Symplectic Geometry

* New problem set : 7.1 & 9.1 *

Hamiltonian dynamics

↳ Consider (M,ω) a symplectic manifold. In physics, this gives a model for phase space ('half' the coordinates are position, 'half' are momenta).

↳ There is also the hamiltonian $H: M \to \mathbb{R}$ smooth, which tells us the 'energy' of the system in that state.

↳ $dH \in \Omega^1(M)$

Recall: $\omega^\#: TM \to T^*M$ $\Rightarrow \omega^\#: \{\text{v.fields}\} \xrightarrow{\sim} \{\text{1-forms}\}$

$X \mapsto \omega(X, -) =: \iota_X \omega$

Fact $\exists!\, X_H$ v.field on M such that

$$\iota_{X_H}\omega = dH \quad \text{(Hamilton's Equation)}$$

In canonical (symplectic) coordinates:

$$\omega = \sum_i dq_i \wedge dp_i \quad , \quad dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i$$

$$\Rightarrow \boxed{X_H = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}}$$

↳ The symplectic form ω gives us a formulation of the equations of motion coordinate-free

~~Thus~~, equations of motion should ~~be~~ give flows along X_H:

Integral curves of X_H:

$$\gamma : I \longrightarrow M$$
$$t \longmapsto (q(t), p(t))$$

such that $\forall t \quad \dot{\gamma}(t) = X_H \big|_{\gamma(t)}$

↳ In many cases in classical mechanics, you begin with a configuration space of particles, and then M becomes the (co)tangent bundle of this phase space (more on this later).

→ In coordinates,

$$\dot{\gamma}(t) = (\dot{q}(t), \dot{p}(t)) = \sum_i \dot{q} \frac{\partial}{\partial q_i} + \dot{p} \frac{\partial}{\partial p_i}$$

Hamilton's equation: $\forall i$

$$\boxed{\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}}$$

Defn. X_H is the symplectic gradient of H, or the hamiltonian v.field of H

E.g. In $\mathbb{R}^{2n}_{q_1, \ldots, p_n}$, compare with the gradient of H w.r.t. the Euclidean metric:

$$\nabla H = \operatorname{grad} H = \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_n}\right)$$
$$= \sum_i \frac{\partial H}{\partial q_i}\frac{\partial}{\partial q_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial p_i}$$

Defn. The gradient flow for H: $\boxed{\dot{q}_i = \frac{\partial H}{\partial q_i}, \quad \dot{p}_i = \frac{\partial H}{\partial p_i}}$

Note: $X_H \perp \nabla H$

We know $dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i$, so for a v.f. X,

$$\boxed{dH(X) = \langle \operatorname{grad} H, X \rangle}$$

standard inner product on $\mathbb{R}^{2n}$

Compare this to hamilton's eqn: $dH(X) = \omega(X_H, X)$

On any Riemannian manifold (M, g), $H \in C^\infty(M)$

$\Rightarrow$ grad H is the v.f. defined by $dH = g(\operatorname{grad} H, -)$

Further, if (M, ω, g, J) is almost Kähler, then for $H \in C^\infty(M)$,

$$dH = \iota_{\nabla H}\, g = \iota_{\nabla H}\,\omega \circ J = \iota_{-J\nabla H}\,\omega$$

$$\Rightarrow \quad \boxed{X_H = -J \cdot \nabla H}$$

$\boxed{\text{Conservation of energy:}}$

$\boxed{H \text{ is constant along the trajectories of } X_H}$

Proof. Take $\gamma(t) : I \to M$, $\dot{\gamma}(t) = X_H|_{\gamma(t)}$ $\forall t$

$$\frac{d}{dt} H(\gamma(t)) = dH(\dot{\gamma}(t)) \underset{\substack{\uparrow \\ \text{Hamilton's} \\ \text{equation}}}{=} \omega(\underbrace{X_H, \dot{\gamma}(t)}_{\text{same!}}) = 0$$

Let $\psi_t : M \to M$, $t \in \mathbb{R}$ be the flow generated by X_H.

$\hookrightarrow \psi_0 = \mathrm{Id}_M$

$\hookrightarrow \frac{d}{dt} \psi_t(m) = X_H\big|_{\psi_t(m)}$

- This is defined on a flow domain $\mathcal{D} \subseteq \mathbb{R} \times M$ such that $\forall m \in M$ $\{t \mid (t,m) \in \mathcal{D}\}$ is an open interval about 0

- If $\operatorname{supp} X_H$ is compact then $\mathcal{D} = \mathbb{R} \times M$.

