Final Review Solutions MATH 344 COMBINATORICS

$\boxed{1}$ (a) There are 16 letters of 10 types (three t, two each of o, n, i, l, and one each of c, s, u, a, y). There are therefore

$$P(16;3,2,2,2,2,1,1,1,1,1) = \frac{16!}{3!(2!)^4}$$

ways to rearrange these letters.

(b) No consecutive t's means that we are going to fill boxes:

$$\boxed{}\; t \;\boxed{}\; t \;\boxed{}\; t \;\boxed{}$$

The requirement that the t's not be consecutive means that the two middle boxes not be empty. There are $C((13-2)+4-1,(13-2)) = C(14,11) = C(14,3)$ ways to fill these boxes with $13-2 = 11$ non-t's. To specify *which* non-t's are in which box, we multiply $C(14,3)$ by $P(13;2,2,2,2,1,1,1,1,1) = 13!/(2!)^4$. Thus the final answer is

$$\binom{14}{3} \cdot \frac{13!}{(2!)^4}.$$

(c) This is simply part (b) with a specified order of vowels. (As an American, I will consider the y as a vowel, so there are 7 vowels.) Now in the solution to part (b), we simply consider all 7 vowels as one "type" of letter, so we place the 13 letters via $P(13;7,2,2,1,1) = 13!/[7!\cdot(2!)^2]$. Thus the final answer is

$$\binom{14}{3} \cdot \frac{13!}{7!(2!)^2}.$$

$\boxed{2}$ This is a pigeonhole principle problem, and thus we shouldn't really expect to see this on the exam. But here's one way to solve it.

Let's write the 100 integers as $\{a_1, a_2, \dots, a_{100}\}$. Divide each of the 200 numbers $\pm a_j$ by 197 and find their remainder. (That is, find the 200 residue classes $\pm a_j$ mod 197.) There are only 197 possible remainders (residue classes): 0 through 196, so by the pigeonhole principle there must be two with the same remainder.

This isn't enough, however. Let's consider some cases:

Case I: If two of the integers are divisible by 197, then so is their sum.

Case II: If exactly one integer is divisible by 197, then discard this integer. There are still 99 other integers – giving 198 remainders – so *still* one remainder appears from two of the $\pm a_j$. (Note that it cannot be from $+a_j$ and $-a_j$, as then 197 divides $2a_j$ but not a_j, an impossibility.) The difference of these two with the same remainder is divisible by 197.

Case III: If none of the integers are divisible by 197, we can proceed as in Case II, but without needing to discard one integer.

$\boxed{3}$ This is a fairly standard inclusion-exclusion problem. The universe $\mathcal{U}$ will be the possible ways for the books to be distributed. Let A_k be the ways in which woman k gets her own book, so we're asked for $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$. The inclusion-exlusion theorem says that

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) = N - S_1 + S_2 - S_3 + S_4 - S_5 + \cdots ,$$

where $N = N(\mathcal{U}) = 10!$ and $S_1 = \sum_j N(A_j)$, $S_2 = \sum_{jk} N(A_jA_k)$, and so on. Notice that if we specify that k women get their own books, then we're simply counting distributions of $10 - k$ books to $10 - k$ people. Thus

$$S_k = \sum_{i_1,i_2,\ldots,i_k} N(A_{i_1}A_{i_2}\cdots A_{i_k}) = \binom{5}{k} \cdot (10-k)!$$

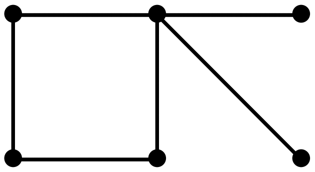
if $1 \le k \le 5$, and $S_k = 0$ for $k > 5$. (The $C(5,k)$ chooses which k women keep their book.) Thus the answer is

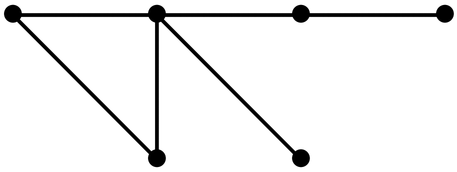
$$10! - \binom{5}{1} \cdot 9! + \binom{5}{2} \cdot 8! - \binom{5}{3} \cdot 7! + \binom{5}{4} \cdot 6! - \binom{5}{5} \cdot 5!$$

$\boxed{4}$ As suggested, group the six people into two sets: {Bob & friends} and {strangers to Bob}. Suppose there are three other people other than Bob in {Bob & friends}. If two of them know each other, then (together with Bob) there is a collection of three people that know each other. On the other hand, if none of the three know each other, this is a collection of three mutual strangers.

