MAT 247S - Problem Sets 8–9

Problem Set 8(questions 1–15), due Thursday April 2; questions 3b), 4, 8b), 10a) and 15b) will be marked.

Problem Set 9, questions 16–29; not to be handed in.

1. Let T be a linear operator on a finite-dimensional complex vector space. Suppose that $T^{r+2} = 4\,T^r$ for some positive integer r. Prove that T is diagonalizable if and only if $T^3 = 4\,T$. (*Hint*: What can you say about the minimal polynomial of T?)

2. Let $V = P_2(\mathbb{R})$. Let $\beta = \{\, t+1,\, t-1,\, t^2-1 \,\}$, and let $\beta^* = \{\, f_1, f_2, f_3 \,\}$ be the dual basis for V^*.
   a) Compute the values $f_j(a + b\,t + c\,t^2)$ for $a,\, b,\, c \in \mathbb{R}$ and $1 \le j \le 3$.
   b) Let $f : V \to \mathbb{R}$ be defined by $f(p(t)) = p(2) + p'(0)$, $p(t) \in V$. (Here, $p'(t)$ is the derivative of $p(t)$.) Show that $f \in V^*$. Express f as a linear combination of the basis vectors f_1, f_2 and f_3.

3. Let $V = P_2(\mathbb{R}) = \{\, p(t) = a + b\,t + c\,t^2 \mid a,\, b,\, c \in \mathbb{R} \,\}$. Let V^* be the dual space of V. Define f_1, f_2 and $f_3 \in V^*$ by:

$$f_1(a+b\,t+c\,t^2) = -a+b, \quad f_2(a+b\,t+c\,t^2) = 2\,a+b, \quad f_3(a+b\,t+c\,t^2) = -b-c, \quad a,\, b,\, c \in \mathbb{R}.$$

   a) Prove that $\{\, f_1,\, f_2,\, f_3 \,\}$ is a basis for V^*.
   b) Find the basis $\beta = \{\, p_1(t),\, p_2(t), p_3(t) \,\}$ for V having the property that the dual basis β^* for V^* is equal to $\{\, f_1,\, f_2,\, f_3 \,\}$.

4. Let V be a finite-dimensional inner product space. Let $\beta = \{\, x_1, \ldots, x_n \,\}$ be a basis for V, and let $\beta^* = \{\, f_1, \ldots, f_n \,\}$ be the basis for V^* that is dual to β. For $1 \le j \le n$ let $y_j \in V$ be such that $f_j(x) = \langle x, y_j \rangle$, $x \in V$. Set $\beta' = \{\, y_1, \ldots, y_n \,\}$.
   a) Prove that β' is a basis for V.
   b) Prove that $\beta' = \beta$ if and only if β is orthonormal.

5. Let V be a finite-dimensional inner product space, and let $T : V \to V$ be linear. If $x \in V$, define $f_x \in V^*$ by $f_x(y) = \langle y, x \rangle$, $x \in V$. Prove that the adjoint T^* of T is the unique linear operator on V that satisfies $T^t(f_x) = f_{T^*(x)}$ for all $x \in V$.

6. #5, §2.6

7. # 7, §2.6

8. Determine whether H is a subgroup of G. If H is not a subgroup of G, find x and $y \in H$ such that $xy^{-1} \notin H$. If H is a subgroup, answer the following two questions (and justify your answers): *(i)* Is H an abelian group? *(ii)* Is H a normal subgroup of G?
   a) Let V be a finite-dimensional vector space and let $G = GL(V)$ be the group of invertible linear operators on V (with the group multiplication given by composition of operators). Let $H = \{\, T \in GL(V) \mid N(T - I_V) \ne 0 \,\}$.
   b) Let $G = GL_2(\mathbb{Q})$ and $H = \left\{ \begin{pmatrix} a & 3b \\ b & a \end{pmatrix} \mid a,\, b \in \mathbb{Q},\ \text{at least one of } a \text{ and } b \text{ is nonzero} \right\}$. (Here, $\mathbb{Q}$ is the field of rational numbers.)
   c) Let $G = GL_2(F)$ and $H = \{\, A \in G \mid A^2 = I \,\}$.
   d) Let $G = \{\, A \in GL_3(F) \mid A_{21} = A_{31} = A_{32} = 0 \,\}$ (that is, G is the group of upper triangular matrices in $GL_3(F)$). Let $H = \{\, A \in G \mid A_{11} = A_{22} = A_{33} = 1 \,\}$.
   e) Let G be the group of upper triangular matrices in $GL_3(F)$ and let H be the set of all diagonal matrices in G.

f) Let $G = GL_3(F)$ and let $H = \{ A \in G \mid A_{12} = A_{21} = A_{23} = A_{32} = 0 \}$.

