MAT247S, 2009 Winter, Problem Set 7 Solution

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All below I denote D_λ to be the dot diagram for eigenvalue λ.

1.(a) D_4 $\begin{matrix}\bullet\\\bullet\\\bullet\\\bullet\end{matrix}$, D_{-1} $\begin{matrix}\bullet&\bullet\\\bullet&\bullet\end{matrix}$.
(b) $\dim(N(T-4I))^j = j$ for $j = 1, 2, 3, 4$, $\dim(N(T-4I))^j = 4$ for $j \geq 4$; $\dim(N(T+I)) = 2$, $\dim(N(T+I))^j = 4$ for $j \geq 2$.
(c) $m_T(t) = (t-4)^4(t+1)^2$.

2 (a) (1 mark) Compute the characteric polynomial $ch_T(t) = (t-4)^8$.
(3 marks) Compute the dimension of nullspaces: $\dim(N(T-4I)) = 3$, $\dim(N(T-4I))^2 = 6$, $\dim(N(T-4I))^3 = 8$
(2 marks) Dot diagram D_4 $\begin{matrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\\\bullet&\bullet&\end{matrix}$.

(b) (3 marks) $\begin{pmatrix} 4&1&&&&&&\\ &4&1&&&&&\\ &&4&&&&&\\ &&&4&1&&&\\ &&&&4&1&&\\ &&&&&4&&\\ &&&&&&4&1\\ &&&&&&&4 \end{pmatrix}$ (c) $(t-4)^3$

3.(a) $\begin{pmatrix} i&1&&&&&&&&\\ &i&&&&&&&&\\ &&i&&&&&&&\\ &&&i&&&&&&\\ &&&&-1&&&&&\\ &&&&&-1&&&&\\ &&&&&&-1&&&\\ &&&&&&&\sqrt{3}&1&\\ &&&&&&&&\sqrt{3}&1\\ &&&&&&&&&\sqrt{3} \end{pmatrix}$. (b) $(t-i)^2(t+1)(t-\sqrt{3})^3$.

4.(a) (total 7 marks)
For D_1, (1 mark) nullity$(T-I) = 3$ implies first row has 3 dots.
(2 marks) $N(T-I) \subseteq R(T-I)^2$ implies each column has at least 3 dots. But $\dim(K_1) = 9$. So each column has exactly 3 dots.
(1 mark) D_4 $\begin{matrix}\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\\\bullet&\bullet&\bullet\end{matrix}$

For D_{-1}, (1 mark) rank$(T+I)^3 =$ rank$(T+I)^2$ implies 3rd row has no dot.
(1 mark) rank$(T+I)^2 =$ rank$(T+I) - 1$ implies 2nd row has 1 dot.
(1 mark) D_{-1} $\begin{matrix}\bullet&\bullet&\bullet\\\bullet&&\end{matrix}$

(b) (3 marks) $\begin{pmatrix} 1&1&&&&&&&&&&& \\ &1&1&&&&&&&&&& \\ &&1&&&&&&&&&& \\ &&&1&1&&&&&&&& \\ &&&&1&1&&&&&&& \\ &&&&&1&&&&&&& \\ &&&&&&1&1&&&&& \\ &&&&&&&1&1&&&& \\ &&&&&&&&1&&&& \\ &&&&&&&&&-1&1&& \\ &&&&&&&&&&-1&& \\ &&&&&&&&&&&-1& \\ &&&&&&&&&&&&-1 \end{pmatrix}$. (c) (3 marks) $(t-1)^3(t+1)^2$

5.(a) (total 6 marks) Let $p(t) = \prod_i (t-\lambda_i)^{k_i}$, where λ_i are the eigenvalues of T. We have $f(t) = \prod_i (t-\lambda_i)^{2k_i}$. We know $\dim(K_{\lambda_i})$ = deg of $t-\lambda_i$ in f. Hence $\dim(K_{\lambda_i})$ is even.

(b) (total 6 marks) If T is diagonalizable, then all $k_i = 1$, and so $p(t)$ has $n/2$ distinct factors, hence T has $n/2$ distinct eigenvalues. The converse is just reversing the steps above.

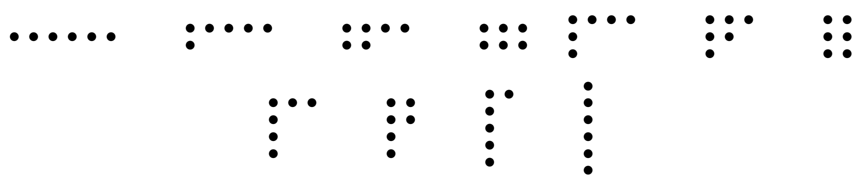
6. Notice T_1 and T_2 have the same Jordan form if and only if they have the same dot diagram.
For D_3, the condition implies for each T_i there are totally 3 dots and 2 on the first column. There is only one such diagram (which must be $\begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix}$) and so D_3 are the same for both T_i.
For D_i the condition implies for each T_i there are totally 6 dots and 3 on the first column. Try all possibilities (or see Q.10.(a)) one can see if the number of dots on the first row is fixed, the diagram is uniquely determined. Now recall the number of dots on the first row is just $\dim(N(T_i - iI))$.

7. (total 9 marks) (3 marks) First choose a basis β for V from cycles of generalized eigenvectors. Then decompose the set $\beta = \bigsqcup_{i=1}^{p} \beta_i$ such that $\bigsqcup_{i=1}^{k} \beta_i$ is a basis for $N(T-3I)^k$. Here p is the degree of the minimal polynomial of T.
(4 marks) Now take another set of vectors $\bigcup_{i=1}^{p} \gamma_i$ for $\gamma_i = (T+I)^i \beta_i$. I claim these sets are linearly independent. For if $\sum_{j=1}^{n} a_j (T+I)^{i_j} v_j = 0$, apply $(T-3I)^{p-1}$ to kill all $v_j \notin \beta_p$. Then $\sum_{j=1}^{|\beta_p|} a_j (T+I)^p (T-3I)^{p-1} v_j = 0$, with the v_j in the summand are all in β_j. Now recall $T+I$ is indeed invertible (one of the exercise of the last problem set), so $\sum_{j=1}^{|\beta_p|} a_j (T-3I)^{p-1} v_j = 0$. Also notice v_j are the end-vectors of different cycles, so they are linearly independent. Hence all coefficients a_j for each summand $(T+I)^{i_j} v_j$ (with $v_j \in \beta_p$) are 0. Continue the process, namely apply $(T-3I)^{p-2}$, $(T-3I)^{p-3}$, $\ldots$, one can argue that indeed all a_i are zero.
(2 marks) It remains to show γ_i is in $N(T^2 - 2T - 3I)^i$, but it is clear since $T^2 - 2T - 3I = (T+I)(T-3I)$. Therefore γ_i is a basis of $N(T^2 - 2T - 3I)^i$. Under such basis, $T^2 - 2T$ has Jordan-form the same as that of T.

8.(a) Since $T^i(x) = 0$ implies $T^{i+1}(x) = T(T^i(x)) = T(0) = 0$.
(b) Just apply Corollary on P.51.
(c) Choose an order of β_p so that the last vectors of β_{i+1} are those in β_i, then $[T]_\beta$ is of the form $\begin{pmatrix} 0 & * & \cdots & \cdots & * \\ & 0 & * & \cdots & * \\ & & \ddots & & \\ & & & 0 & * \\ & & & & 0 \end{pmatrix}$. Here each entry is a block for β_j, and $*$ can be anything .
(d) It is clear from the form in part (c).

9. Such a matrix has all 0 on the lower triangle, 1 or 0 on the 'first upper diagonal', and 0 on remaining entries. But restrict on blocks one can just assume all 1 on the 'first upper diagonal'. One observes a power of such matrix just move the 'diagonal with entries 1' upper by 1, and sufficient large power kills the matrix.
For example, take the simplest case $\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}^3 = 0$.

10.(a) Using Ex 8 we know the characteristic polynomial of each T_i splits, and therefore their Jordan-forms can be defined over F. Both T_i having the same minimal polynomial means their dot diagrams having the same number of dots on the first column, and T_i having the same dimension on nullspace means the diagrams have the same number of dots on the first row. Now construct all dot diagrams of order 6



One see that if the number of dots of first row and first column are fixed, the diagram is uniquely determined.
(b) For dot diagram of order 7 it is not the case as that of order 6. We have

11. Again the characteristic polynomial splits, hence we can consider Jordan form. Now $N(T - \lambda_j I) = N(T - \lambda_j I)^2$
$\Leftrightarrow$ For each λ_j, the dot diagram has only 1 row
$\Leftrightarrow$ Each block for λ_j is diagonal
$\Leftrightarrow$ T is diagonal.

12.(a) By Q.11 we just check $N(S - \lambda_j I) \supseteq N(S - \lambda_j I)^2$. If $(S - \lambda_j I)^2(x) = (\lambda_1 - \lambda_j)^2 v_1 + \cdots + (\lambda_k - \lambda_j)^2 v_k = 0$, then since each v_i are linearly independent, we have $(\lambda_i - \lambda_j)^2 = 0$, so $\lambda_i - \lambda_j = 0$. We have $(S - \lambda_j I)(x) =$

$(\lambda_1 - \lambda_j)v_1 + \cdots + (\lambda_k - \lambda_j)v_k = 0$, hence proved.
(b) Since T and S commutes with the projection on each K_{λ_i}, it suffices to prove the assertion on each K_{λ_j}. Since $S_{K_{\lambda_j}} \equiv \lambda_j I$ and $(T-S)_{K_{\lambda_j}} = T - \lambda_j I$ is nilpotent, all the statements are clear.