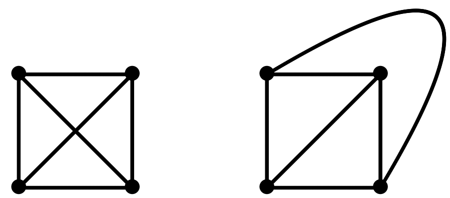
MATH 344 MIDTERM EXAM SOLUTIONS WINTER, 2004

1 (a) *Suppose $P_k(G)$ is the chromatic polynomial for a graph G. If $P_k(G)$ is zero for $k = 3$, then $P_k(G)$ is zero for $k = 2$ as well.*

True Recall that $P_k(G)$ is the number of ways to properly colour G with k colours. If there are zero ways to colour G with 3 colours, how can there be anything but zero ways to colour G with 2 colours?

(b) *The complete graph K_4 is planar.*

True Here are two pictures:



The picture on the left is the usual picture of K_4, while the picture on the right is a planar representation.

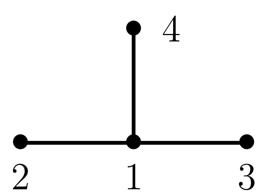
(c) *If a connected graph G with $\mathbf{e} > 1$ satisfies $\mathbf{e} \leq 3\mathbf{v} - 6$, then G is planar.*

False The conclusion is the other way: If a connected graph G with $\mathbf{e} > 1$ is planar, then it satisfies $\mathbf{e} \leq 3\mathbf{v} - 6$. The stated direction has some standard counterexamples: $K_{3,3}$ has $\mathbf{v} = 6$ and $\mathbf{e} = 9$ (so $\mathbf{e} \leq 3\mathbf{v} - 6$), but is non-planar.

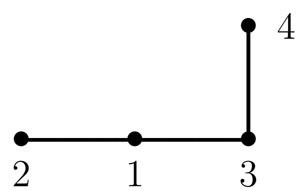
(d) *The graph to the right has an Euler path.*

True The graph has only two vertices of odd degree, so it has an Euler path.

(e) *The Prufer sequence $\{1, 3\}$ corresponds to the labeled graph below:*



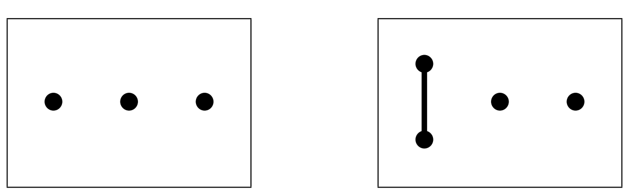
False The Prufer sequence $\{1, 3\}$ correspondes to the graph



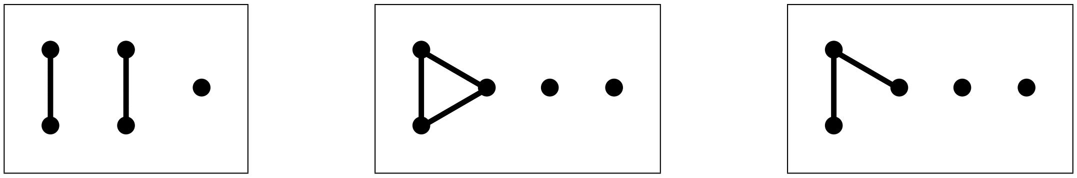
The graph given in the problem corresponds to the Prufer sequence $\{1, 1\}$.

2 We list all nonisomorphic undirected graphs with exactly three components and up to six vertices. It is clear that, in order to have three components, a graph must have at least three vertices. Thus we can enumerate the possible graphs by the number of vertices.