If there *aren't* three other people (other than Bob) in {Bob & friends}, then there are at least three people in {strangers to Bob}. If they all know each other, then they're a group of mutual friends. If two of them don't know each other, then these two, with Bob, are a group of three mutual strangers.

$\boxed{5}$ The two possibilities are





Suppose x is the vertex of degree 4. If two of the vertices adjacent to x are connected, then we have vertices of degree 4, 2, and 2 – the remaining vertices must have degree 1, 1, and 2. Thus the last vertex must be as shown on the right.

If none of the two vertices adjacent to x are connected to each other, then two of them must be connected to a common vertex. Why? We're running out of vertices: we need three vertices of degree 2 and we can add only one vertex. Thus you can argue that we must have the graph on the left.

$\boxed{6}$ (a) [Courtesy of Jordan, a student] First order the 10 numbers in $\{4,4,4,4,3,3,3,2,2,1\}$ relative to one another. There are $P(10;4,3,2,1)$ ways to do this.

For each of the above orderings, you want to squeeze in three fives and one glued-together (as Tucker would put it) "55". You have 11 spaces to choose from to do this. For example, if one of your orderings from above is 1223334444, then you want to pick 4 of the blank spaces from

$$\_\,1\,\_\,2\,\_\,2\,\_\,3\,\_\,3\,\_\,3\,\_\,4\,\_\,4\,\_\,4\,\_\,4\,\_$$

to fill in. There are $C(11,4)$ ways to choose 4 blanks out of the possible 11.

Once you've chosen 4 of those blank spaces, you need to order your three single 5's and one "55" relative to one another. Of the 4 spaces, choose 1 to put your "55" in, then put the single 5's in the rest. That's $C(4,1)$.

The final answer is

$$P(10;4,3,2,1)\cdot\binom{11}{4}\cdot\binom{4}{1}$$

ways to order the numbers.

(b) One way to approach this is to place the 2's and 3's (let's call them T), then fill the spaces in between them:

$$\boxed{}\,T\,\boxed{}\,T\,\boxed{}\,T\,\boxed{}\,T\,\boxed{}\,T\,\boxed{}$$

There are 10 non-T's (neither 2 nor 3), but 4 of them need to go between the T's. Thus, there are $C((10-4)+6-1,(10-4)) = C(11,6) = C(11,5)$ ways to distribute the non-T's, and $P(10;5,4,1) = 10!/[5!\cdot 4!]$ ways to arrange them. There are $P(5;3,2) = 5!/[3!\cdot 2!]$ ways to distribute the T's, so the final answer is

$$\binom{11}{5}\cdot\frac{10!}{5!\cdot 4!}\cdot\frac{5!}{3!\cdot 2!} = \frac{11!\cdot 10!}{6!\cdot 5!\cdot 4!\cdot 3!\cdot 2!}.$$

(c) Consider the coefficient of x^r in $(1-x)^n(1+x)^n$. The coefficients of these factors are

$$(1-x)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^k$$
$$(1+x)^n = \sum_{j=0}^{n}\binom{n}{j}x^j.$$

Multiply these together to get

$$(1-x)^n(1+x)^n = \sum_{r=0}^{2n}\left[\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k}\right]x^r$$

Thus what we want to simplify is the x^r coefficient of $(1-x)^n(1+x)^n = (1-x^2)^n$. This expands to

$$(1-x^2)^n = \sum_{k=0}^{n}\binom{n}{k}(-x^2)^k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{2k}.$$

