University of Toronto
THE FACULTY OF ARTS AND SCIENCE

FINAL EXAMINATIONS, APRIL/MAY 2004

MAT 344H1S
Introduction to Combinatorics

Examiner: Peter M. Garfield
Duration: 3 hours

Last Name: ____________________

First Name: ____________________

Signature: ____________________

Student Number: | | | | | | | | | |

Question	1	2	3	4	5	6
Marks	20	20	15	20	15	20
Score						

Question	7	8	9	10	11	Total
Marks	15	20	15	15	25	200
Score						

- No books or aids of any kind may be used. Calculators are not allowed.
- Describe your arguments clearly – the method or proof (not the final answer) is what is important!
- If you need more room, use the backs of the pages and indicate to the examiner that you have done so.
- There are 11 pages on this exam (not counting this cover page). Please be sure that you have a complete exam.
- Good luck!

1 (20 marks) For all parts of this problem, you are given a collection of 40 different balls; one each of 40 different colours.

(a) (5 marks) How many ways are there to choose 10 of these balls?

(b) (5 marks) How many ways are there to put 10 balls into each of 4 (different) baskets?

(c) (5 marks) How many ways are there to put 10 balls into each of 3 (different) baskets?

(d) (5 marks) How many ways are there to put 10 balls into each of 3 (identical) piles? (Merely re-arranging the piles doesn't distinguish ways to put the balls into the piles.)

2 (20 marks) How many ways are there to pick 10 coins if:

(a) (6 marks) There are large piles of (identical) pennies, nickels, dimes, and quarters?

(b) (7 marks) There are large piles of (identical) pennies, nickels, and dimes, but only 4 (identical) quarters?

(c) (7 marks) There are large piles of (identical) pennies, nickels, and dimes, but only 4 (identical) quarters (as in part (b)), and in addition we require the selection includes at least one of each type of coin?

3 (15 marks) Show that $K_{3,3}$ is non-planar without appealing to Kuratowksi's theorem. (Recall that Kuratowksi's theorem says that a graph is non-planar if and only if it contains a subgraph that is a $K_{3,3}$ or K_5 configuration.)

4 (20 marks)

(a) (10 marks) Show that the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

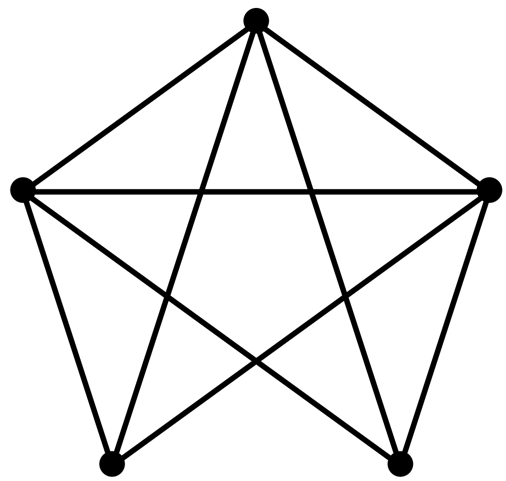
holds by using a combinatorial argument. (By a combinatorial argument, we mean something like a block-walking or committee selection argument, *not* an algebraic computation.)
Hint: It may help you to think of k as $C(k,1)$, and n as $C(n,1)$.

(b) (10 marks) Let $n = 2m$ be an even integer. Show that

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + (-1)^k\binom{n}{k}^2 + \cdots + (-1)^n\binom{n}{n}^2 = (-1)^m\binom{2m}{m}.$$

Hint: Consider the coefficient of x^n in $(1-x^2)^n = (1-x)^n(1+x)^n$.

5 (15 marks) Let G be the following graph.



Find the chromatic polynomial $P_k(G)$ of G.

6 (20 marks)

(a) (10 marks) Find a generating function for the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = r.$$

(b) (10 marks) Find the coefficient of x^7 in the generating function from part (a), and explain what it means.

(c) (10 marks) Find a generating function for the number of non-negative integer solutions to

$$3e_1 + 5e_2 + 11e_3 = r$$

if we require that $2 \leq e_1 \leq 5$, $e_2 \geq 0$, and $1 \leq e_3 \leq 4$.

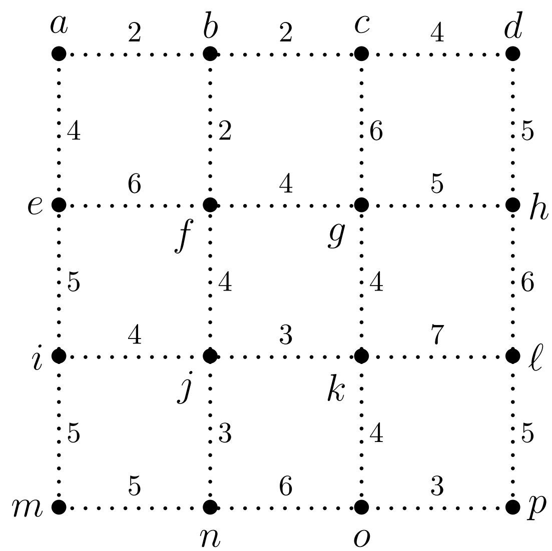
7 (15 marks) Let p, q, and r be distinct (that is, different) prime numbers. How many integers n, with $1 \leq n \leq pqr$, are relatively prime to pqr? (Recall that two integers are relatively prime if they have no common integer factors larger than 1. In particular, this means that 1 is relatively prime to all positive integers.)

8 (20 marks) Five men put their hats in a box, then each man picks a hat at random.

(a) (5 marks) How many ways are there for each man to pick one hat?

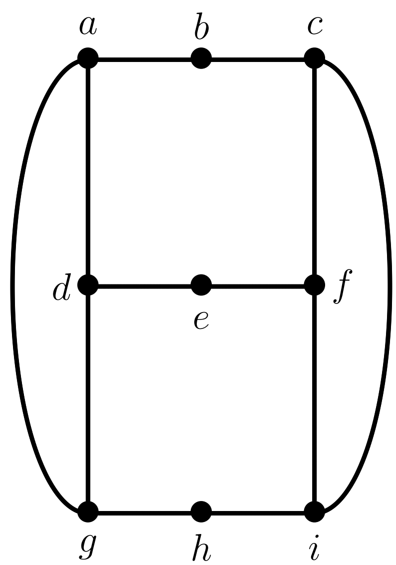
(b) (15 marks) How many of these ways result in no man picking his own hat? (This means that none of the five men picks his own hat.)

9 (15 marks) Here is a graph G with a cost associated to each edge of G. (It is drawn dotted, rather than solid, for your convenience.)



Find a minimal spanning tree for G.

10 (15 marks) Does there exist a Hamilton circuit in the graph below?



Explain your reasoning.

11 (25 marks)

(a) (10 marks) How many ways are there to arrange the letters in the word VISIBILITY with no pair of consecutive Is?

(b) (15 marks) How many ways are there to arrange the letters in the word EXAMINATION so that the consonants appear in alphabetical order and, in addition, no two vowels are adjacent?