Recall : Let (V,ω) be a symplectic vector space.

↳ g an inner product $V \otimes V \to \mathbb{R}$
  ↳ symmetric
  ↳ bilinear
  ↳ positive-definite ($\Rightarrow$ nondegenerate)

↳ J a complex structure $V \to V$
  ↳ "multiplication by i"
  ↳ $J^2 = -\mathrm{Id}$

• compatibility: $g(u,v) = \omega(u, Jv)$

E.g. On $\mathbb{C}^n = \mathbb{R}^{2n}$ with the standard symplectic, inner product, and complex structures:

↳ Basis: $e_1, e_2, \_\_, e_n, f_1, f_2, \_\_, f_n$

↳ Dual basis: $e_1^*, e_2^*, \_\_, e_n^*, f_1^*, \_\_, f_n^*$

↳ $e_j^*(z_1, \_\_, z_n) = \mathrm{Re}\, z_j$

$f_j^*(z_1, \_\_, z_n) = \mathrm{Im}\, z_j$

↳ Written in this basis, the standard g, ω, and J are:

↳ $g = \sum_{j=1}^{n} (e_j^* \otimes e_j^* + f_j^* \otimes f_j^*)$

Represented by $I_{2n} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

$\hookrightarrow \omega_{std} = \sum_{j=1}^{n} e_j^* \wedge f_j^*$

Represented by $\begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}$

$\hookrightarrow Je_i = f_i \ , \ Jf_j = -e_j$

Represented by $\begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}$

Recall: Given $\varphi, \eta \in V^*$

$\hookrightarrow (\varphi \otimes \eta)(u,v) = \varphi(u)\eta(v)$

$\hookrightarrow (\varphi \wedge \eta) = \varphi \otimes \eta - \eta \otimes \varphi$

(since $V \wedge W \hookrightarrow V \otimes W$)

The compatibility $J \leftrightarrow g$ via $g(u,v) = \omega(u, Jv)$ defines a linear isomorphism

$$\begin{array}{ccc} \mathrm{End}(V) & \longleftrightarrow & (V \otimes V)^* \\ \wr\!\| & & \wr\!\| \\ \{\text{linear } V \to V\} & \longleftrightarrow & \{\text{bilinear } V \times V \to \mathbb{R}\} \\ & \Big\Downarrow \text{restriction} & \\ \mathcal{J}(V,\omega) & \longleftrightarrow & \mathcal{G}(V,\omega) \end{array}$$

Defn. Let $X \subseteq \mathbb{R}^N$, $Y \subseteq \mathbb{R}^k$

↳ $f: X \to \mathbb{R}^k$ is smooth if for each $x \in X$, $\exists$ open neighbourhood $x \in V \subseteq \mathbb{R}^N$ and a smooth (C^∞) extension of $f|_{X \cap V}$ to V $F: V \to \mathbb{R}^k$ (i.e. $F|_{X \cap V} = f|_{X \cap V}$).

↳ $f: X \to Y$ is smooth if $\iota \circ f$ is smooth, where $\iota: Y \hookrightarrow \mathbb{R}^k$ denotes inclusion.

↳ $f: X \to Y$ is a diffeomorphism if f has a smooth inverse $f^{-1}: Y \to X$.

Remark: The above gives us a way to talk about diffeomorphisms of objects which are not necessarily submanifolds.

Defn. $X \subseteq \mathbb{R}^N$ is an n-dimensional embedded submanifold if, for all $x \in X$, $\exists$ neighbourhood $U \subseteq X$ (in X) and an open $\Omega \subseteq \mathbb{R}^n$ along with a diffeomorphism (in the above sense) $\varphi: U \to \Omega$

Remark: In the above case, $\{\varphi: U \to \Omega\}$ is an atlas.

Thus, ~~it is enough to se~~

$$\mathcal{J}(V,\omega) \overset{J\leftrightarrow g}{\longleftrightarrow} \mathcal{G}(V,\omega)$$

is a diffeomorphism. So, it is enough to show one of them are an embedded submanifold to show that both are.

