MAT 247S - Problem Set 2

Due Thursday January 29th

NOTE: Questions 1a), 2b), 3, 8 and 12e) will be marked.

1. For each of the following inner product spaces V, let $T : V \to V$ be the orthogonal projection of V on the subspace W. Compute $T(v)$ for all vectors v in V.
   a) Let $V = P_2(\mathbb{R}) = \{ f(x) = ax^2 + bx + c \mid a, b, c \in \mathbb{R} \}$ be the vector space of polynomials of degree at most two, having real coefficients. Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V that satisfies

   $$\begin{array}{lll} \langle 1, 1 \rangle = 2 & \langle 1, x \rangle = 2 & \langle 1, x^2 \rangle = -2 \\ \langle x, x \rangle = 4 & \langle x, x^2 \rangle = -2 & \langle x^2, x^2 \rangle = 3 \end{array}$$

   Let $W = \operatorname{Span}\{ x^2, x \}$.

   b) Let $V = \mathbb{C}^4$ with the standard inner product. Let

   $$W = \{ x = (x_1, x_2, x_3, x_4) \in V \mid \sqrt{2}x_1 - x_3 = 0,\ x_1 - ix_2 + x_4 = 0 \}.$$

   c) Let $n \geq 2$, $V = M_{n\times n}(\mathbb{R})$, with the inner product $\langle A, B \rangle = \operatorname{trace}(AB^*)$, A, $B \in V$, and $W = \{ A \in V \mid A = A^* \}$.

2. Let W_1 and W_2 be finite-dimensional subspaces of an inner product space V. Assume that $W_1 \cap W_2 = \{\mathbf{0}\}$. Let

   $$W = W_1 + W_2 = \{ v \in V \mid v = w_1 + w_2 \text{ for some } w_1 \in W_1 \text{ and } w_2 \in W_2 \}.$$

   a) Prove that if β is a basis for W_1 and γ is a basis for W_2, then $\beta \cup \gamma$ is a basis for W.

   For parts b) and c), let T_j be orthogonal projection of V on W_j, $j = 1, 2$, and let U be orthogonal projection of V on W. Let $T = T_1 + T_2$.

   b) Prove that if $T = U$, then $W_1 \subset W_2^{\perp}$ and $T_1T_2(v) = T_2T_1(v) = \mathbf{0}$ for all $v \in V$.
   c) Suppose that $W_1 \subset W_2^{\perp}$. Prove that $T = U$.

3. Let W be a nonzero subspace of a finite-dimensional inner product space V. Let $T \in \mathcal{L}(V)$ be the orthogonal projection of V on W. Suppose that $U \in \mathcal{L}(V)$. Let

   $$U(W) = \{ U(x) \mid x \in W \} \quad \text{and} \quad U(W^{\perp}) = \{ U(y) \mid y \in W^{\perp} \}.$$

   Suppose that U is invertible and $U(W)^{\perp} = U(W^{\perp})$. Prove that UTU^{-1} is the orthogonal projection of V on the subspace $U(W)$.

4. Let $V = P_2(\mathbb{R})$, with the same inner product as in question 1a). Find the polynomial $g(x) \in V$ such that $\langle f , g \rangle = f'(0) - f(1)$ for all $f(x) \in V$.

5. Let $V = M_{n\times n}(\mathbb{C})$ with inner product $\langle A, B \rangle = \operatorname{trace}(AB^*)$, A, $B \in V$. Fix $C \in V$. Define a linear operator $T_C : V \to V$ by $T_C(A) = CA$.
   a) Find the adjoint $(T_C)^*$ of T_C
   b) In the case $n = 3$, find all $C \in V$ that have the property that $(T_C)^* = -iT_C$.

6. Suppose that $\beta = \{ x_1, \dots, x_n \}$ is an orthonormal basis for an inner product space V. Let $T : V \to V$ be a linear operator. Define a function $U : V \to V$ by

$$U(x) = \sum_{j=1}^{n} \langle x, T(x_j) \rangle x_j, \qquad x \in V.$$

   a) Prove that U is linear.
   b) Prove that $U = T^*$.

7. Let V be a finite-dimensional inner product space. For each x, $y \in V$, let $T_{x,y}$ be the linear operator on V defined by $T_{x,y}(z) = \langle z\, , \, y \rangle \, x$. Show that
   (i) $T^*_{x,y} = T_{y,x}$.
   (ii) $\operatorname{trace}(T_{x,y}) = \langle x\, , \, y \rangle$.
   (iii) $T_{x,y} T_{u,v} = T_{x, \langle y\, , \, u \rangle v}$.
   (iv) Under what conditions is $T_{x,y}$ self-adjoint? Explain your answer fully.

8. Let $V = P_1(\mathbb{C}) = \{ a + b\, x \mid a, \, b \in \mathbb{C} \}$ be the complex vector space of polynomials in the variable x, of degree at most 1, with complex coefficients. Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V and $\beta = \{ x, x + 1 \}$ is an orthonormal basis for V. Let $T : V \to V$ be the linear operator on V defined by $T(a + b\, x) = -a + b + (ai - a + b)x$, a, $b \in \mathbb{C}$.
   a) Find $T^*(a + b\, x)$ for all a, $b \in \mathbb{C}$.
   b) For each $c \in \mathbb{C}$, let $U_c = T + cT^*$. Find all complex numbers c such that $U_c = (U_c)^*$. (Please explain your answer fully.)

9. Let $T : V \to V$ be a linear operator on a real inner product space V.
   a) Let $U = T - T^*$. Prove that $U^* = -U$ and $\langle U(x), x \rangle = 0$ for every vector $x \in V$.
   b) Suppose that V is finite-dimensional and has dimension at least 2. Using part a), prove that there exists a **nonzero** linear operator $U : V \to V$ such that $\langle U(x), x \rangle = 0$ for every vector $x \in V$.

10. Suppose that $f(t) = a_n t^n + \cdots + a_1 t + a_0$, where $a_0, \dots, a_n \in F$ and $F = \mathbb{R}$ or $F = \mathbb{C}$. For $T \in \mathcal{L}(V)$, define $f(T) = a_n T^n + \cdots + a_1 T + a_0 I_V$ and $\bar{f}(T) = \bar{a_n} T^n + \cdots + \bar{a_1} T + \bar{a_0} I_V$. Assume that V is a finite-dimensional inner product space. Let $T \in \mathcal{L}(V)$ and let $U = f(T)$. Prove that $U^* = \bar{f}(T^*)$.

11. §6.3, #13.

12. Let T be an invertible linear operator on a finite-dimensional inner product space V.
   a) Prove that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.
   b) Suppose that $T^* = cT^{-1}$ for some scalar $c \in F$. Show that T is normal.
   c) Show that if T is as in part b), then $\lambda\bar{\lambda} = c$ for every eigenvalue λ of T. Explain why this implies that c is a positive real number when $F = \mathbb{C}$, and also when $F = \mathbb{R}$ and the characteristic polynomial of T has at least one real root.
   d) If $F = \mathbb{R}$ and $T^* = cT^{-1}$, then it is possible that T has no real eigenvalues. In this case, how would you show that c is positive?
   e) Let c be a positive real number. Prove that $T^* = cT^{-1}$ if and only if for every orthonormal basis $\{ x_1, \dots, x_n \}$ for V, the set $\{ \sqrt{c}^{\,-1} T(x_1), \dots, \sqrt{c}^{\,-1} T(x_n) \}$ is also an orthonormal basis for V. (Here, $\sqrt{c}$ denotes the positive square root of c.)