MAT 247H1S - ALGEBRA II

Term Test - Solutions to questions 4 and 5

4. Let W be a finite-dimensional subspace of an inner product space V. Let $T \in \mathcal{L}(V)$ be orthogonal projection on the subspace W.
   a) Prove that $\|T(x)\| \leq \|x\|$ for all $x \in V$. (Recall that $\|x\| = \sqrt{\langle\, x, x\,\rangle}$, $x \in V$.)
   b) Assume that $W \neq \{\mathbf{0}\}$ and $W \neq V$. Prove that there exists a vector $x \in V$ such that $0 < \|T(x)\| < \|x\|$.

Solution of 4a). Let $x \in V$. As shown in the notes on orthogonal projections, $x - T(x) \in W^{\perp}$. Because $T(x) \in W$, we see that $\langle x - T(x), T(x)\rangle = 0$. It follows that

$$\begin{aligned}\langle\, x, x\,\rangle &= \langle x - T(x) + T(x), x - T(x) + T(x)\rangle \\ &= \langle x - T(x), x - T(x)\rangle + \langle x - T(x), T(x)\rangle + \langle T(x), x - T(x)\rangle + \langle T(x), T(x)\rangle \\ &= \langle x - T(x), x - T(x)\rangle + 0 + 0 + \langle T(x), T(x)\rangle \\ &.\end{aligned}$$

We have shown that

$$\|T(x)\|^2 = \|x\|^2 - \|x - T(x)\|^2.$$

This implies $\|T(x)\|^2 \leq \|x\|^2$. Taking non-negative square roots, we get $\|T(x)\| \leq \|x\|$.

Alternate solution of 4a): Let $d = \dim(W)$. Let $\{\, x_1, \ldots, x_d\,\}$ be an orthonormal basis for W. Then $T(x) = \sum_{j=1}^{d} \langle\, x, x_j\,\rangle x_j$, $x \in V$. (In what follows, we are using properties of inner products, as well as $\langle x_j, x_k\rangle = 0$ whenever $j \neq k$, and $\langle x_j, x_j\rangle = 1$.) Observe that

$$\begin{aligned}\langle\, T(x), T(x)\,\rangle &= \langle\, \sum_{j=1}^{d} \langle\, x, x_j\,\rangle x_j\,,\, \sum_{k=1}^{d} \langle\, x, x_k\,\rangle x_k\,\rangle \\ &= \sum_{j=1}^{d}\sum_{k=1}^{d} \langle\, x, x_j\,\rangle \overline{\langle\, x, x_k\,\rangle} \langle\, x_j, x_k\,\rangle \\ &= \sum_{j=1}^{d} \langle\, x, x_j\,\rangle \overline{\langle\, x, x_j\,\rangle}.\end{aligned}$$

Now we assume that $n = \dim V < \infty$. As shown in the notes on orthogonal projections, since $\{\, x_1, \ldots, x_d\,\}$ is a basis for W, there exist unit vectors $x_{d+1}, \ldots, x_n$, $n = \dim V$ such that $\beta = \{\, x_1, \ldots, x_d, x_{d+1}, \ldots x_n\,\}$ is an orthonormal basis for V. (In fact, we can simply take $\{\, x_{d+1}, \ldots, x_n\,\}$ to be an orthonormal basis for $W^{\perp}$.)

$$\begin{aligned}\langle\, x, x\,\rangle &= \langle\, \sum_{j=1}^{n} \langle\, x, x_j\,\rangle x_j\,,\, \sum_{k=1}^{n} \langle\, x, x_k\,\rangle x_k\,\rangle \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \langle\, x, x_j\,\rangle \overline{\langle\, x, x_k\,\rangle} \langle\, x_j, x_k\,\rangle \\ &= \sum_{j=1}^{n} \langle\, x, x_j\,\rangle \overline{\langle\, x, x_j\,\rangle}.\end{aligned}$$

We now have

$$\|x\|^2 - \|T(x)\|^2 = \sum_{j=d+1}^{n} \langle\, x, x_j \,\rangle\overline{\langle\, x, x_j \,\rangle}.$$

Since $\langle\, x, x_j \,\rangle\overline{\langle\, x, x_j \,\rangle} \geq 0$, we find that $\|x\|^2 - \|T(x)\|^2 \geq 0$, which implies that $\|x\| \geq \|T(x)\|$.