Observe: $Sp(V,\omega) \curvearrowright V$ linearly, so we obtain actions $Sp(V,\omega) \curvearrowright End(V)$ and

$$Sp(V,\omega) \curvearrowright (V\otimes V)^* \cong \{\text{bilinear } V\times V \to \mathbb{R}\}$$

These actions are given by:

For $A \in Sp(V,\omega)$, $L \in End(V)$,

$$A.L = A_* L = A \circ L \circ A^{-1}$$

For $A \in Sp(V,\omega)$, $g: V\otimes V \to \mathbb{R}$

$$A.g = (A^{-1})^* g = (u\otimes v \mapsto g(A^{-1}u, A^{-1}v))$$

$Sp(V,\omega)$ has subrepresentations

↳ $\{J: V\to V \mid J^2 = -Id_V\}$

↳ $\{\text{inner products } V\otimes V \to \mathbb{R}\}$

Further, the linear isomorphism $End(V) \cong (V\otimes V)^*$ is $Sp(V,\omega)$-equivariant. The subsets $\mathcal{J}(V,\omega)$, $\mathcal{G}(V,\omega)$ are $Sp(V,\omega)$-invariant.

If you have a complex structure at every tangent space, you get an almost complex structure, which plays a big role in symplectic geometry and introduces the notion of a holomorphic curve.

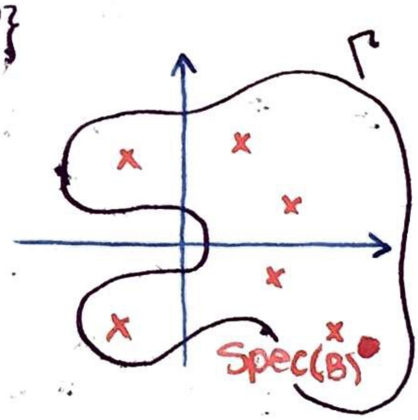
Powers of an operator

Let V be an $\mathbb{R}$-vector space. Given $\alpha \in \mathbb{R}$, we can define B^α on

$$\{B : V \to V \text{ linear} \mid \operatorname{Spec}(B) \subseteq \mathbb{C} \setminus (-\infty, 0]\}$$

$$\operatorname{Spec}(B) = \{z \mid \nexists (z\mathbb{1} - B)^{-1}\}$$

via the Cauchy integral,



$$B^\alpha := \frac{1}{2\pi i} \oint_{z \in \Gamma} z^\alpha (z\mathbb{1} - B)^{-1} dz$$

On $\mathbb{C} \setminus (-\infty, 0]$, choose the branch of z^α such that $x > 0 \Rightarrow x^\alpha > 0$.

Choose Γ which envellopes $\operatorname{Spec}(B)$.

↳ For $\alpha \in \mathbb{C}$, we get $B^\alpha : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$

↳ If $\alpha \in \mathbb{R}$, we get $B^\alpha : V \longrightarrow V$.

Similarly, we can define $\exp(B)$, $\mathrm{Log}(B)$, ...

less of a headache; exp is entire, so can just use the power series

same headache as on previous page.

Properties: ($\alpha \in \mathbb{R}$)

- If B is diagonalizable with positive eigenvalues, $\lambda_1, \_\_\_, \lambda_n$, then B^α is mutually diagonalizable with eigenvalues $\lambda_1^\alpha, \_\_\_, \lambda_n^\alpha$
- $\mathrm{Ker}\, \mathrm{ad}_B \subseteq \mathrm{Ker}\, \mathrm{ad}_{B^\alpha}$ (i.e. if B commutes with A, then it also commutes with B^α)
- $\forall A$, $(ABA^{-1})^\alpha = AB^\alpha A^{-1}$
- $(B,\alpha) \longmapsto B^\alpha$ is smooth.
- With respect to any inner product $\langle \cdot, \cdot \rangle$ on V:
  - If B is symmetric, so is B^α
  - If B is symmetric and positive definite, so is B^α.