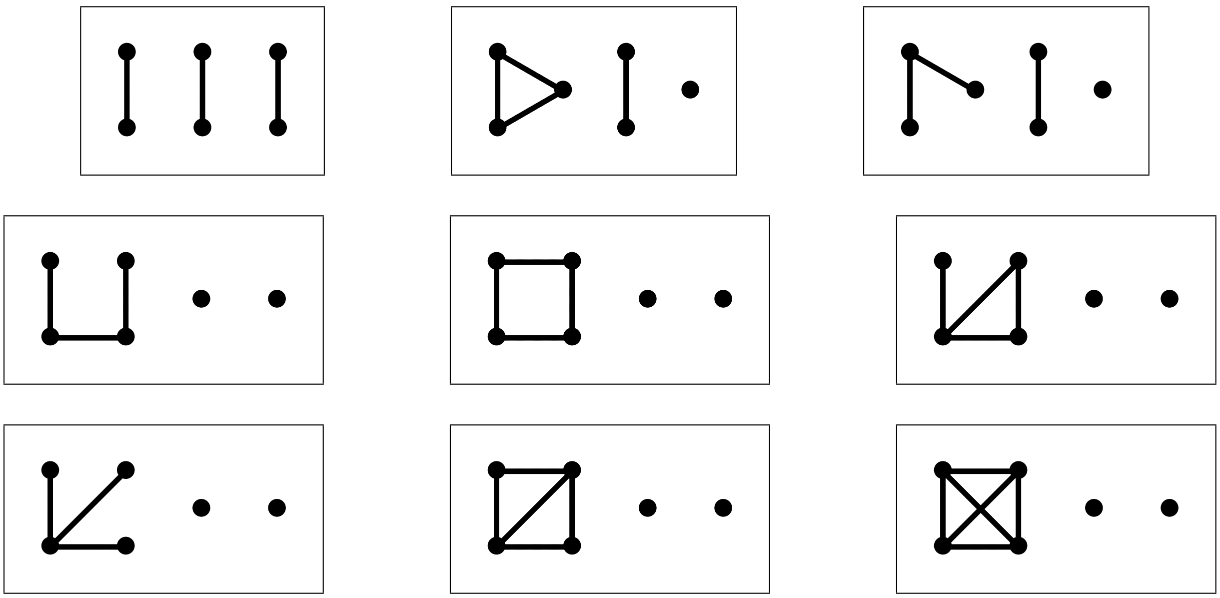
We begin with three or four vertices, where there is only one possible graph in which case:



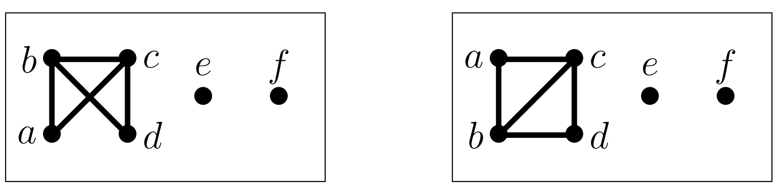
If there are five vertices, they can be grouped in two possible ways: $\{2, 2, 1\}$ and $\{3, 1, 1\}$. There are three possible graphs:



Finally, six vertices may be arranged as $\{2, 2, 2\}$, $\{3, 2, 1\}$, or $\{4, 1, 1\}$. There are nine possibilities:



Thus there are a total of 14 graphs (up to isomorphism) that have six or fewer vertices and three components. Notice that some figures that *look* different are, in fact, isomorphic. For example, the following two graphs are isomorphic (which is why the one on the left does not appear above):



3 (a) Recall that, in an m-ary tree, the number of leaves ℓ, internal vertices i, and total vertices $n = \ell + i$ are related simply:

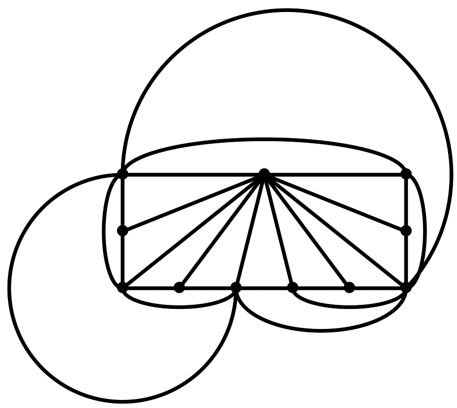
- Given n, $i = (n-1)/m$ and $\ell = \frac{(m-1)n+1}{m}$.
- Given i, $n = mi + 1$ and $\ell = (m-1)i + 1$.
- Given ℓ, $n = (m\ell - 1)/(m-1)$ and $i = \frac{\ell-1}{m-1}$.