9. Let G be a group. Suppose that $x^2 = e$ for every $x \in G$. Prove that G is abelian.

10. Let G be a group.
   a) Let H be a nonempty finite subset of G. Prove that H is a subgroup of G if and only if $xy \in H$ for all x and $y \in H$.
   b) Find an example of a group G and a nonempty subset H of G that has the property that $xy \in H$ for all x and $y \in H$, and H is not a subgroup of G.

11. Let U_n be the group of $n \times n$ unitary matrices. Prove that an element A of U_n has finite order if and only if all eigenvalues of A lie in the set $\{ z \in \mathbb{C} \mid z^\ell = 1 \text{ for some positive integer } \ell \}$.

12. Let G be a group. Let x and y be elements of G. Show that y and xyx^{-1} have the same order.

13. Let $G = GL_2(\mathbb{R})$. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
   a) Compute the order of A and the order of B.
   b) Let H be a subgroup of G that contains A and B. Prove that H has infinite order.
   c) Let $H_1 = \{ C \in G \mid C \text{ has finite order } \}$. Prove that H_1 is not a subgroup of G.

14. Let p be a prime and let $\mathbb{F}_p$ a field that contains p elements (that is, $\mathbb{F}_p$ is the "integers modulo p"). Compute the order of the finite group $GL_n(\mathbb{F}_p)$. (*Hint*: If $A \in GL_n(\mathbb{F}_p)$, then each row of A belongs to $\mathbb{F}_p^n$, and the rows of A must form a linearly independent subset of $\mathbb{F}_p^n$. Thus the first row of A must be a nonzero element of $\mathbb{F}_p^n$, the second row of A must be an element of $\mathbb{F}_p^n$ that is not a scalar multiple of the first row, the third row cannot belong to the span of the first two rows,....)

15. In each case, determine whether the indicated function $\varphi : G \to G'$ is a homomorphism. If φ is not a homomorphism, find x and $y \in G$ such that $\varphi(xy) \neq \varphi(x)\varphi(y)$. If φ is a homomorphism identify the kernel of φ and the image of φ.
   a) Let F be a field and let $G = G' = GL_n(F)$. Assume that $n \geq 2$ and let $\varphi(A) = AA^t$.
   b) Let $G = \left\{ A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{C},\ ad \neq 0 \right\}$, with matrix multiplication. Let
   $$G' = \{ (c, z) \mid c \in \mathbb{R},\ c \neq 0,\ z \in \mathbb{C},\ z \neq 0 \},$$
   with multiplication given by $(c, z) \cdot (c', z') = (cc', zz')$. Let $\varphi(A) = (|a|, d^4)$, $A \in G$.
   c) Let G be as in part b). Define $\varphi : G \to G$ by
   $$\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \begin{pmatrix} a^2 & -b \\ 0 & d \end{pmatrix}, \qquad a,\ b,\ d \in \mathbb{C},\ ad \neq 0.$$

16. Let $G = GL_n(F)$, where F is a field. Let J be a fixed element of G and define $\varphi : G \to G$ by $\varphi(A) = J(A^t)^{-1}J^{-1}$.
   a) Show that φ is a group isomomorphism.
   b) Let $H = \{ A \in G \mid \varphi(A) = A \}$. Prove that H is a subgroup of G.
   c) Suppose that $B \in G$ and $J_1 = BJB^t$. Define $\varphi_1(A) = J_1(A^t)^{-1}J_1^{-1}$, $A \in G$. Applying parts b) and c) to φ_1 instead of φ, we see that φ_1 is a group isomorphism and $H_1 = \{ A \in G \mid \varphi_1(A) = A \}$ is a subgroup of G. Prove that H and H_1 are isomorphic groups. (*Hint*: Prove that $A \in H_1$ if and only if $B^{-1}AB \in H$. Then use this fact to define a group isomorphism from H_1 to H.)

17. Show that the groups $G = GL_1(\mathbb{Q})$ and $G' = GL_1(\mathbb{R})$ are not isomorphic.

Let D_n be the dihedral group of order $2n$, $n \geq 3$. Let r be a rotation in D_n of order n. Let s be a reflection in D_n.