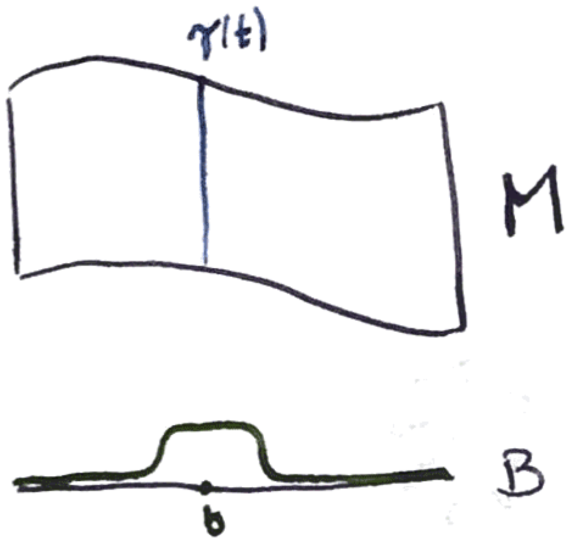
- Assume $\exists \pi : M \to B$ smooth such that

$$\underbrace{d\pi(X_H)}_{\frac{d}{dt}\pi(\gamma(t))} \equiv 0 \quad \Rightarrow \quad \pi \text{ is constant along trajectories of } X_H$$

so flows stay within level sets of π.

Assume π is proper, or more generally that the restriction $\pi| : \operatorname{supp} X_H \to B$ is proper. Then $D = \mathbb{R} \times M$.

<u>Proof Idea</u>: Consider $(\pi^* \rho) \circ X_H$ where $\rho : B \to \mathbb{R}$ is compactly supported and $\equiv 1$ near $b \in B$. ($b = \pi(\gamma(0))$, say).



<u>In particular</u>, if $H \in C^\infty(M)$, is proper, then the hamiltonian flow $\psi_t : M \to M$, $(t,m) \mapsto \psi_t(m)$ is defined $\forall t, m$.

$\boxed{\text{Claim}}$ The hamiltonian flow is symplectic, i.e. $\forall t \quad \psi_t^* \omega = \omega$.

<u>Recall</u>: $\forall$ diff. forms α, $\frac{d}{dt} \psi_t^* \alpha = \psi_t^* \mathcal{L}_{X_H} \alpha$

$\psi_0 = \mathrm{id}_M \Rightarrow \psi_t^* \omega = \omega$ for $t = 0$.

$$\frac{d}{dt} \psi_t^* \omega = \psi_t^* (\mathcal{L}_{X_H} \omega)$$

$$\mathcal{L}_{X_H} \omega \overset{\text{Cartan's magic}}{\underset{\text{formula}}{=\!=}} d \underbrace{\iota_{X_H} \omega}_{dH} + \iota_{X_H} \underbrace{d\omega}_{0} = 0$$

$\boxed{\text{Corollary}}$ The Hamiltonian flow is measure-preserving.
Liouville-measure

<u>Proof</u>. Let $n = \frac{1}{2} \dim M$. $\psi_t^* \frac{\omega^n}{n!} = \frac{(\psi_t^* \omega)^n}{n!} = \frac{\omega^n}{n!}$

Resulting qualitative properties of $\psi_t : M \to M$

↳ Energy conservation: ψ_t is <u>not</u> <u>ergodic</u> on M (i.e. $\forall$ measurable invariant sets A, either A or $M \setminus A$ has measure 0).

<u>But</u>, the restriction of a flow to a level set $H^{-1}(c)$, $c \in \mathbb{R}$ <u>can</u> be ergodic.

↳ volume-preserving: no "sinks" or "sources"

Note: $X_{H_1} = X_{H_2} \iff dH_1 = dH_2 \iff H_2 = H_1 + \text{constant}.$

$H \mapsto$ the flow of X_H gives ~~$C^\infty(\mathbb{R})$e~~

$$\frac{C^\infty(M)}{\underbrace{\mathbb{R}}_{\text{constant functions}}} \hookrightarrow \begin{array}{l}\text{1-parameter}\\ \text{groups of}\\ \text{symplectomorphisms}\end{array}$$

so long as M is compact. If M is <u>not</u> compact,

$$C_c^\infty(M) \hookrightarrow \begin{array}{l}\text{1-parameter}\\ \text{subgroups of}\\ \text{symplectomorphisms.}\end{array}$$

Thus, $\mathrm{Symp}(M,\omega) := \{\text{symplectomorphisms } M \to M\}$

is HUGE!