Thus the x^r coefficient is

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k} = \begin{cases}(-1)^k\binom{n}{k} & \text{if } r = 2k\\ 0 & \text{if } r \text{ is odd}\end{cases}$$

$\boxed{7}$ This is the coefficient of x^{25} in the generating function

$$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^5,$$

divided by 6^5, the number of possible rolls of the six dice. Another way to look at this is to solve the Diophantine equation

$$e_1 + e_2 + e_3 + e_4 + e_5 = 25, \qquad 1 \le e_k \le 6,$$

or, equivalently,

$$e_1 + e_2 + e_3 + e_4 + e_5 = 20, \qquad 0 \le e_k \le 5.$$

If the restriction $e_k \le 5$ was removed, then there would be $C(20 + 5 - 1, 20) = C(24, 20) = C(24, 4)$ solutions. We can then exclude the restricted solutions via inclusion-exclusion. Let A_i be the solutions with $e_i \ge 6$, so we're interested in $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5)$. We compute:

$$\begin{aligned}
N(A_i) &= \binom{(20-6)+5-1}{(20-6)} = \binom{18}{14} = \binom{18}{4}\\
N(A_iA_j) &= \binom{(20-12)+5-1}{(20-12)} = \binom{12}{8} = \binom{12}{4}\\
N(A_iA_jA_k) &= \binom{(20-18)+5-1}{(20-18)} = \binom{6}{2} = \binom{6}{4}\\
N(A_iA_jA_kA_\ell) &= 0.
\end{aligned}$$

Thus, by the inclusion-exclusion theorem, our answer is

$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4\overline{A}_5) = \binom{24}{4} - \binom{5}{1}\cdot\binom{18}{4} + \binom{5}{2}\cdot\binom{12}{4} - \binom{5}{3}\cdot\binom{6}{4}.$$

$\boxed{8}$ Again, this is a pigeonhole principle problem that is unlikely to appear on our exam.

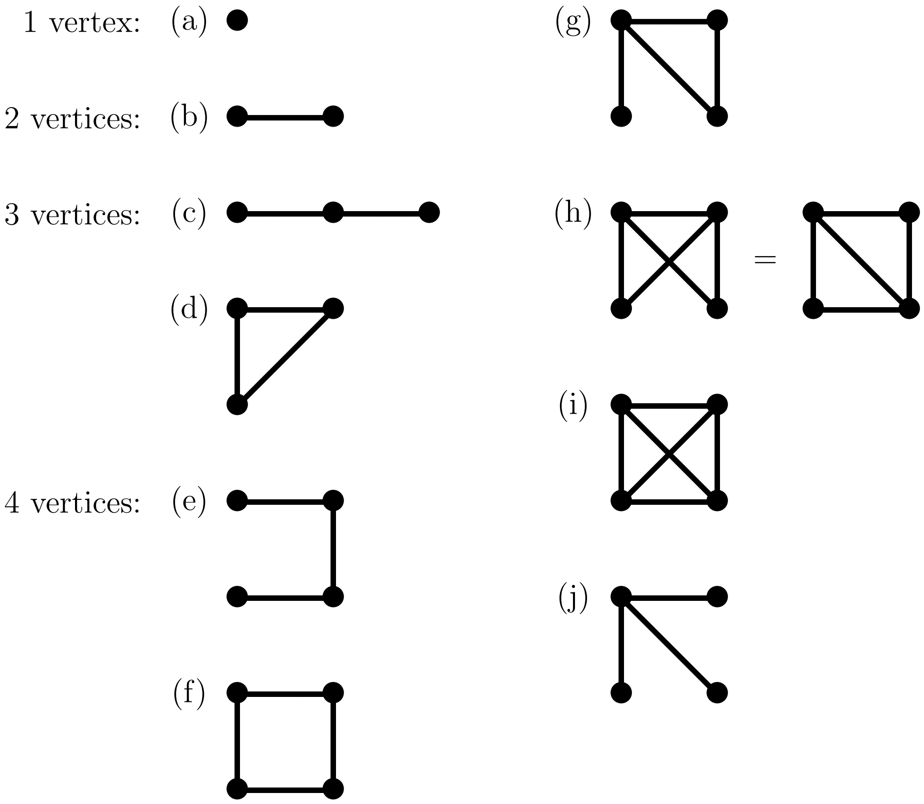
As suggested, let's sum a_E as E ranges over all the 2-sets of S. (A *k-set* of S is a subset of S having k elements.) Recall that a_E is the number of sets in $\mathcal{P}$ that contain the set E (so a_E may be anything from 0 to 10, the number of sets in $\mathcal{P}$). So what is the following sum?

