# Symplectic & hamiltonian vector fields

Let $(M,\omega)$ be a symplectic manifold.

$$\begin{array}{ccc} \mathfrak{X}(M) & \underset{}{\overset{\text{bijection}}{\rightleftarrows}} & \Omega^1(M) \\ X & \overset{\omega^\#}{\longmapsto} & \iota_X\omega \end{array}$$

vector fields

1-forms

Hamilton's equation: $\boxed{dH = \iota_{X_H}\omega}$

Through the bijection of $\omega^\#$:

$$\left\{\begin{array}{c}\text{Hamiltonian}\\ \text{vector fields}\end{array}\right\} \rightleftarrows \{\text{exact 1-forms}\}$$

$$\left\{\begin{array}{c}\text{symplectic}\\ \text{vector fields}\end{array}\right\} \rightleftarrows \{\text{closed 1-forms}\}$$

because $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X \underbrace{d\omega}_{0} = d(\iota_X\omega)$

So, $H^1(M) \cong \dfrac{\{\text{symplectic v. fields}\}}{\{\text{hamiltonian v. fields}\}}$

For $M$ connected (so constant functions $\cong \mathbb{R}$); we get an exact sequences

$$0 \to \mathbb{R} \to C^\infty(M) \to \mathfrak{X}_{ham}(M) \to 0$$

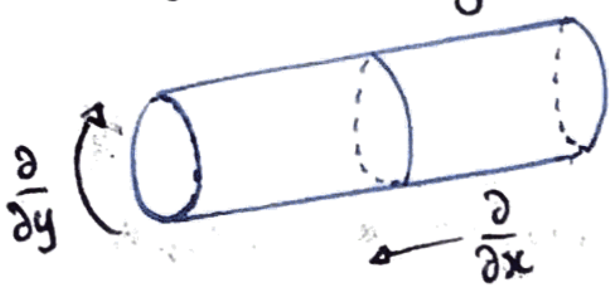
$$0 \to \mathbb{R} \to C^\infty(M) \to \mathfrak{X}_{symp}(M) \to H^1_{dR}(M) \to 0$$

We can upgrade this to maps of Lie algebras.

Eg. The following is a non-hamiltonian symplectic vector field:

$M = \mathbb{R} \times \mathbb{R}/\mathbb{Z}$, $\omega = dx \wedge dy$ with coordinates

$x$ and $y \pmod 1$




$\frac{\partial}{\partial y}$ is hamiltonian generated by $x$.

$\frac{\partial}{\partial x}$ is symplectic but not hamiltonian.

## Lie brackets of vector fields

$$[\xi, \eta] := \xi\eta - \eta\xi$$

as a derivation on $C^\infty(M)$.

If $\frac{d}{dt}\psi_t = \eta \circ \psi_t$. Then $[\xi, \eta] = \frac{d}{dt}\Big|_{t=0} (\psi_{t,*}\xi) =: \pm \mathcal{L}_\eta \xi$

convention choice

Now it's time for a new magic formula:

$$\iota_{[\xi,\eta]} = [\iota_\xi, \mathcal{L}_\eta] = [\mathcal{L}_\xi, \iota_\eta]$$

True for:
- → $df$
- → given $\alpha, \beta$, it follows for $\alpha \wedge \beta$

follows from $\mathcal{L}_X = d\iota_X + \iota_X d$

For a 1-form $\alpha$:

$$(d\alpha)(u,v) = u\,\alpha(v) - v\,\alpha(u) - \alpha([u,v])$$

differentiation

For a 2-form $\alpha$:

$$(d\alpha)(u_0,u_1,u_2) = u_0\,\alpha(u_1,u_2) + u_1\,\alpha(u_2,u_0) + u_2\,\alpha(u_0,u_1) - \left(\alpha([u_0,u_1],u_2) + \alpha([u_1,u_2],u_0) + \alpha([u_2,u_0],u_1)\right)$$

## Poisson brackets of functions on $(M,\omega)$

**Defn.** $\{f,g\} := X_g f$ where $dg = \iota_{X_g}\omega$

**Claim** This is a Lie bracket and $g \longmapsto X_g$ is an anti-Lie homomorphism.

**Proof.** Antisymmetry: $\{f,g\} = X_g f = \iota_{X_g} df = \iota_{X_g}\iota_{X_f}\omega = \omega(X_f,X_g)$

**Lemma & Proof.**

$$\omega([X_f,X_g],X_h) = -[X_f,X_g]h = -X_f X_g h + X_g X_f h = -\{\{h,g\},f\} + \{\{h,f\},g\}$$

Pf... Jacobi: Let $C := \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$.