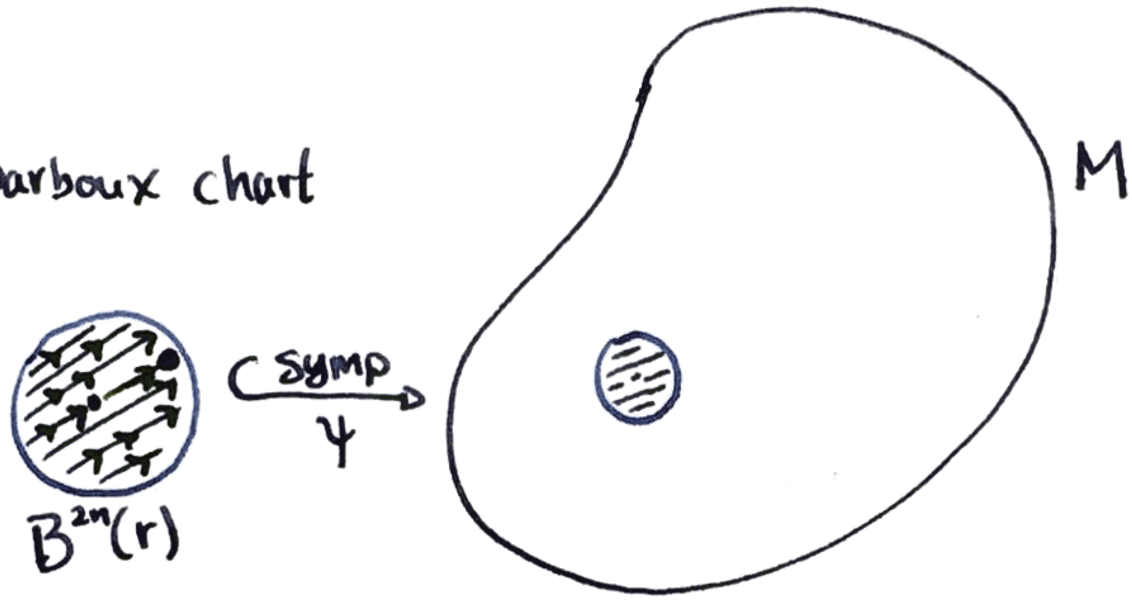
Lemma Let (M,ω) be a connected symplectic manifold. Then $\mathrm{Symp}(M,\omega) \circlearrowright M$ transitively.

Proof. Define $p \sim q \iff \exists$ symplecto $\psi : M \to M$ s.t. $\psi(p) = q$.

It is enough to show that equivalence classes are open.

It's enough to show: in each Darboux chart, $\exists$ nbhd of 0 st. for each point in this neighbourhood, it is equivalent to 0.

Fix a Darboux chart



Take a point $y = (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_n, \hat{p}_n) \in B^{2n}(r)$

Then $y = \psi_1(0)$, where $\psi_t : \mathbb{R}^{2n} \circlearrowleft$ is $x \mapsto x + ty$.

It's generated by $\sum \hat{q}_i \frac{\partial}{\partial q_i} + \hat{p}_i \frac{\partial}{\partial p_i} = X_H$. For

$$H : \mathbb{R}^{2n} \to \mathbb{R}$$
$$(q_1, \ldots, p_n) \mapsto \sum_i -b_i q_i + a_i p_i$$

Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}$ be smooth, $\mathrm{supp} \subseteq B^{2n}(r)$ s.t. $\rho(ta_1, tb_1, \ldots, ta_n, tb_n) = 1 \ \forall t \in [0,1]$.

Then $\begin{cases} \rho \cdot H \circ \psi^{-1} & \text{on the Darboux chart } \psi \\ 0 & \text{outside} \end{cases}$ is a smooth function

on M whose time 1 flow takes $\psi(0)$ to $\psi(y)$.

E.g. $M = S^2$, cylindrical coordinates $\underbrace{\theta}_{\bmod 2\pi}, z$

$\omega = d\theta \wedge dz$

$H(\theta, z) = z$ height function

$dH = dz$, $\Rightarrow X_H = \frac{\partial}{\partial \theta}$ $\Rightarrow$ The Hamiltonian flow is rotations (defined on all of M).

This has the interpretation of momentum rather than energy.