MAT247 - Problem Set 2 Solution

Problem 1

a) We can easily pick $\{x, x^2\}$ as a basis. Using Gram-Schmidt to get a orthonormal basis. Take inner product of T with this basis we will get $T(v) = (a - \frac{1}{2}c)x^2 + (b + \frac{1}{4}c)x$
b) We have $\{(1, -i, \sqrt{2}, 0)^T, (1, 0, \sqrt{2}, -1)^T\}$, apply Gram-Schmidt on this to get our ON basis. Finally we get $T(v) = \frac{1}{4}(x_1 + \sqrt{2}x_3 + x_4) * (1, 0, \sqrt{2}, 1)^T + \frac{1}{28}(x_1 + 4ix_2 + \sqrt{2}x_3 + 3x_4) * (1, -4i, \sqrt{2}, 3)^T$
c) We talked about this basis in the tutorial. This basis is consists of matrices with exactly one entry of 1 on the diagonal (zeros everywhere else) and also matrices with exactly two 1s on the off diagonal (zeros on diagonal), where the two 1s satisfy the symmetric condition. With this basis, it's not hard to show that $T(v) = \frac{1}{2}(A + A^T)$

Problem 2

a) This part is obvious, just write out a vector as a linear combination of the two basis. One can easily see the union is the basis for W.
b) Let $v = w_1 + w_2$ then $U(v) = w_1 + w_2 + T_1(w_2) + T_2(w_1)$. Thus we have $T_1(w_2) = 0$ and $T_2(w_1) = 0$ so $w_1 \in w_2^{\perp}$. Take two basis and write $T_1(v) = \sum_{i=0}^{n} < v, x_i > x_i$ and $T_2(v) = \sum_{y=0}^{m} < v, y_j > y_j$ we can easily see that $T_1(T_2(v)) = 0$ and similarly the reverse.
c) Following part b, use basis for T_1 and T_2, the union is a basis for W. We see that with the basis expansion T = U.

Problem 3

Take $x \in U(W)$, $UTU^{-1}(x) = UT(y) = U(y) = x$ where $y = U^{-1}(x), y \in W$. Now take $x \in U(W)^{\perp} = U(W^{\perp})$ then, $UTU^{-1}(x) = UT(y) = U(0) = 0$. Hence, UTU^{-1} is the orthogonal project on $U(w)$.

Problem 4

Write $f(x) = ax^2 + bx + c$ and $g(x) = dx^2 + ex + f$, take the inner product and solve for the unknown variables. We will get $d = -2, e = 1/2, f = -3$. Thus, $g(x) = -2x^2 + \frac{1}{2}x - 3$

Problem 5

$$< T_C(A), B >= tr(CAB^*) = tr(B^*CA) = tr(AB^*C) = tr(A(C^*B)^*) =< A, C^*B >$$

so we must have ${T_C}^*(A) = C^*A$.

Also, it is clear that $(T_C)^* = -iT_C$ if and only if $C^* = -iC$ (evaluate the operators at the identity matrix), which is true iff $\overline{c_{ji}} = -ic_{ij}$, where $C = (c_ij)$. Note that this in particular implies the diagonal entries are of the form $\lambda(1+i)$ for $\lambda \in \mathbb{R}$. So, for say $n = 3$, we get all matrices of the form

$$\begin{bmatrix} \lambda_1(1+i) & c_{12} & c_{13} \\ i\overline{c_{12}} & \lambda_2(1+i) & c_{23} \\ i\overline{c_{13}} & i\overline{c_{23}} & \lambda_3(1+i) \end{bmatrix}$$

Problem 6

The linearity of U follows immediately from the linearity of the inner product. Now for $x, y \in V$, consider

$$\begin{aligned} < x, U(y) >= \sum_{j=1}^{n} < x, < y, T(x_j) > x_j >= & \sum_{j=1}^{n} \overline{< y, T(x_j) >} < x, x_j > \\ = & \sum_{j=1}^{n} < T(x_j), y >< x, x_j > \\ = & < T(\sum_{j=1}^{n} < x, x_j > x_j), y > \\ = & < T(x), y > \end{aligned}$$

Since this was true for any $x, y \in V$, by the uniqueness of the adjoint, $U = T^*$.

Problem 7

(i) We have

$$\begin{aligned} < T_{x,y}(z), w > = << z, y > x, w > = < z, y >< x, w > &= < z, \overline{< x, w >} y > \\ &= < z, < w, x > y > \\ &= < z, T_{y,x}(w) > \end{aligned}$$

for any $z, w \in V$ so by uniqueness of the adjoint, $T_{x,y}{}^{*} = T_{y,x}$.

(ii) Let $\beta = x_1, \ldots, x_n$ be an orthonormal basis for V. Then

$$\begin{aligned} tr(T_{x,y}) = \sum_{j=1}^{n} < T_{x,y}(x_j), x_j > = \sum_{j=1}^{n} << x_j, y > x, x_j > \\ = < x, \sum_{j=1}^{n} < y, x_j > x_j > = < x, y > \end{aligned}$$

(iii) This is proved in a similar way. (ie. evaluate both operators on an arbitrary vector and notice that you get the same answer...)