18. Let G' be a group and let $\varphi : D_n \to G'$ be a homomorphism.
    a) Prove that the image of φ is an abelian group if and only if $\varphi(x)^2 = e'$ for all $x \in D_n$.
    b) Suppose that n is odd and $\varphi(r) \neq e'$. Prove that the image of φ is a nonabelian group.
    c) Suppose that $n = 9$ and $\varphi(r^3) \neq e'$. Prove that φ is one-to-one.
19. Which of the following maps $\varphi : D_n \to D_n$ are homomorphisms? Which are isomorphisms? What is the kernel of each one that is a homomorphism?
    a) $\varphi(r^j) = r^{-j}$ and $\varphi(r^j s) = r^{-j}s$, $0 \leq j \leq n-1$.
    b) $\varphi(r^j) = r^j$ and $\varphi(r^j s) = r^{j+2}s$, $0 \leq j \leq n-1$.
    c) $\varphi(r^j) = r^{-j}$ and $\varphi(r^j s) = r^j s$, $0 \leq j \leq n-1$.
    d) $\varphi(r^j) = r^{2j}$ and $\varphi(r^j s) = r^{2j}s$, $0 \leq j \leq n-1$.
20. Let G be a finite group and let $\varphi : G \to GL_n(\mathbb{C})$ be a homomorphism. Show that $\varphi(x)$ is diagonalizable for every $x \in G$.
21. Let H be a normal subgroup of D_n. Suppose that $r^j s \in H$ for some $j \in \{0, \ldots, n-1\}$.
    a) Prove that $r^2 \in H$.
    b) Prove that $H = D_n$ when n is odd.
22. If x, $y \in D_n$, let $\langle x, y\rangle$ be the subgroup of D_n generated by x and y (that is, the smallest subgroup of D_n that contains both x and y).
    a) Show that if n is even, then $\langle r^2, s\rangle$ is a normal subgroup of D_n.
    b) Show that if n is even and $n \geq 6$, then $\langle r^2, s\rangle$ is isomorphic to $D_{n/2}$.
    c) Show that $\langle rs, r^2 s\rangle = D_n$.
23. Show that the elements of the 2×2 unitary group U_2 have the form

$$\begin{pmatrix} z & w \\ -e^{i\theta}\bar{w} & e^{i\theta}\bar{z} \end{pmatrix},$$

    where z, $w \in \mathbb{C}$, $\theta \in \mathbb{R}$, and $z\bar{z} + w\bar{w} = 1$. (Here, $e^{i\theta} = \cos\theta + i\sin\theta$.)
24. Let $\varphi : D_8 \to GL_2(\mathbb{C})$ be a homomorphism. Suppose that $\varphi(r) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.
    a) Prove that the image of φ is a nonabelian group.
    b) Prove that $\varphi(s) = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$ for complex numbers a and b satisfying $a^2 - b^2 = 1$.
    c) Find all of the elements in the kernel of φ.
    d) Prove that the image of φ is isomorphic to D_4. (*Hint*: Define a specific map ψ from the image of φ to D_4. Note that $\psi(\varphi(r))$ must have order 4 and $\psi(\varphi(s))$ must have order 2 and cannot commute with $\varphi(r)$. After defining ψ, show that it is an isomorphism.)
25. Let $\varphi : D_8 \to GL_2(\mathbb{C})$ be a map. Suppose that $\varphi(r) = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$. Verify that $\varphi(r)$ has order 8 (as an element of $GL_2(\mathbb{C})$). Show that φ is not a homomorphism. (*Hint*: Consider what

properties $\varphi(s)$ must have if φ is a homomorphism and show that there is no matrix having those properties.)

26. Prove that D_m is isomorphic to a subgroup of D_n if and only if m divides n.

27. Let D_n be the dihedral group of order $2n$ ($n \geq 3$). Let $r \in D_n$ be an element of order n, and let $s \in D_n$ be an element of order 2 such that $rs = sr^{-1}$.

    For parts a) and b), determine whether the indicated subset H of D_6 is a subgroup of D_6. If H is not a subgroup, demonstrate which property fails. If H is a subgroup, prove that it is a subgroup. Also, if H is a subgroup, determine whether or not H is an abelian group, and explain your answer fully.

    a) $H = \{ e, r^3, rs, sr^2 \} \subset D_6$.
    b) $H = \{ g \in D_6 \mid g^2 = e \}$.
    c) Suppose that H is a normal subgroup of D_n and there exists j with $0 \leq j \leq n-1$ such that the element $r^j s$ belongs to H. Prove that the element r^2 also belongs to H. (*Note*: $r^0 = e$).
    d) Find all normal subgroups of D_6 that contain the element rs.

28. Let D_4 be the dihedral group of order 8, realized as the group of symmetries of a square centered at the origin in $\mathbb{R}^2$, with vertices $\pm(1,1)$, $\pm(1,-1)$. Let r be counterclockwise rotation of the plane (about the origin) through $\pi/2$. Let s be reflection about the x-axis.

    a) Let φ be a map from D_4 to the group $GL_1(\mathbb{C}) = \mathbb{C}^\times$ of nonzero complex numbers. Prove that if $\varphi(r) = i$, then φ cannot be a homomorphism.

    For parts b) and c), let $GL_2(\mathbb{C})$ be the group of invertible 2×2 complex matrices (relative to matrix multiplication). Suppose that $\varphi : D_4 \to GL_2(\mathbb{C})$ is a group homomorphism such that $\varphi(r) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

    b) Prove that φ is one-to-one.
    c) Prove that $\varphi(s) = \begin{pmatrix} 0 & c^{-1} \\ c & 0 \end{pmatrix}$ for some nonzero complex number c.

29. Let G be an abelian group. Let H be the set of elements in G that have finite order. Prove that H is a subgroup of G.