$$\sum_{\text{all 2-sets } E} a_E$$

For each 4-set $\{a, b, c, d\}$ in $\mathcal{P}$, this sum counts all of its subsets E with $E \subset \{a, b, c, d\}$. There are $C(4, 2) = 6$ such subsets *for each element of* $\mathcal{P}$; therefore the sum is $6 \cdot 10 = 60$.

Since S has only 11 elements, there are only $C(11, 2) = 55$ possible 2-sets. Since the sum above is 60, there is at least one 2-set that is counted more than once; that is, there is at least one 2-set that is a subset of at least two elements of $\mathcal{P}$.

$\boxed{9}$ There are ten. acccording to my (quick) calculations:



$\boxed{10}$ This is a proof by induction. (As such, it's not really appropriate *by itself* as a problem for our midterm. Proofs by induction on graphs (or anything else combinatorial), however, are fair game.)

When $n = 1$, $n^5 - n = 0$ is divisible by 10. By induction that if $a_n = n^5 - n$ is divisible by 10, then so is $a_{n+1} = (n+1)^5 - (n+1)$. Thus it is enough to show that

$$a_{n+1} - a_n = \big((n+1)^5 - (n+1)\big) - \big(n^5 - n\big)$$

is divisible by 10. This is simply

$$\begin{aligned} a_{n+1} - a_n &= \binom{5}{0}n^5 + \binom{5}{1}n^4 + \binom{5}{2}n^3 + \binom{5}{3}n^2 + \binom{5}{4}n + \binom{5}{5} - n - 1 - n^5 + n \\ &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - 1 - n^5 \\ &= 5n^4 + 10n^3 + 10n^2 + 5n. \end{aligned}$$

Finally, $a_{n+1} - a_n$ is thus divisible by 10 precisely when $5n^4 + 5n = 5(n^4 + n)$ is divisble by 10, or $n^4 + n$ is divisible by 2. If n is even, then so is n^4, so $n^4 + n$ is even. On the other hand, if n is odd, so is n^4, so $n^4 + n$ is even. Hence $n^4 + n$ is always even, so $5(n^4 + n)$ is always divisible by 10. Therefore $a_{n+1} - a_n$ is divisible by 10, and so $a_n = n^5 - n$ is always divisible by 10.

$\boxed{11}$ (a) The answer is

$$P(15;5,4,3,2,1)=\frac{15!}{5!\cdot 4!\cdot 3!\cdot 2!\cdot 1!}.$$

(b) As in 1(b) or 6(b), we place the a's, then position the other letters in boxes surrounding them:

$$\boxed{}\ a\ \boxed{}\ a\ \boxed{}\ a\ \boxed{}\ a\ \boxed{}$$

The requirement that the a's not be consecutive means that the four middle boxes not be empty. There are therefore $C((10-4)+6-1,(10-4))=C(11,6)=C(11,5)$ ways to fill the six boxes with the $10-4=6$ remaining non-a's. To specify *which* non-a's are in which box, we multiply $C(11,5)$ by $P(10;4,3,2,1)=10!/[4!\cdot 3!\cdot 2!]$. Thus the final answer is

$$\binom{11}{5}\cdot\frac{10!}{4!\cdot 3!\cdot 2!}.$$

(c) This is the same question as part 6(b).

$\boxed{12}$ Another pigeonhole principle problem that isn't really appropriate for our final exam.

Let the 27 numbers be $\{a_1,\ldots,a_{27}\}$, and consider the new set of 54 numbers $\{a_1,\ldots,a_{27},a_1+13,a_2+13,\ldots,a_{27}+35\}$. Two of our original numbers differ by 13 if two numbers in this new set coincide.