(iv) Suppose $T_{x,y} = T_{x,y}{}^{*} = T_{y,x}$. If x or y are zero, then $T_{x,y} = T_{y,x} = 0$ trivially, so suppose $y \neq 0$. Then $< y, y > x = < y, x > y$, so $x = \lambda y$ for some $\lambda \in F$. But we can see that

$$\lambda = \frac{< y, x >}{< y, y >} = \frac{\overline{\lambda} < y, y >}{< y, y >} = \overline{\lambda},$$

so λ must be real. Hence $T_{x,y}$ is self-adjoint if and only if y is zero, or $x = \lambda y$ for some $\lambda \in \mathbb{R}$.

Problem 8

a) Straight forward calculation gives us $T * (a + bx) = ai - bi + (a - bi + ai)x$

b) Since we have 2 by 2 matrices, we get two equation on the values of c. Namely $i + c = 1 + i\overline{c}$ and $1 - ic = -i + \overline{c}$. Solving this equations and letting $c = a + bi$, we see that we only require $a = 1 + b$, thus any complex number of the form $c = (1 + b) + bi, b \in \mathbb{R}$ will work.

Problem 9

a) If $U = T - T^*$, then $U^* = (T - T^*)^* = T^* - T^{**} = T^* - T = -U$, and for any $x \in V$,

$$< U(x), x > = < T(x), x > - < T^*(x), x > = < T(x), x > - < x, T(x) > = 0$$

since V is assumed to be real.

b) Suppose now that V is finite-dimensional, and has dimension $n \geq 2$. In light of the above, it suffices to find an operator T which is not self-adjoint. Here's one: take any orthonormal basis $\{x_1, \ldots, x_n\}$, and set $T(x_1) = x_1 + x_2, T(x_j) = x_j$ for $j \neq 1$. Clearly $T \neq T^*$ (check the matrix of T if you like), so $U = T - T^*$ is non-zero, and $< U(x), x > = 0 \ \forall x \in V$.

Problem 10

$$\begin{aligned} U^* = (f(T))^* &= (a_nT^n + \cdots + a_0Id)* = (a_nT^n)^* + \cdots + (a_1T)^* + (a_0Id)^* \\ &= \overline{a_n}(T^*)^n + \cdots + \overline{a_1}T^* + \overline{a_0}Id \\ &= \overline{f}(T^*) \end{aligned}$$

Problem 11

a) Clearly, $N(T) \subseteq N(T^*T)$. On the other hand, if $x \in N(T^*T)$, then

$$0 = < T^*T(x), x > = < T(x), T(x) > = ||T(x)||^2,$$

hence $x \in N(T)$, and since $rank(T) + dim(N(T)) = dim(V)$, we have $rank(T^*T) = rank(T)$.

b) Note that $x \in N(T)$ if and only if $x \in R(T^*)^\perp$ (this follows easily from definitions). Therefore,

$$rank(T) = dim(V) - dim(N(T)) = dim(V) - dim(R(T^*)^\perp) = dim(R(T^*)) = rank(T^*).$$

If we apply part a) to T^*, we have that $rank((T^*)^*T^*) = rank(TT^*) = rank(T^*) = rank(T)$.

c) Just put these results together for the matrix A, (which defines a linear operator on $V = F^n$).

Problem 12

a) For any $x, y \in V$, note that

$$< x, y > = < T \cdot T^{-1}(x), y > = < T^{-1}(x), T^*(y) > = < x, (T^{-1})^* \cdot T^*(y) > .$$

Letting x range over an (orthonormal) basis, say, we have $(T^{-1})^* \cdot T^* = Id$. Similarly, $T^* \cdot (T^{-1})^* = Id$, so T^* is invertible with inverse given by $(T^{-1})^*$.

b) We have $TT^* = c = T^*T$, so T is normal by definition.

c) Suppose that there is an eigenvalue λ for T (i.e. if by chance $F = \mathbb{R}$, suppose that there actually is a root of the characteristic polynomial), with eigenvector x. Note that since T is invertible, λ is necessarily nonzero. Then

$$\lambda\overline{\lambda} < x, x > = < T(x), T(x) > = < T^*T(x), x > = c < x, x >$$

Since $< x, x > \neq 0$, we have $c = \lambda\overline{\lambda} = |\lambda|^2$, which is a positive real number.

d) We actually didn't need to assume that T has an eigenvalue. Consider, for $x \neq 0$,

$$< T(x), T(x) > = < T^*T(x), x > = c < x, x >$$

Since T is injective, $T(x) \neq 0$, so $c = \frac{||T(x)||^2}{||x||^2} > 0$.

e) Let $\{x_1, \ldots, x_n\}$ be an orthonormal basis, and note that for $c > 0$, we have

$$\begin{aligned} < \sqrt{c}^{-1}T(x_i), \sqrt{c}^{-1}T(x_j) > = \frac{1}{c} < T^*T(x_i), x_j > = \delta_{ij} \ \forall i, j \\ \Longleftrightarrow T^*T(x_i) = cx_i \ \forall i \\ \Longleftrightarrow T^*T = cId \\ \Longleftrightarrow T^* = cT^{-1} \end{aligned}$$

where δ_{ij} is the usual Kronecker delta (1 if $i = j$, 0 otherwise).