(See the Corollary on page 97 of Tucker.) In this case we are told that $\ell = 65$ and $m = 5$, then asked for i. Using the appropriate formula from above, we see that $i = \frac{\ell-1}{m-1} = \frac{65-1}{5-1} = 16$.

One can also derive the formula using only the formula $n = mi + 1$ (from Theorem 2 on page 96 of Tucker) and $n = \ell + i$ (which says that all vertices are leaves or internal). From this we get $\ell + i = mi + 1$, or $i = (\ell - 1)/(m-1)$.

(b) The graph with 11 vertices that has the largest number of edges is K_{11}, the complete graph with 11 vertices. Each of these 11 vertices has degree 10; that is, each is connected to all the other vertices. Thus $2\mathbf{e} = 11 \cdot 10$, the total vertex degree of the graph. Solving for $\mathbf{e}$, the number of edges, we see that $\mathbf{e} = 55$. (This is a specific case of the fact that K_n has $n(n-1)/2$ edges.)

(c) A *planar* graph with 11 vertices is restricted by the fact that $\mathbf{3} \leq 3\mathbf{v} - 6 = 3 \cdot 11 - 6 = 27$. But is there really a planar graph with 11 vertices and 27 edges? Yes, there is: one such graph is shown below:



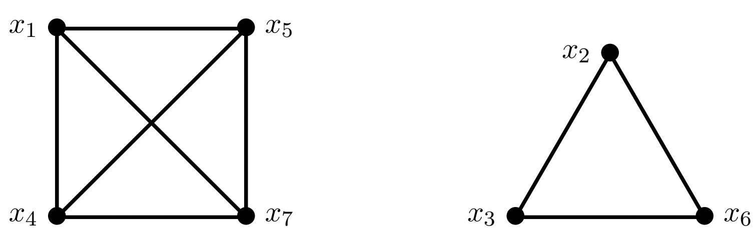
4 Since we are told that the minimal cost tour contains c_{21} (the leg of the tour from 2 to 1), we may eliminate row 2 ("from 2") and column 1 ("to 1"). Also, to eliminate the subcircuit $2-1-2$, we set $c_{12} = \infty$. Thus we're left with:

	2	3	4	5
1	∞	5	3	6
3	5	∞	5	7
4	4	4	∞	8
5	1	2	4	∞

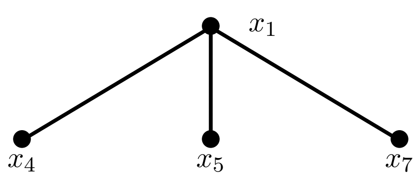
Remove 3 from row 1, 5 from row 3, 4 from row 4, and 1 from row 5; then a following 2 from column 5, we get the tree of Figure 1.