If we exclude the numbers 1 through 13 and 52 to 64, we are left with 28 possibilities (14 through 51) for these 54 numbers. Of course, 13 may be 1 through 13, and 13 may be 52 through 64, so there may only be 28 numbers to fit into these 28 slots. So there are two cases:

Case 1: Our 54 numbers include 1 through 13 and 52 through 64. This means our original set was $\{1,2,\ldots,13,x,39,40,41,\ldots,51\}$, where the x is our 27th number (not 1 through 13 or 39 to 51). If $14\le x\le 26$, then x and $x-13$ are both in the original set. On the other hand, $27\le x\le 38$, then x and $x+13$ are in the original set. In either situation, two of our original numbers differ by 13.

Case 2: Our 54 numbers do not include 1 through 13 and 52 through 64. In this case, we have at least $54-12-13=29$ numbers from 14 to 51, and so we have a duplicate. That is, two of our original numbers (which were assumed to be distinct) must differ by 13.

$\boxed{13}$ If there was no restriction on boys getting more than 5 pieces of candy, then the total number would be $C(6+30-1,30)=C(35,30)=C(35,5)$ (this is assuming all the children are different, but that the candies are all the same). How many ways are there to distribute the candies so that one boy gets *at least* 6 pieces? Yes, that's right, it's turned into an inclusion-exclusion problem.

Let A_k be the set of distributions in which boy k gets more than 5 pieces, so we're interested in $N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4)$, or

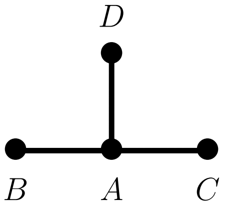
$$N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4)=N-S_1+S_2-S_3+S_4.$$

(Note $S_k=0$ for $k\ge 5$, since there are only four A_k's.) When computing S_k, we need to count the ways to give 6 or more candies to k boys. This is the number of ways to distribute

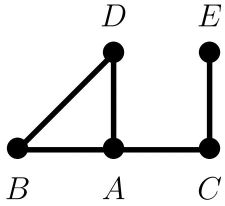
$r = 30 - 6k$ (identical) candies to all $n = 6$ (different) children, or $C(6+(30-6k)-1, 30-6k) = C(35-6k, 30-6k) = C(35-6k, 5)$. Then $S_k = C(4,k)C(35-6k,5)$, giving an answer of

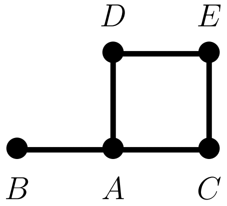
$$\begin{aligned} N(\overline{A}_1\overline{A}_2\overline{A}_3\overline{A}_4) &= N - S_1 + S_2 - S_3 + S_4 \\ &= \binom{35}{5} - \binom{4}{1}\binom{29}{5} + \binom{4}{2}\binom{23}{5} - \binom{4}{3}\binom{17}{5} + \binom{4}{4}\binom{11}{5}. \end{aligned}$$

14 Let us begin with a vertex of degree 3 and the three adjacent vertices:



Without loss of generality, assume that C is connected to the fifth vertex E. If B and D are connected, then we get the graph on the left, below. If B and D are *not* connected, then one of them is degree 1 and the other is connected to E, as in the graph on the right:





Since we haven't made any choices (up to isomorphism), all graphs (with 5 vertices of degrees 1, 2, 2, 2, 3) must be one of these two.

15 One way to solve this problem is via generating functions. That is, we're looking for the coefficient of x^{84} of

$$g(x) = \left(x^{16} + x^{18} + x^{20} + x^{22} + \cdots\right)\left(x^9 + x^{12} + x^{15} + x^{18} + x^{21} + x^{24}\right)\left(1 + x^7 + x^{14} + x^{21}\right).$$

(Here the exponent in the first term gives us the value of $2x$, the second $3y$, and the third $7z$.) It is not too hard, from here, to count that the coefficient of x^{84} is 12. (Note that the terms from the second and third term in the product must either both be even or both be odd. This means that the first term must be at least x^{42}.)