This is a special case of symplectic reduction.

Recall: A map f is holomorphic $\Longleftrightarrow$ df is complex-linear at each point.

For $f: \mathbb{C}^m \to \mathbb{C}^n$, this is a theorem.

For $f: (M_1, J_1) \to (M_2, J_2)$ a map between almost cpx. manifolds, this is a definition.

Special Case: $f: \underset{\text{Riemann surface}}{(\Sigma, j)} \to \underset{\text{almost cpx. mfld.}}{(M, J)}$

Here, holomorphic $\Leftrightarrow$ $\boxed{df \circ j = J \circ df}$

$\Leftrightarrow$ holomorphic curve

$\Leftrightarrow$ pseudoholomorphic curve

$\Leftrightarrow$ J-holomorphic curve.

And now it's time for linear algebra...

Let $A: \mathbb{C}^n \to \mathbb{C}^m$ be real linear.

Note: $df \circ j = J \circ df \iff \boxed{df + J \circ df \circ j = 0}$

$$(df + Jdfj)j = dfj - Jdf = -J \cdot J(dfj - Jdf)$$

$$= -J(df + Jdfj)$$

So $df + Jdfj$ is complex antilinear.

With this observation, $A = A' + A''$ where

$A' = \frac{1}{2}(A - iAi)$ — complex linear

$A'' = \frac{1}{2}(A + iAi)$ — complex antilinear

Remark: The value of (almost) complex structures is they give us a way of finding friends: $\forall u$, nonzero,

$$\omega(u, Ju) > 0$$

If (M, ω, J) is almost Kähler and $f: M \to \mathbb{C}$ is holomorphic (such f rarely exists), given a regular value $u \in \mathbb{C}$ of f, $f^{-1}(u) \subseteq M$ is an almost complex, hence symplectic, submanifold.

Exercise 5.1 Given a real linear map $A: \mathbb{C}^n \to \mathbb{C}$, if $|A''| < |A'|$, then $\mathrm{codim}_{\mathbb{R}}(\ker A) = 2$, and $\ker A$ is symplectic.

NB: I ran out of red ink; this is still an exercise, tho.

Notation: $g \circ h := h \circ g$

Notation: For $f: (M_1, J_1) \to (M_2, J_2)$, define

$$\partial f := \frac{1}{2}(df - J_1 \circ df \circ J_2)$$

$$\bar{\partial} f := \frac{1}{2}(df + J_1 \circ df \circ J_2)$$

Consequence (of exercise) Given $f:(M,J,\omega)\to\mathbb{C}$, if $|\bar\partial f|<|\partial f|$, at each point of $f^{-1}(\{0\})$, then $f^{-1}(\{0\})$ is a symplectic submanifold.
(0, it follows, is regular).

Note: On $\mathbb{C}^n$, write $f=f(z_1,\_\_\_,z_n)$. Then

$$\partial f=\sum_{k=1}^{n}\frac{\partial f}{\partial z_k}dz_k \quad \text{and} \quad \bar\partial f=\sum_{k=1}^{n}\frac{\partial f}{\partial \bar z_k}d\bar z_k$$

Where $z_k=x_k+iy_k$, $\bar z_k=x_k-iy_k$

So $dz_k=dx_k+idy_k$, $d\bar z_k=dx_k-idy_k$

At each point $q\in M=\mathbb{C}^n$, $\{dz_k,d\bar z_k\}_k$ forms a basis to the $\mathbb{C}$-v.s.

$$\mathrm{Hom}_{\mathbb{R}}(T_qM,\mathbb{C})\cong\mathrm{Hom}_{\mathbb{C}}(T_qM\otimes\mathbb{C},\mathbb{C})$$

The dual basis is given by:

$$\frac{\partial}{\partial z_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}-i\frac{\partial}{\partial y_k}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar z_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}+i\frac{\partial}{\partial y_k}\right)$$

Observe: $\sum_{k=1}^{n}dx_k\wedge dy_k=\frac{i}{2}\sum_{k=1}^{n}dz_k\wedge d\bar z_k=\frac{i}{2}\partial\bar\partial\sum_{k=1}^{n}z_k\bar z_k$

Fubini-Study form (symplectic) ω_{FS} on $\mathbb{P}^n(\mathbb{C})$

$$\begin{tikzcd} S^{2n+1} \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathbb{C}^{n+1} \\ \mathbb{P}^n(\mathbb{C}) & \end{tikzcd} \qquad \pi^*\omega_{FS} = i^*\omega_{\mathbb{C}^{n+1}}$$

observe: $(dx + i dy) \wedge (dx - i dy) = -i dx \wedge dy + i dy \wedge dx$
$= -2i dx \wedge dy$

Recall: $dx \wedge dx = 0$

Office Hour

For an almost complex manifold (M, J), TFAE:

① ↳ J comes from a complex atlas

② ↳ On $TM \otimes_{\mathbb{R}} \mathbb{C} = \underbrace{T^{1,0}M}_{\substack{i\text{-eigenspace} \\ \shortparallel \\ \operatorname{span}\{\frac{\partial}{\partial z_k}\}}} \oplus \underbrace{T^{0,1}M}_{\substack{(-i)\text{ eigenspace} \\ \shortparallel \\ \operatorname{span}\{\frac{\partial}{\partial \bar{z}_k}\}}}$,

then $T^{1,0}M$ is **involutive**, i.e. $\Gamma(T^{1,0}M)$ has a Lie algebra structure inherited from $\Gamma(TM \otimes_{\mathbb{R}} \mathbb{C})$.

③ ↳ The **Nijenhuis tensor** vanishes: $N_J = 0$

② ⟺ ③ "easy"

① ⟺ ③ Newlander-Nirenberg.

Think of the Lie bracket of v.f.'s as differentiation of one v.f. along the other:

$$[X,Y] = \mathcal{L}_X(Y)$$

Defn. $N_J : TM^{\otimes 2} \longrightarrow TM$

$$N_J(X,Y) := [JX,JY] - J([JX,Y] + [X,JY]) - [X,Y]$$
$$= [JX,JY] - J[JX,Y] + J^2[X,Y] - J[X,JY]$$

A priori, this definition does not give a tensor (i.e. defined fibrewise, independant of the extension of X_p and Y_p to a nbhd of p). You can show this, however.