Guest Lecture

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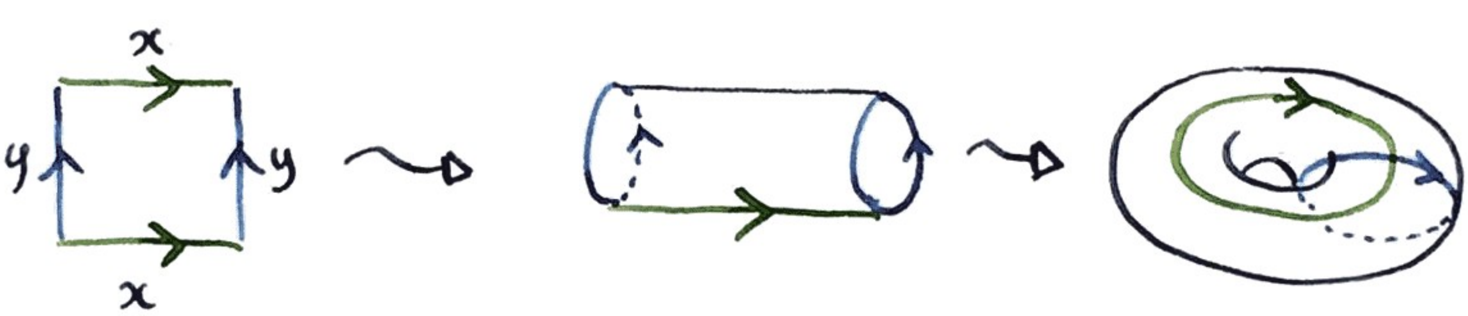
{ Flat Connections on 2-manifolds and representations of the fundamental group of orientable 2-manifolds }

Denote by Σ_g the orientable 2-manifold of genus g.

↳ $g=0 \Rightarrow \Sigma_0 \cong S^2$

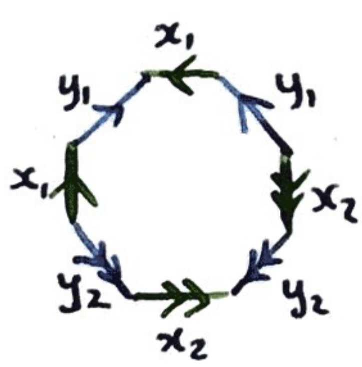
↳ $g=1 \Rightarrow \Sigma_1 \cong T^2 \cong (S^1)^{\times 2}$

Σ_1 :



$$\pi_1(\Sigma_1) = \langle x, y \mid x y x^{-1} y^{-1} = e \rangle$$

Σ_2 :



$$\pi_1(\Sigma_2) = \langle x_1, y_1, x_2, y_2 \mid 1 = x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \rangle$$

Look for representations of $\pi_1(\Sigma_g)$ in a compact Lie group, for example $SU(2)$ or $U(1)$.

Representations of $\pi_1(\Sigma_1)$ in $\mathcal{U}(1)$:

$$\rho : \left.\begin{array}{l} x \longmapsto A \\ y \longmapsto B \end{array}\right\} \in \mathcal{U}(1) \quad \text{such that } AB = BA$$

(this is true as $\mathcal{U}(1)$ is abelian).

If we want representations into a nonabelian G; ~~then A lies in some maximal torus T, so B must lie in T as well.~~ A and B commute, hence lie in the same maximal torus T.

↳ The space of representations is all conjugates of $T \times T$.

Representations of $\pi_1(\Sigma_g)$ in $\mathcal{U}(1)$: $\mathcal{U}(1)^{\times 2g}$

in a nonabelian group: (say $g=2$):

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} = \mathbb{1}.$$

Quotient this by the diagonal action of G by conjugation; call it $\mathcal{M}$. This is a manifold (except on a set of measure 0). Indeed at smooth points it has a symplectic structure.

$g=1$: $\mathcal{M} = \mathcal{U}(1)^{\times 2}$ the 2-torus; we know the symplectic structure.

For $\mathrm{Hom}(\Sigma_g, \mathcal{U}(1)) = \mathcal{U}(1)^{\times 2g}$, this also has a(n obvious) symplectic structure.

Alternative description in terms of connections

(on a trivial principal G-bundle over Σ_g).

↳ $\Sigma_g \times G$

Defn. A **connection** on $\Sigma_g \times G$ is a $\mathfrak{g} := \mathrm{Lie}(G)$-valued 1-form on Σ_g.

For $G = U(1)$, the connection is a 1-form ($\mathrm{Lie}\, U(1) = \mathbb{R}$); write it as C. The curvature is denoted F_C

↳ If G is abelian, $F_C = dC$ (a 2-form)

↳ If not, then $F_C = dC + C \wedge C$ (matrix groups)

Think of C as a matrix (C_i^j) of 1-forms. Then

↳ $dC = (dC_i^j)$

↳ $(C \wedge C)_i^j = C_i^k \wedge C_k^j \quad (= \sum_k C_i^k \wedge C_k^j)$

Defn. A connection is **flat** if $dC = 0$

Defn. The **gauge group** $\mathcal{G} := C^\infty(\Sigma_g, G)$ acts on the space of connections by pointwise conjugation,

$\gamma \in \mathcal{G}: \quad \gamma . C = \gamma^{-1} \cdot C \cdot \gamma$

Gauge transformations modify the choice of trivialization.

The action of $\mathcal{G}$ takes flat connections to other flat connections.

We have a symplectic form on $\mathcal{A}$:= space of all connections

$$\omega(C^1, C^2) := \int_{\Sigma_g} \mathrm{Tr}(C^1 \wedge C^2)$$

$C^1 \wedge C^2$ is a ~~$\mathfrak{g}$~~ matrix-valued 2-form; $\mathrm{Tr}(C^1 \wedge C^2)$ is a 2-form.

$\mathcal{A}$ is a vector space! The description of ω is the same at all points of $\mathcal{A}$ (view elements of $\mathcal{A}$ also as tangent vectors; think of the $\mathbb{R}^{2n}$ case).

It is closed: in $\mathbb{R}^2$ $f(x,y)dx \wedge dy$ gives a 2-form. If f is constant, it is closed. The analogue of f for $\mathcal{A}$ is also constant.

ω is invariant under the action of $\mathcal{G}$; ~~so~~ moreover it descends to give a symplectic form on (flat connections)$/\mathcal{G}$, which <u>is</u> finite-dimensional.

Current Topics in this area:

There are analogues (obtained by taking a 2-manifold with 1 boundary component and requiring the loop around the boundary to be a nontrivial element of $Z(G)$) which are smooth manifolds.

E.g. $G = SU(2)$
$g = 2$

 $= \Sigma_g' = \Sigma_g \setminus D^2$

Here, $\pi_1(\Sigma_g') = \langle x_1, x_2, y_1, y_2 \rangle$

For $\rho : \pi_1(\Sigma_g') \to G$

$$x_i \mapsto A_i$$
$$y_i \mapsto B_i$$

If we require $A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} = -\mathbb{1}$, then we obtain a <u>smooth</u> manifold M'.

↳ The cohomology groups of M' were explored by (Atiyah-Bott 1982)

↳ —— " —— ring —— " —— (J-Kirwan 1998)

↳ Hamiltonian flows on M' (J-Weitsman 1992—)

[See MAT1312 course notes on Lisa's homepage]

! Flat connections give rise to representations, since ~~they are~~ holonomy is invariant under homotopy. Therefore

$$\frac{\text{flat connections}}{\mathcal{G}} \cong \frac{\text{representations of } \pi_1(\Sigma_g)}{\text{conjugation}}$$