

MAT 247S - Problem Set 6

Due Thursday March 19

Questions 3b), 4c), 4d), and 6 will be marked.

1. Let V be a finite-dimensional vector space and let $T \in \mathcal{L}(V)$. Suppose that γ is a cycle of generalized eigenvectors corresponding to an eigenvalue λ of T . Let ℓ be the length of γ and let $W = \text{span}(\gamma)$.
 - a) Prove that W is T -invariant.
 - b) Prove that the characteristic polynomial of the restriction T_W of T to W is equal to $(-1)^\ell(t - \lambda)^\ell$.
 - c) Prove that the minimal polynomial of T_W is $(t - \lambda)^\ell$.
2. Let V be a vector space and let $T \in \mathcal{L}(V)$. Suppose that γ_1 and γ_2 are two cycles of generalized eigenvectors of T corresponding to an eigenvalue λ of T . Let x_1 and x_2 be the initial vectors of γ_1 and γ_2 , respectively. Prove that if $x_1 \neq x_2$, then $\gamma_1 \cap \gamma_2 = \emptyset$.
3. Suppose that V is a finite-dimensional vector space over a field F , $T \in \mathcal{L}(V)$, and $\lambda \in F$ is an eigenvalue of T . Let K_λ be the generalized eigenspace of T corresponding to the eigenvalue λ . Let $g(t) = a^d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t + a_0 \in P(F)$.
 - a) Show that K_λ is $g(T)$ -invariant.
 - b) Let $U = g(T)|_{K_\lambda}$ be the restriction of $g(T)$ to the subspace K_λ . Prove that U is invertible if and only if $g(\lambda) \neq 0$. Do not assume that $g(t)$ splits over F . (*Hint*: One implication can be proved by generalizing the argument used in the proof of Theorem 7.1, part (b).)
4. Suppose that V is an n -dimensional vector space over a field F , $T \in \mathcal{L}(V)$, and the characteristic polynomial $f(t)$ of T and the minimal polynomial $p(t)$ of T are given by:

$$f(t) = (-1)^n (t - \lambda_1)^{m_1} (t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$$

$$p(t) = (t - \lambda_1)^{\ell_1} (t - \lambda_2)^{\ell_2} \cdots (t - \lambda_k)^{\ell_k},$$

where $\lambda_1, \dots, \lambda_k \in F$ are such that $\lambda_i \neq \lambda_j$ if $1 \leq i \neq j \leq k$, and $1 \leq \ell_j \leq m_j$, $1 \leq j \leq k$. Let K_{λ_j} be the generalized eigenspace for T corresponding to the eigenvalue λ_j , $1 \leq j \leq k$. As shown in class (see also §7.1 of the text),

- (i) $V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \cdots \oplus K_{\lambda_k}$.
- (ii) $K_{\lambda_j} = N((T - \lambda_j I_V)^{m_j})$, $\dim K_{\lambda_j} = m_j$, and K_{λ_j} is T -invariant, $1 \leq j \leq k$.
- (iii) If $c \in F$ and $c \neq \lambda_j$, then the restriction of $T - c \cdot I_V$ to K_{λ_j} is invertible.
 - a) Suppose that $h(t) \in P(F)$ (that is, $h(t)$ is a polynomial with coefficients in F). Prove that each subspace K_{λ_j} is $h(T)$ -invariant, and then show that $h(T) = T_0$ if and only if the restriction of $h(T)$ to every K_{λ_j} is the zero operator.
 - b) Prove that $K_{\lambda_j} = N((T - \lambda_j \cdot I_V)^{\ell_j})$.
 - c) Prove that the minimal polynomial of the restriction $T|_{K_{\lambda_j}}$ of T to K_{λ_j} is equal to $(t - \lambda_j)^{\ell_j}$.
 - d) Prove that any cycle of generalized eigenvectors of T corresponding to the eigenvalue λ_j has length at most ℓ_j . Show that at least one such cycle has length ℓ_j .

5. Let V be an n -dimensional vector space over a field F . Assume that $n \geq 2$. Suppose that $c \in F$.
- Suppose that d is an integer and $1 \leq d \leq n$. Prove that there exists $T \in \mathcal{L}(V)$ such that the minimal polynomial $p(t)$ of T is equal to $(t - c)^d$.
 - Suppose that $T \in \mathcal{L}(V)$ has minimal polynomial $p(t) = (t - c)^{n-1}$. Prove that the characteristic polynomial of T is equal to $(-1)^n(t - c)^n$. (Note: As a first step, show that the characteristic polynomial of T must split over F .)
 - Suppose that T_1 and $T_2 \in \mathcal{L}(V)$ both have minimal polynomial $p(t) = (t - c)^{n-1}$. Prove that there exists an invertible $U \in \mathcal{L}(V)$ such that $T_2 = UT_1U^{-1}$.
 - Show by example that it is possible for T_1 and $T_2 \in \mathcal{L}(V)$ to have the same minimal polynomial when T_2 is not of the form UT_1U^{-1} for any choice of invertible $U \in \mathcal{L}(V)$.
6. Let β be the standard basis for \mathbb{C}^4 . Suppose that $T \in \mathcal{L}(V)$ and

$$[T]_{\beta} = \begin{pmatrix} 0 & i & 1 & -1 \\ i & 0 & 1 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

In each part below, please explain your answer fully and show all of your work.

- Find the characteristic polynomial and all eigenvalues of T .
 - Compute $\text{rank}(T - \lambda I_V)$ for every eigenvalue λ of T .
 - Find the minimal polynomial of T .
 - Find the Jordan canonical form of T .
 - For one eigenvalue λ of T , find a basis for the generalized eigenspace K_{λ} made up of cycles of generalized eigenvectors of T corresponding to λ .
7. Let $V = P_3(\mathbb{R})$ and let $\beta = \{x^3, x^2, x, 1\}$. Suppose that $T \in \mathcal{L}(V)$ and

$$[T]_{\beta} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}.$$

Find a Jordan canonical basis for T and find the Jordan canonical form of T .

8. Suppose that V is a 3-dimensional real vector space and $T \in \mathcal{L}(V)$ has Jordan canonical form

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Find the Jordan canonical form of $T^2 + T - I_V$.