$$0 = d\omega(X_f, X_g, X_h) = \sum_{\text{cyclic}} X_f\,\omega(X_g, X_h) - \omega([X_f, X_g], X_h)$$

$$= \{\{g,h\},f\} - \left(-\{\{h,g\},f\} + \{\{h,f\},g\}\right) \quad + \text{cyclic sums}$$

$$= C - (-C + C) = C.$$

Antilie-homomorphism:

$$([X_f, X_g] + X_{\{f,g\}})h = X_f X_g h - X_g X_f h + X_{\{f,g\}} h$$

$$= \{\{h,g\},f\} - \{\{h,f\},g\} + \{h,\{f,g\}\}$$

$$= -\left(\{\{g,h\},f\} + \{\{h,f\},g\} + \{\{f,g\},h\}\right)$$

$$= 0 \text{ by Jacobi.} \quad \blacksquare$$

Also, if $\xi, \eta$ are symplectic vector fields, then $[\xi,\eta]$ is the hamiltonian vector field ~~for~~ $X_H$ for

$$H = \omega(\eta, \xi).$$

Proof. $dH \overset{?}{=} \iota_{[\xi,\eta]}$

**Proof.** 

$$\begin{aligned}
\iota_{[\xi,\eta]}\omega &= [\iota_\xi, \mathcal{L}_\eta]\omega \\
&= \iota_\xi \underbrace{\mathcal{L}_\eta \omega}_{0} - \underbrace{\mathcal{L}_\eta}_{d\iota_\eta + \iota_\eta d} \iota_\xi \omega \\
&= -d\iota_\eta \iota_\xi \omega - \iota_\eta \underbrace{d\iota_\xi \omega}_{d\iota_\xi\omega + \iota_\xi \underbrace{d\omega}_{0} = \mathcal{L}_\xi \omega = 0} \\
&= -d\omega(\xi,\eta) \\
&= d\omega(\eta,\xi). \quad \blacksquare
\end{aligned}$$

More generally,

**Defn.** Let $M$ be a manifold ~~and~~. A **Poisson bracket** on $M$ is an operation

$$\begin{aligned}
C^\infty(M) \otimes C^\infty(M) &\longrightarrow C^\infty(M) \\
f \otimes g &\longmapsto \{f,g\}
\end{aligned}$$

which is a Lie bracket,

and $\forall f,g,h \in C^\infty(M)$

$$\boxed{\{f, gh\} = \{f,g\}h + g\{f,h\}}$$

E.g. On a symplectic manifold,

$$\{f, gh\} = -X_f(gh) = -\big((X_f g)h + g(X_f h)\big)$$
$$= -(\{g,f\}h + g\{h,f\})$$
$$= \{f,g\}h + g\{f,h\}$$

E.g. On any manifold, $\{\cdot,\cdot\} \equiv 0$ works

E.g. $(M,\omega) \times N$

You should think of symplectic v. fields as the Lie algebra of the group of symplectomorphisms.
Likewise, we ~~should~~ can think of hamiltonian v. fields as corresponding to a 'hamiltonian group' $\subseteq$ Symplecto$(M)$.

## Time-dependant hamiltonian flows

$H_t : M \to \mathbb{R}$, $t \in [a,b]$ smooth family

(i.e. $\begin{array}{c} (t,m) \longmapsto H_t(m) \\ [a,b]\times M \longrightarrow \mathbb{R} \end{array}$ smooth)

Hamilton's equation says $\boxed{dH_t = \iota_{X_t}\omega}$

If we're lucky, we get a flow $\psi_t : M \to M$ such that $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ and $\underbrace{\psi_0 = Id}_{\substack{\text{makes sense} \\ \text{if } 0 \in [a,b].}}$

e.g. if $\overline{\{(t,m) \mid X_t|_m \neq 0\}} \subseteq [a,b] \times M$ is compact.

Defn. A hamiltonian isotopy is a smooth family of diffeos $\psi_t : M \to M$, $t \in [a,b]$ such that $\exists$ time-dependant hamiltonian $H_t : M \to \mathbb{R}$, $t \in [a,b]$ such that $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ $\forall t \in [a,b]$ and $\iota_{X_t}\omega = dH_t$.

↳ This is a hamiltonian isotopy from $\psi_a$ to $\psi_b$.

↳ WARNING! With this definition, if $\{\psi_t\}$ is a hamiltonian isotopy for $\{H_t\}$, then so is $\{\psi_t \circ f\}$ for any diffeo $f : M \to M$.
Usually one insists $a=0$ and $\psi_0 = Id$.

**Lemma** If $\exists t_0 \in [a,b]$ such that $\psi_{t_0}$ is a symplectomorphism, then $\forall t$, $\psi_t$ is a symplecto.