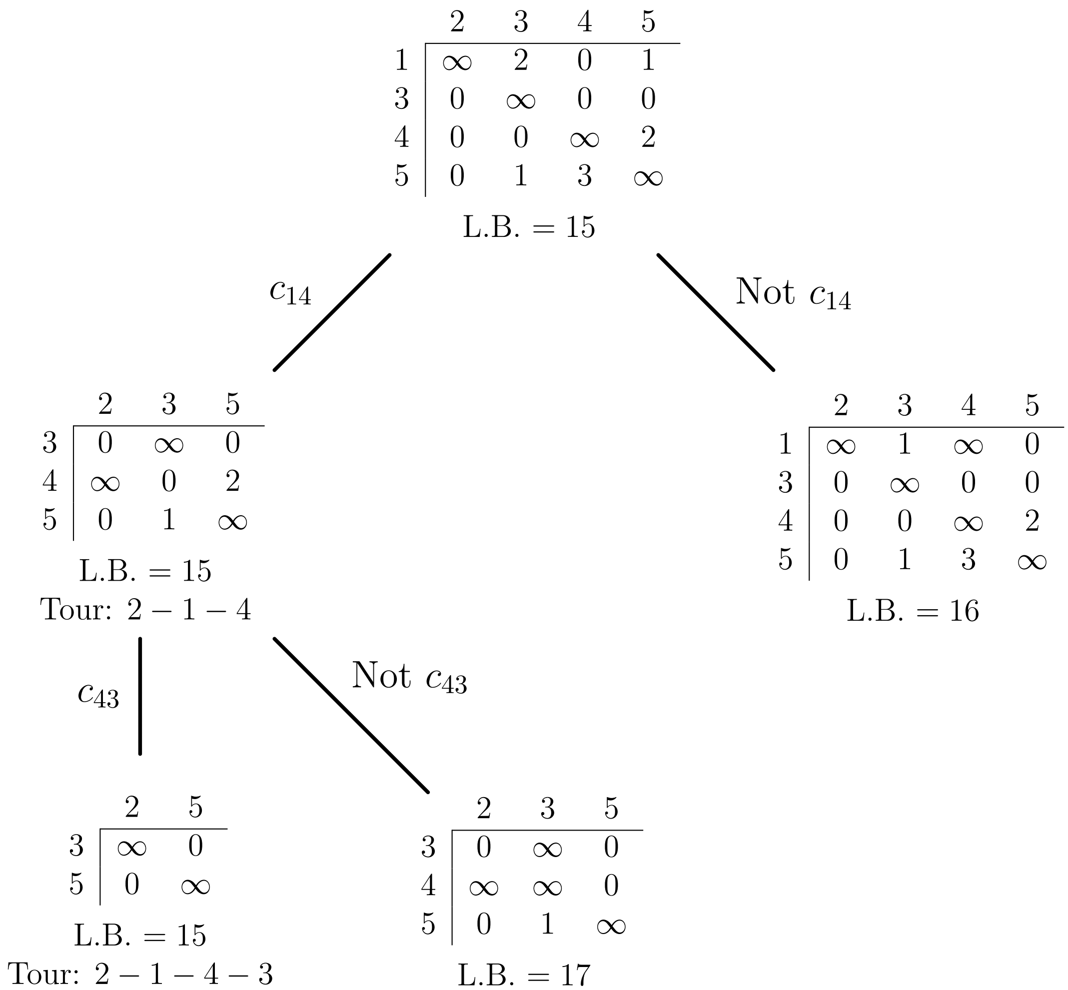
The result is the tour $2-1-4-3-5-2$ that has the minimum cost: 15.

5 One way to do this is to simply draw the graph. This is tedious and time-consuming, but will work:



Another (simpler) way is to construct a spanning tree. We start with an arbitary vertex (say, x_1) as the root, and construct a tree using the breadth-first method:





	2	3	4	5
1	∞	2	0	1
3	0	∞	0	0
4	0	0	∞	2
5	0	1	3	∞

L.B. = 15

	2	3	5
3	0	∞	0
4	∞	0	2
5	0	1	∞

L.B. = 15
Tour: $2-1-4$

	2	3	4	5
1	∞	1	∞	0
3	0	∞	0	0
4	0	0	∞	2
5	0	1	3	∞

L.B. = 16

	2	5
3	∞	0
5	0	∞

L.B. = 15
Tour: $2-1-4-3$

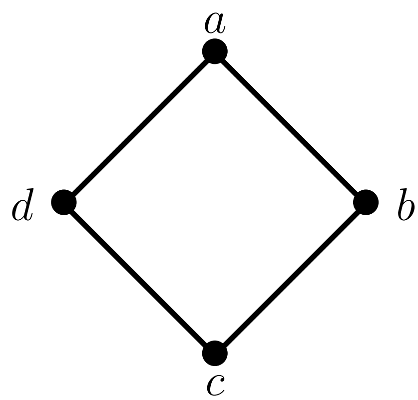
	2	3	5
3	0	∞	0
4	∞	∞	0
5	0	1	∞

L.B. = 17

Figure 1: Decision Tree For Problem 4

The tree cannot be extended past this, since (as is easy to check) none of x_4, x_5, or x_7 is connected to one of the remaining vertices (x_2, x_3, and x_6).

6 I have labeled the vertices in the graph below:



If we just start in and say that there are k choices of colours for a, then $k-1$ choices for each of b and d, then we end up with a problem at c. This is: there are $k-1$ choices if b and d are the same colour, and only $k-2$ choices if b and d are different colours. So we'll break up the problem into two cases:

Case I: Vertices b and d are different colours
Now we get k choices for a, $k-1$ choices for b, and $k-2$ choices for d (as d is different from both a and b). Finally, c has $k-2$ colour choices, as it must be different from both b and d (two different colours).

