

## Quiz Four Solutions

1 (This problem is parts (b) and (c) of Problem 9 of Tucker, Section 2.4. The three facts (A) through (C) that you may assume are problems 6(ab) and 9(a) of this section.)

(a) The key here is that  $\chi(\overline{G}) \geq q$ , so by fact (B):

$$\chi(G)\chi(\overline{G}) \geq \chi(G)q \geq n.$$

So why is  $\chi(\overline{G}) \geq q$ ? Remember that  $q$  is the size of the largest independent set in  $G$ . Since none of the vertices in this independent set are connected in  $G$ , they must *all* be connected in  $\overline{G}$ . Thus there is a  $K_q$  subgraph inside  $\overline{G}$ , so we need at least  $q$  colours in order to colour  $\overline{G}$ . That is,  $\chi(\overline{G}) \geq q$ .

(b) We will use a handy fact:  $a^2 + b^2 \geq 2ab$ . (This follows simply by rearranging the inequality  $(a - b)^2 \geq 0$ .) We'll then compute:

$$\begin{aligned} (\chi(G) + \chi(\overline{G}))^2 &= \chi(G)^2 + 2\chi(G)\chi(\overline{G}) + \chi(\overline{G})^2 \\ &= (\chi(G)^2 + \chi(\overline{G})^2) + 2\chi(G)\chi(\overline{G}) \\ &\geq 4\chi(G)\chi(\overline{G}) \quad \text{by our handy fact} \\ &\geq 4n \quad \text{by part (a).} \end{aligned}$$

Now take the square root of both sides to get  $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$ , which is the desired inequality.

- 2 (a) (This is Tucker Section 3.1, Problem 10.) In an  $m$ -ary tree, we know that the largest number of leaves  $\ell$  is  $\ell = m^h$  (Theorem 3). The Corollary to this theorem then implies that  $n = (m\ell - 1)/(m - 1) = (m^{h+1} - 1)/(m - 1)$ .

Another way to figure this out is to simply add. A height  $h$   $m$ -ary tree with the maximum number of vertices is one in which every vertex that isn't in level  $h$  has  $m$  children. Thus level 0 has 1 vertex (the root), level 1 has  $m$  vertices, level 2 has  $m^2$  vertices, and so on, to level  $h$  which has  $m^h$  vertices. The tree as a whole therefore has

$$1 + m + m^2 + \cdots + m^h = \frac{m^{h+1} - 1}{m - 1}$$

vertices.

- (b) The minimum number of vertices in a height  $h$   $m$ -ary tree is the one in which only one vertex per level has children. Thus, at each level from 1 to  $h$ , there are  $m$  vertices. Adding the one vertex at level 0 (the root), we have  $1 + hm$  vertices in the tree.