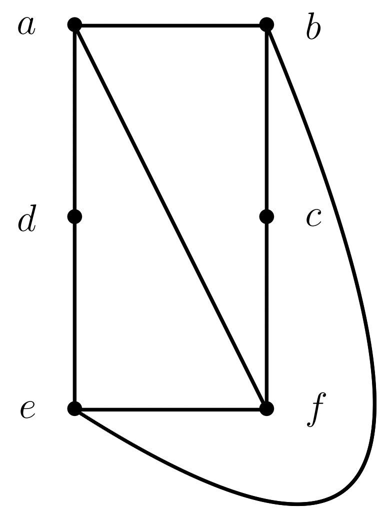
(f) *What is the chromatic number of the graph?*

The chromatic number of any bipartite graph is 2: just colour the vertices on the left one colour, and the vertices on the right another colour.

(g) *Is the graph planar?*

Yes. One simple way to see this is to switch the positions of c and d, then draw one of (a, f) and (b, e) inside and the other outside:



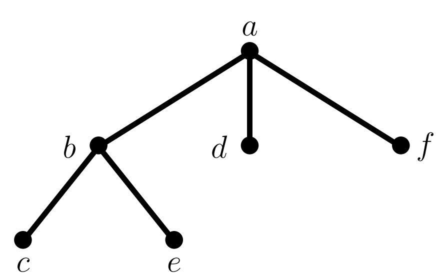


(h) *Find a spanning tree for the graph using the breadth-first method.*

(i) *Find a spanning tree for the graph using the depth-first method.*

As usual, the two methods produce different trees. (Your answers may vary; it depends which vertex you choose to be root.)

Breadth-first:



Depth-first: