

MAT247S, 2009 Winter, Problem Set 3 Solution

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1.(a) Just check $AA^* = A^*A$. (b) Let $\omega = \exp(\pi i/4)$ be the 8-th roots of unity. A possible orthonormal basis is $\left\{ \begin{pmatrix} 0 \\ \omega \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\omega \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\omega \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \omega \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, with eigenvalues $\omega, \omega, -\omega, -\omega$ respectively.

2.(a) (i) yes (ii) yes (iii) no
(b) (i) no (ii) yes (iii) no
(c) (i) no (ii) yes (iii) yes

3. No question printed

4. (a) Direct computation.

(b) Notice the following facts

(i) If A is self-adjoint, then iA is anti-self-adjoint ($A^* = -A$).

(ii) If A is both self-adjoint and anti-self-adjoint then $A = 0$.

So if $T_1 + iT_2 = T = U_1 + iU_2$, then $T_1 - U_1 = i(T_2 - U_2)$, with LHS self-adjoint and RHS anti-self-adjoint. So both are 0 and hence $T_1 = U_1$ and $T_2 = U_2$.

(c) T normal iff $T^*T = TT^*$ iff $(T_1 + iT_2)(T_1 + iT_2)^* = (T_1 + iT_2)^*(T_1 + iT_2)$ iff $(T_1 + iT_2)(T_1 - iT_2) = (T_1 - iT_2)(T_1 + iT_2)$ (since T_i are self-adjoint) iff $i(T_2T_1 - T_1T_2) = i(T_1T_2 - T_2T_1)$ iff $T_1T_2 = T_2T_1$.

5. No. For example, Take $V = F^2$, standard inner product, $U(1, 0) = U(0, 1) = (1, 0)$. So U satisfies the given condition, but U is not one-one, so not invertible and not unitary.

6.(a) (15 points, 5 for each part, work is needed) (i) no (ii) yes (iii) no

(b) Using the basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ for $M_{2 \times 2}(\mathbb{R})$, one can compute $T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. (i) Yes (ii) Yes (iii) Yes

7.a) Under the standard basis, $T = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$ which is symmetric, hence diagonalizable (Theorem 6.20 of book).

(b) The matrix of the given inner product is $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. So $T^* = B^{-1}\overline{T}^t B = \begin{pmatrix} 2 & 3+3i \\ 0 & -2i \end{pmatrix}$.

(c) Suppose $[T]_\beta$ is diagonal, then it is normal. But one can check $TT^* - T^*T \neq 0$. This yields contradiction.

8.(a)(b) Notice the orthogonal projections T_1, T_2 are self-adjoint since (under suitable orthogonal basis) they are of the form UDU^{-1} where U is

unitary and D is diagonal with entries 0 or 1. Using this one can compute $UU^* - U^*U = \dots = \text{Im}(c_1\bar{c}_2)(T_1T_2 - T_2T_1)$. Hence if $c_1\bar{c}_2$ is real or $T_1T_2 = T_2T_1$, we have $UU^* = U^*U$, i.e. U normal.

(c) (10 points, 4 for giving example and 6 for justifying the answer) Solution is open. Just make sure $c_1\bar{c}_2$ is not real and $T_1T_2 \neq T_2T_1$. For instance, since $\dim_{\mathbb{C}}V = 2$, we have $T_1T_2 \neq T_2T_1$ if and only if either $W_1 \neq W_2$ or W_1 is not orthogonal to W_2 .

9.(a) (2 points) $TT^* = T(T - 2iI) = T^2 - 2iT = (T - 2iI)T = T^*T$

(b) (3 points) Since λ is eigenvalue for T , $\bar{\lambda}$ is eigenvalue of T^* and $\lambda - \bar{\lambda} = 2\text{Im}(\lambda)$ is eigenvalue for $T - T^* = 2i$. Hence $\text{Im}(\lambda) = 1$ and $\lambda - i$ is real.

(c) (5 Points) Suppose $T + I$ is not invertible, then it is not one-one. So there is non-zero $v \in V$ that $(T + I)v = 0$. This shows -1 is an eigenvalue for T , which contradicts part (b). Similarly for $T + (-2i + 1)I = T^* + I$.

(d) (5 Points) Write $U = (T^* + I)(T + I)^{-1}$, which is of the form A^*A^{-1} with $A = T + I$ invertible. Such matrix U is unitary.

10. (10 points) Since T is normal, by Theorem 6.16 in the book there is orthonormal basis $\beta = \{v_1, \dots, v_n\}$ so that $Tv_i = \lambda_i v_i$, i.e. v_i are eigenvectors for T . Since T is invertible, all λ_i are no zero. Let $T_1v_i = \frac{\lambda_i}{|\lambda_i|}v_i$ and $T_2v_i = |\lambda_i|v_i$ for all i . Since we have defined T_1, T_2 on the basis, they are defined on V as linear operators. We write λ for λ_i and v for v_i below and check:

(i) $T_1^*v = \frac{\bar{\lambda}}{|\lambda|}v$ and so $T_1T_1^*v = \frac{\bar{\lambda}}{|\lambda|} \frac{\lambda}{|\lambda|}v = 1 \cdot v$. Similarly $T_1^*T_1v = 1 \cdot v$. Hence $T_1^*T_1 = T_1T_1^* = 1_V$, i.e. T_1 unitary

(ii) $T_2^*v = |\bar{\lambda}|v = |\lambda|v = T_2v$, so T_2 self-adjoint

(iii) $T_1T_2v = T_1|\lambda|v = |\lambda|T_1v = |\lambda|(\frac{\lambda}{|\lambda|})v = \lambda v = Tv$. Similarly $T_2T_1v = Tv$.

(2 points for applying the Theorem stated, 2 for using the hint to find T_1, T_2 , and 2 each for (i) - (iii).)