MAT247S, 2009 Winter, Problem Set 5 Solution

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1. By finite dimensionality, one can write $W = \oplus_{i=0}^{n} F(T^n(x))$ for some finite n. Then (a) and (b) are straightforward.

2. As hinted, write $V = \oplus_{i=0}^{n} F(T^n(x))$ and $U(x) = g(T)(x)$ as in Ex 1, then for $k = 0, \ldots, n$ we have $UT^k(x) = T^kU(x)$ (since $UT = TU$) $= T^k g(T)(x) = g(T)T^k(x)$ (since T and $g(T)$ clearly commute). We have shown $U = g(T)$ on the basis $x, T(x), \ldots, T^n(x)$, hence they are equal.

3. (8 marks) Let $W \neq 0$ be a T-invariant subspace of V. Let m_T and m_{T_W} be the minimal polynomial for T and T_W respectively. Then we have m_{T_W} divides m_T since $m_T(T_W) = (m_T(T))_W = 0$. Also the characteristic polynomial of T splits, so does its factor m_T, and also m_{T_W}. As $W \neq 0$ we know $\deg(m_{T_W}) \geq 1$, which means m_{T_W} has at least a linear factor $t - \lambda$. So $\ker(T - \lambda I)$ (or $N(T - \lambda I)$ in the book) is non zero by definition of m_{T_W} being minimal. Hence $0 \neq v \in \ker(T - \lambda I)$ is an eigenvector. (If m_{T_W} does not split, there may not exist any eigenvector for T_W !!)

2 marks for showing m_{T_W} divides m_T.
2 for showing m_{T_W} splits
2 for showing $W \neq 0 \Rightarrow \deg(m_{T_W}) \geq 1$. (Most of you did not mention this.)
2 for showing eigenvalue and eigenvector exists.

4. (7 marks total) Let the minimal polynomial of T be $m_T(t) = \sum_{i=0}^{d} a_i t^i$ with $a_d = 1$. So we have $m_T(T) = \sum_{i=0}^{d} a_i T^i = T_0$, in particular $\sum_{i=0}^{d} a_i T^i(x) = 0$. So $T^d(x) = -\sum_{i=0}^{d-1} a_i T^i(x)$ (2 marks). Using this iteration we have $W = \operatorname{span}\{x, T(x), \ldots, T^{d-1}(x)\}$ (2 marks), i.e. W is spanned by d vectors, hence $\dim(W) \leq d$ (3 marks).

5. (10 marks) (Here I only check $j = 3$. The method also applies for $j = 1, 2$ and indeed for any j as long as we have eigenvectors of j distinct eigenvalues.)
First show $\{y_3, T(y_3), T^2(y_3)\}$ are linearly independent. Suppose $uy_3 + vT(y_3) + wT^2(y_3) = 0$ for some $u, v, w \in F$. Now writing $T^k(y_3) = T^k(x_1 + x_2 + x_3) = \lambda_1^k x_1 + \lambda_2^k x_2 + \lambda_3^k x_3$, we have

$$(u + v\lambda_1 + w\lambda_1^2)x_1 + (u + v\lambda_2 + w\lambda_2^2)x_2 + (u + v\lambda_3 + w\lambda_3^2)x_3 = 0. \tag{1}$$

Recall eigenvectors corresponding to different eigenvalues are linearly independent. Hence above equation (1) gives

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

One check the matrix above has determinant $(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\neq 0$ as all λ_i distinct. The only solution for (u,v,w) is $(0,0,0)$. We have shown $\{y_3, T(y_3), T^2(y_3)\}$ linearly independent.
To claim $\dim(W_3)=3$, it suffices to show any $T^k(y_3)$ is a linear combination of $\{y_3, T(y_3), T^2(y_3)\}$ for $k\geq 3$, i.e. there exists u,v,w so that

$$uy_3 + vT(y_3) + wT^2(y_3) = T^k(y_3). \tag{3}$$

Again writing (3) in terms x_1, x_2, x_3 as in (1), we have the linear equations

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \\ \lambda_3^k \end{pmatrix}. \tag{4}$$

Again since the determinant of the matrix is non zero, we can solve for (u,v,w) which cannot be all 0.

2 marks for setting up equations (2), (4).
3 for showing the determinant is non-zero.
5 for showing $\{y_3, T(y_3), T^2(y_3)\}$ being a basis, i.e. linearly independent and spanning.

6. Let $W_k = \operatorname{span}_F\{I_V, T, \ldots, T^k\}$. Then W_k is an increasing sequence of F-subspace of W. Since V is, and so are $L(V)$ and W, finite dimensional, there is k so that $W_k = W_{k+1} = \cdots = W$. By taking minimal such k we have $\dim(W) = k+1$. Now $W_k = W_{k+1}$ means $T^{k+1} = \sum_{j=0}^{k} a_j T^k$ for some $a_j \in F$. This is a polynomial equation for T, so its minimal polynomial $m_T(x)$ must have degree $d \leq k+1 = \dim(W)$. Now if $d \lneq k+1$, then it is easily seen that $W_{d-1} = W_d = \cdots$, contradicting the minimality of k. So $d = k+1 = \dim(W)$.