Proof. $\frac{d}{dt}\psi_t^*\omega = \psi_t^*(\underbrace{\mathcal{L}_{X_t}\omega}_{\text{hamiltonian.}}) = 0 \Rightarrow \psi_t^*\omega = \psi_{t_0}^*\omega = \omega.$ ■

currently a set

**Defn.** The hamiltonian group is

$$\mathrm{Ham}(M,\omega) := \{\psi \mid \exists \text{hamiltonian isotopy from Id to } \psi\}$$

**The famous Arnold Conjecture**

For $(M,\omega)$ compact symplectic, $(\psi : M \to M) \in \mathrm{Ham}(M,\omega)$, then

$$\#\{m \mid \psi(m)=m\} \underset{(*)}{\geq} \min_{f \in C^\infty(M)} \#\mathrm{crit}\, f \geq 2$$

when $\dim M \geq 2$

$C^1$-small case:

$\exists C^1$-nbhd $\Omega$ of Id in $C^\infty(M,M)$ such that $\forall \psi \in \Omega \cap \mathrm{Ham}(M,\omega)$, $(*)$ holds.

Proof idea: For such $\psi$ $\exists$ time indep. ~~flow~~ $H$ such that $\psi$ is the time 1 flow of $X_H$. Then $\{m \mid \psi(m)=m\} \subseteq \{m \mid X_H|_m = 0\} = \mathrm{Crit}\, H$.

General case (many cases): Floer techniques.

This is false for non-hamiltonian symplectomorphisms, e.g. translations of $(\mathbb{R}/\mathbb{Z})^2$

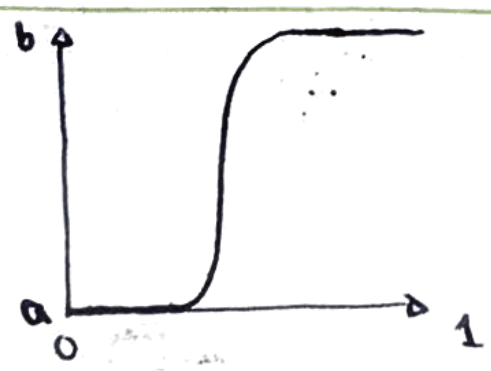
# Reparametrization of hamiltonian isotopes

Suppose $\{\psi_t\}$ is generated by $\{H_t\}$. Let $t = t(\tau)$

Then $\hat{\psi}_\tau := \psi_{t(\tau)}$ is gen by $\hat{H}_\tau = H_{t(\tau)} \frac{dt}{d\tau}$

**Corollary** If $\exists$ ham. isotopy $\{\psi_t\}_{t\in[a,b]}$ from $\psi$ to $\psi'$
Then $\exists$ ham. isotopy $\{\hat{\psi}_\tau\}_{\tau\in[0,1]}$ from $\psi$ to $\psi'$
that is generated by $\hat{H}_\tau$ st. $\hat{H}_\tau = 0$ for $\tau$ near 0 and 1.

**Proof**. Reparameterize:




## Right translation:

If $\{\psi_t\}_{t\in[a,b]}$ is a ham. isot. from $\psi_a$ to $\psi_b$
gen. by $\{H_t\}_{t\in[a,b]}$, then $\{\psi_t \circ f\}_{t\in[a,b]}$ is a
ham. isot. from $\psi_a \circ f$ to $\psi_b \circ f$ gen. also by $\{H_t\}_{t\in[a,b]}$.

## Concatenation:

Let $\{\psi_t^i\}_{t\in[0,1]}$ be ham. isot.'s from $\psi^i$ to $\tilde{\psi}^i$ w. $\tilde{\psi}^1 = \psi^2$.
gen. resp. by $\{H_t^i\}_{t\in[0,1]}$ st. $H_t^i \equiv 0$ near $t=0, t=1$.

Define

$$\psi_\tau := \begin{cases} \psi_\tau^1, & 0 \le \tau \le 1 \\ \psi_{\tau-1}^2, & 1 \le \tau \le 2 \end{cases}$$

Then $\Psi$ is a ham. isot. from $\psi^1$ to $\tilde{\psi}^2$ gen. by.

$$\tilde{H}_\tau = \begin{cases} H^1_\tau & \tau \in [0,1] \\ H^2_{\tau-1} & \tau \in [1,2] \end{cases}$$

$Ham(M,\omega)$ is a group. Let $\psi, \varphi \in Ham(M,\omega)$

$$\left.\begin{array}{lll} \{\psi_t\}_{t\in[0,1]} & \psi_0 = Id, & \psi_1 = \psi \\ \{\varphi_t\}_{t\in[0,1]} & \varphi_0 = Id & \varphi_1 = \varphi \end{array}\right\} \text{gen by ham's that vanish near } t=0 \ \& \ t=1.$$

Then $\tilde{\psi}_t := \begin{cases} \psi_\tau & , \tau \in [0,1] \\ \varphi_{\tau-1} \circ \psi, & \tau \in [1,2] \end{cases}$ is a ham. isot. from Id to $\varphi \circ \psi$

and $\{\varphi_{1-t} \circ \varphi^{-1}\}$ is a ham. isot. from Id to $\varphi^{-1}$.

~~Then $\tilde{\psi}$ is a ham. isot~~.