MAT247S, 2009 Winter, Problem Set 4 Solution

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1.(a) $P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.
(b) Let $T_{i_1,\dots,i_k}$ = orthogonal projection on the space spanned by $\{v_{i_1}, \dots, v_{i_k}\}$, where v_j is the j-column of P, then $T = 4T_1 + (-2)T_{23}$.

2.(a) We did this in the last Problem Set.
(b) Let $\omega = \exp(\pi i/4)$ 8-th roots of 1, then $P = \begin{pmatrix} 0 & -\omega & 0 & \omega \\ \omega & 0 & -\omega & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}^*$, $D = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$.
(c) $T = \omega T_{12} + (-\omega)T_{34}$. (See Q.1(a) for notation)

3.(a) Check $AA^* = A^*A$.
(b) (11 marks) First to find the eigenvalues of A, expand $\det(A - \lambda I) = \cdots$ and solving $\lambda = 2i$ and $-2i$ (each has multiplicity 2). (2 marks)
To find an orthogonal basis, one may consider the matrix equation $Av = \lambda v$ by substituting $\lambda = 2i$ and $\lambda = -2i$ respectively, and solve for v. One possible solution is

$$Av = 2iv \quad \Rightarrow \quad v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and

$$Av = -2iv \quad \Rightarrow \quad v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Form the matric $U = (v_1, v_2, v_3, v_4)$ and show $U^*U = 1$, i.e. U is unitary. (If your solutions do not form an orthonormal basis, try to use Gram-Schmidt or whatever method.)
(2 marks for solving equations, 2 for finding an orthogonal basis, and 1 for finding an normalized one.)
Now one can show $U^*AU = D$, and take $P = U^*$ we get $PAP^* = D$. We have

$$P = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}$$

(2 for P and 2 for D)
(c) (3 marks) Let $T_{i_1,\dots,i_k}$ = be the orthogonal projection on the space spanned by $\{v_{i_1}, \dots, v_{i_k}\}$, then $T = (2i)T_{12} + (-2i)T_{34}$.

4. (10 marks) ($\Rightarrow$) (6 marks) Since T is unitary, there is an orthonormal basis $\{v_1, \dots, v_n\}$ so that $[T] = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ in diagonal form. Since $T^2 = -I_V$, the characteristic polynomial of T is $\lambda^2 + 1$, whose roots are $\lambda = i$ or $-i$. Say $\lambda_1 = \cdots = \lambda_k = i$ and $\lambda_{k+1} = \cdots = \lambda_n = -i$, then $W =$

span$\{v_1, \dots, v_k\}$ and $W^\perp = \text{span}\{v_{k+1}, \dots, v_n\}$.
(2 for showing T can be diagonalized, 2 for showing the eigenvalues are i and $-i$, 2 for defining W and $W^\perp$.)
($\Leftarrow$) (4 marks) Take an orthonomal basis $\{v_1, \dots, v_k\}$ for W and $\{v_{k+1}, \dots, v_n\}$ for $W^\perp$. Then $\{v_1, \dots, v_n\}$ is an orthonormal basis for V. Under such basis, $[T] = \text{diag}(i, \dots, i, -i, \dots, -i)$. It is easily seen that $[T][T]^* = I$, so $[T]$ and hence T is unitary.
(2 for writing T as diagonal form, 2 for showing T is unitary.)

5. Since T is normal, T is diagonalizable. Since $T^m = 0$, any eigenvalue λ of T satisfies $\lambda^m = 0$. So any $\lambda = 0$ and $T = 0$.

6.(a) (8 marks) Take an orthogonal basis $\{v_1, \dots, v_{2m}\}$ for the inner product space V with dimension $n = 2m$, then define $T : V \to V$, $v_{2i-1} \mapsto v_{2i}$ and $v_{2i} \mapsto -v_{2i-1}$ for $i = 1, \dots, m$. Since I have defined T on a basis of V, it is defined on whole V by linearity. It is easy to show $T^2 : v_i \mapsto -v_i$ for all i, hence $T^2 = -I_V : V \to V$. To check T is orthogonal, consider under the given basis,

$$A = [T] = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$$

is an orthogonal matrix, by showing $A^T A = I$.
(3 for defining T, 3 for showing it is orthogonal, and 2 for showing $T^2 = -1$.)
(b) (8 marks) Since $\langle T(x), x\rangle = \langle T(x), -(-I_V)(x)\rangle = -\left\langle T(x), T^2(x)\right\rangle = -\langle x, T(x)\rangle$ (by orthogonality of T) $= -\langle T(x), x\rangle$, we must have $\langle T(x), x\rangle = 0$.