Solution of 4b): Because $W \neq \{\mathbf{0}\}$, there exists a nonzero $y \in W$. Because $W \neq V$, we know that $\dim W^{\perp} \geq 1$, so there exists a nonzero $z \in W^{\perp}$. Let $x = y + z$. Then

$$\begin{aligned}\langle\, x, x \,\rangle &= \langle y, y\rangle + \langle y, z\rangle + \langle z, y\rangle + \langle z, z\rangle \\ &= \langle y, y\rangle + \langle z, z\rangle = \langle T(x), T(x)\rangle + \langle z, z\rangle.\end{aligned}$$

Because $y = T(x) \neq \mathbf{0}$ and $z \neq \mathbf{0}$, the fourth property of inner product tells us that $\|T(x)\| > 0$ and $\|z\| > 0$. Since $\|z\|^2 = \|x\|^2 - \|T(x)\|^2$, it follows that $\|x\|^2 > \|T(x)\|^2$. So we have $0 < \|T(x)\| < \|x\|$.

5. Let V be a finite-dimensional complex inner product space. Let $T \in \mathcal{L}(V)$.
   a) Suppose that $T + T^* = I_V$. Prove that

   $$\langle\, T(x), x \,\rangle + \langle\, x, T(x) \,\rangle = \langle\, x, x \,\rangle, \quad \text{for all } x \in V.$$

   b) Suppose that

   $$\langle\, T(x), x \,\rangle + \langle\, x, T(x) \,\rangle = \langle\, x, x \,\rangle \quad \text{for all } x \in V.$$

   Do not assume that T is normal. Prove that $T + T^* = I_V$.

Solution to 5a). Let $x \in V$. Then

$$\langle\, T(x), x \,\rangle + \langle\, x, T(x) \,\rangle = \langle\, x, T^*(x) \,\rangle + \langle\, x, T(x) \,\rangle\langle\, x, (T^* + T)(x) \,\rangle = \langle\, x, I_V(x) \,\rangle = \langle\, x, x \,\rangle.$$

Solution to 5b). Suppose that

$$\langle\, T(x), x \,\rangle + \langle\, x, T(x) \,\rangle = \langle\, x, x \,\rangle$$

for all $x \in V$. (*Note*: This assumption does not tell us whether or not T is normal.) Rewriting the left side using part of the solutions to 5a), we find that

$$\langle\, x, (T^* + T)(x) \,\rangle = \langle\, x, x \,\rangle$$

for all $x \in V$. Subtracting $\|x\|^2$ from both sides and using properties of inner products, we get

$$\langle\, x, (T^* + T - I_V)(x) \,\rangle = 0, \qquad x \in V.$$

Let $U = T^* + T - I_V$. Then $U^* = (T^*)^* + T^* - I_V^* = T + T^* - I_V = T^* + T - I_V = U$. Therefore U is self-adjoint. Note that

$$\langle U(x), x\rangle = \langle x, U^*(x)\rangle = \langle x, U(x)\rangle = 0, \qquad x \in V.$$

As shown in class, if $U = U^*$ and $\langle U(x), x\rangle = 0$ for all $x \in V$, then $U = T_0$. In that case, $T + T^* - I_V = T_0$. Adding I_V to both sides, we get $T + T^* = I_V$. (*Note*: If $T' \in \mathcal{L}(V)$ and $\langle T'(x), x\rangle = 0$ for all $x \in V$, it is possible to have $T' \neq T_0$ when T' is not self-adjoint. There were several examples on problem sets.)

If you did not think of quoting the result from class, you could argue as follows. First, show that U is self adjoint (as above), and $\langle U(x), x\rangle = 0$ for all $x \in V$. Then, according to theorems proved in class, there exists an orthonormal basis $\beta = \{\, x_1, \ldots, x_n \,\}$ for V that consists of eigenvectors of U. There exist scalars $\lambda_1, \ldots, \lambda_n$ such that $U(x_j) = \lambda_j x_j$, $1 \leq j \leq n$. We have $[U]_\beta$ equal to the $n \times n$ diagonal matrix, with diagonal entries $\lambda_1, \ldots, \lambda_n$. Observe that, since x_j is a unit vector,

$$0 = \langle U(x_j), x_j\rangle = \langle \lambda_j x_j, x_j\rangle = \lambda_j\langle x_j, x_j\rangle = \lambda_j \cdot 1 = \lambda_j, \qquad 1 \leq j \leq d.$$

Now we see that the matrix $[U]_\beta$ is equal to the zero matrix. That is, $[U]_\beta = [T_0]_\beta$. Because the map $T' \mapsto [T']_\beta$ from $\mathcal{L}(V)$ to $M_{n\times n}(F)$ is one-to-one, we may now conclude that $U = T_0$. That is, $T + T^* - I_V = T_0$. Adding T_0 to both sides, we get $T + T^* = I_V$.