7.(a) One can show $(p(T))^* = \overline{p}(T^*)$, and using $\langle v, p(T)^*(w)\rangle = \langle p(T)(v), w\rangle = \langle 0, w\rangle = 0$ for any $v, w \in V$, we have $\overline{p}(T^*) = 0$.
(b) Since $\overline{p}(T^*) = 0$, we have the minimal polynomial m_{T^*} divides $\overline{p}$. Now if $\deg(m_{T^*}) \lneq \deg(\overline{p})$, then notice $\overline{m_{T^*}}(T) = \overline{m_{T^*}}(T^{**}) = (m_{T^*}(T^*))^* = 0^* = 0$, we have T satisfies the polynomial $\overline{m_{T^*}}$ whose degree is $\lneq \deg(p)$, contradiction. So $\deg(m_{T^*}) = \deg(\overline{p})$ and hence $m_{T^*} = \overline{p}$ as both are monic.

8. If there is $g = \sum_{i=0}^{n} a_i t^i \in P(\mathbb{R})$ that $g(T) = T_0$, then all polynomials in $V = P(\mathbb{R})$ satisfy a differential equation: $\sum_{i=0}^{n} a_i f^{(i)} = 0$ for all $f \in V$. This is absurd, for example we can take $f(t) = t^n$, then it is easily checked that $g(T)(f) = n! a_n \neq 0$ (where we can assume $a_n \neq 0$ without loss of generality).

9.(a) Since $P^{-1}TP = D$ implies $P^{-1}T^2P = D^2$.
(b) (5 marks) The fact $g(T^2) = 0$ implies T satisfies $g(t^2) = 0$, hence $p(t)$, as the minimal polynomial of T, must divide $g(t^2)$.

(c) (8 marks total) Given T is diagonalizable, suppose $\{\lambda_1, \ldots, \lambda_N\}$ are distinct eigenvalues for T, then
(2 marks) the minimal polynomial $p(t) = \prod_{i=1}^{N}(t - \lambda_i)$.
(2 marks) Given $p(t) = p(-t)$, we have λ is an eigenvalue iff $-\lambda$ is.
(2 marks) We can pair the roots by sign because all $\lambda_i \neq 0$ (given T invertible).

Hence N is even $2n$ and we have $p(t) = \prod_{i=1}^{n}(t^2 - \lambda_i^2)$.
(2 marks) Now T^2 is also diagonalizable, and its eigenvalues are $\{\lambda_i^2 | i = 1, \ldots, n\}$ all distinct, so its minimal polynomial is $g(t) = \prod_{i=1}^{n}(t - \lambda_i^2)$. Hence $p(t) = g(t^2)$.

1 marks will be deducted if you write $T = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ implies $p(t) = \prod_{i=1}^{m}(t - \lambda_i)$. It is because T may have equal eigenvalues.

10.(a) Let m be the minimal polynomial for T, then $m(T) = 0$ implies (for $i = 1, 2$) $m(T_{W_i}) = (m(T))_{W_i} = 0$, i.e. T_{W_i} satisfies $m(t) = 0$. Hence p_i divides m.
(b)(c) Answer is open. For example, one may take T to be diagonal with equal eigenvalue in (b) and different eigenvalues in (c), and W_i be (sum of) different eigenspaces.

7 marks total for (b). 1 each of V, W_1, W_2, T, p, p_1 and p_2.

11.(a) Choose a basis for W_1 and extend to one of V, then $[T]$ is block upper triangular, i.e. of the form $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$, then g_1 is the minimal polynomial of D.
(b) If $h(T)(V) \subseteq W_1$, then $h(D) = 0$, so g_1 divides h as minimal.
(c) Sine $0 = m_T([T]) = \left(\begin{smallmatrix} m_T(A) & * \\ 0 & m_T(D) \end{smallmatrix}\right)$, we have $m_T(D) = 0$ and so g_1 divides m_T. Also because m_T divides ch_T, we have g_1 divides ch_T.
(d) Choose a basis of W_2, extend to one of W_1, and further to one of V. Then A and B are further divided, say $A = \left(\begin{smallmatrix} E & F \\ 0 & H \end{smallmatrix}\right)$, $B\left(\begin{smallmatrix} J \\ K \end{smallmatrix}\right)$. We have $[T] = \left(\begin{smallmatrix} E & F & J \\ 0 & H & K \\ 0 & 0 & D \end{smallmatrix}\right)$. Take g_2 to be the minimal polynomial of the 2×2 lower diagonal block $\left(\begin{smallmatrix} H & K \\ 0 & D \end{smallmatrix}\right)$.