Criterion for $Sp(\mathbb{R}^{2n})$

In coordinates:

$$\omega(u,v) = u^T \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} v$$

$B \in Sp(\mathbb{R}^{2n})$ means $\forall u,v$, $\omega(Bu,Bv) = \omega(u,v)$

$$\Leftrightarrow u^T B^T \underbrace{\begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}}_{\Omega} Bv = u^T \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} v \quad \forall u,v$$

$$\Leftrightarrow B^T \Omega B = \Omega$$

$$\Leftrightarrow \boxed{B^T = \Omega B^{-1} \Omega^{-1}}$$

The Lie algebra:

$$\mathfrak{sp}(V,\omega) := \{ b : V \to V \mid \omega(bu,v) + \omega(u,bv) = 0 \ \forall u,v \}$$

Then $b \in \mathfrak{sp}(V,\omega) \Leftrightarrow \forall u,v \quad u^T b^T \Omega v + u^T \Omega b v = 0$

$$\Leftrightarrow b^T \Omega + \Omega b = 0$$

$$\Leftrightarrow \boxed{b^T = -\Omega b \Omega^{-1}}$$

Exercise

① If $B \in Sp(\mathbb{R}^{2n})$, then so is B^T.
If $b \in \mathfrak{sp}(\mathbb{R}^{2n})$, then so is b^T.

② $\forall \alpha \in \mathbb{R}$ If $B \in Sp(\mathbb{R}^{2n})$ is symmetric, positive definite, then so is B^α.

Remark: The choice of coordinates is less important than the choice of Hermitian structure.

Polar decomposition of matrices

$$\begin{array}{ccc} O(k) \times \left\{ \begin{array}{l} \text{symmetric, positive-} \\ \text{definite matrices} \end{array} \right\} & \xrightarrow[\text{diffeo.}]{\sim} & GL(\mathbb{R}^k) \\ (C, B) & \longmapsto & A = C \cdot B \\ (\underbrace{A\sqrt{A^TA}^{-1}}_{C}, \underbrace{\sqrt{A^TA}}_{B}) & \longleftarrow & A \end{array}$$

Exercise Show $\sqrt{A^TA}$ is symmetric and positive-definite, and $A \cdot \sqrt{A^TA}^{-1}$ is orthogonal.

Claim If k is even,

$$A \in Sp(\mathbb{R}^k) \iff B \text{ and } C \in Sp(\mathbb{R}^k)$$

Proof. exercise (use the criterion for $Sp(\mathbb{R}^k)$).

So by restriction, we get: $(k = 2n)$.

$$\begin{array}{ccc} U(n) \times \left\{ \begin{array}{l} \text{symmetric, pos. definite} \\ \text{matrices in } Sp(\mathbb{R}^k) \end{array} \right\} & \xrightarrow[\text{diffeo}]{\sim} & Sp(\mathbb{R}^k) \\ (C, B) & \longmapsto & C \cdot B \\ & \searrow & B \cdot C \end{array}$$

two diffeomorphisms

This gives us

$$\left\{\begin{matrix}\text{symmetric} \\ \text{pos. definite} \\ \text{mat. in } Sp(\mathbb{R}^k)\end{matrix}\right\} \xrightarrow[\text{diffeo.}]{\sim} Sp(\mathbb{R}^k)/U(n)$$

But also:

$$Sp(\mathbb{R}^{2n})/U(n) \xrightarrow[\text{diffeo.}]{\sim} \mathcal{J}(V,\omega)$$

$$AU(n) \longmapsto A_* \underbrace{J_0}_{\substack{\text{standard complex} \\ \text{structure on } \mathbb{R}^{2n}}}$$

This map is onto by Gram-Schmidt for Hermitian structures.

It still remains to show:

$$\left\{\begin{matrix}\text{symmetric} \\ \text{pos. def.} \\ \text{mat in } Sp(\mathbb{R}^{2n})\end{matrix}\right\} \cong \mathbb{R}^m \quad \text{for some } m.$$