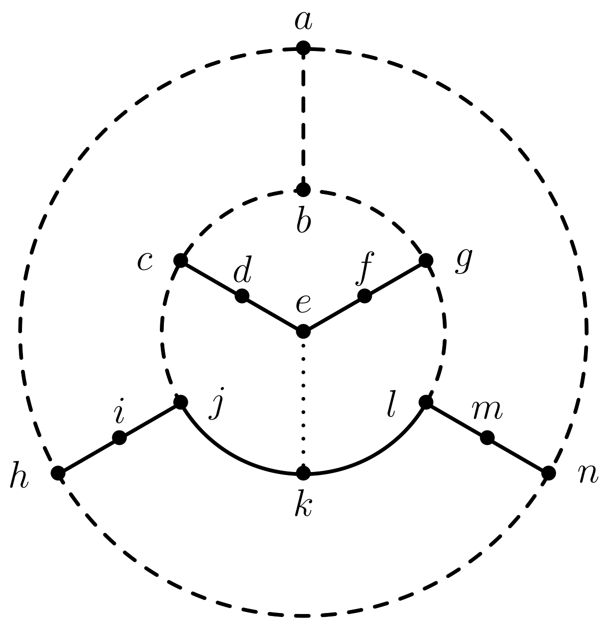
Case II: Vertices b and d are the same colour
There are again k choices for the colour of a and $k-1$ possibilities for b. Now d is determined as it is the same colour as b, and c has $k-1$ colouring options, as it only has to be different than the colour of b and d.

Thus the chromatic polynomial is

$$P_k(G) = k(k-1)(k-2)^2 + k(k-1)^2.$$

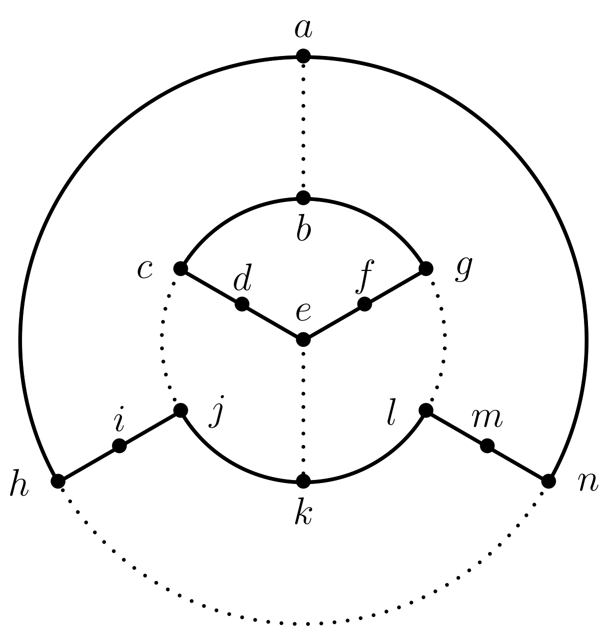
(Notice that this graph is C_4, the circuit of 4 vertices, and is one of the examples that was done in the book and in class.)

7 (15 points) Since the vertices d, f, i, and m have degree 2, all the adjacent edges to these vertices must be in any Hamilton circuit. (This is Tucker's Rule 1.) Now, at vertex e, we have already used two edges, so the third edge (e,k) must be excluded (Rule 3). This means the other two edges at vertex k must be included (Rule 1 again). The current state looks like this:



(Here solid lines are part of the circuit, dotted lines are excluded from the circuit, and dashed lines are *possibly* part of the circuit.)

Now we must also exclude (j, c) and (l, g) (Rule 1 again), so we are forced to include (g, b) and (c, b). This means that (a, b) is out, so (h, a) and (n, a) are in. (And, to finish things off, (h, n) must be excluded as we've used two edges already at both h and n.) We have:



As this is disconnected, this is *not* a Hamilton circuit. We have made no choices, so no Hamilton circuit exists.