MAT247 - Problem Set 1 Solution

Problem 1

a) It is straightforward to verify this is an inner product.
b) Again, easily verified using matrix multiplication.
c) Not an inner product. We have $< x, x >= x_1^2 - 4x_2^2$, take $x_1 = 5$ and $x_2 = 1$, we see that $< \cdot, \cdot >$ is not positive definite. Hence, not an inner product.
d) Not an inner product. Take $f(x) = x(x - i)$ then $< f, f >= 0$, but $f \neq 0$, hence not an inner product.
e) Easy to verify it is an inner product. For condition 4, we use the property that $< \cdot, \cdot >'$ is an inner product, hence we get positive definiteness.

Problem 2

In this question, to be precise, we need to make sure the definition is well-defined. If $x \in span\{v_1, \ldots, v_n\}$, with v_i's in β (i.e $x = \sum_{i=1}^{n} a_i v_i$), then for any other basis vector $w \in \beta$, the only way to represent x as an element of $span\{v_1, \ldots, v_n, w\}$ is by $x = a_1 v_1 + \ldots + a_n v_n + 0 \cdot w$, since β is a basis. Therefore, in computing $< x, y >$, taking any finite collection of basis elements whose span contains both x and y will give the same answer. It is then straightforward to verify that we indeed get an inner product, and in particular, in the case of V is finite-dimensional one recovers the standard inner product.

Problem 3

First, convince yourself that we've been given an inner product. Let $u_1 = (1,1,0), u_2 = (1,0,1), u_3 = (0,1,1)$. We apply the Gram-Schmidt algorithm: take $v_1 = u_1$, and set

$$v_2 = u_2 - \frac{< u_2, v_1 >}{||v_1||^2} v_1 = (1,0,1) - \frac{3}{5}(1,1,0) = (2/5, -3/5, 1)$$

$$v_3 = u_3 - \frac{< u_3, v_1 >}{||v_1||^2} v_1 - \frac{< u_3, v_2 >}{||v_2||^2} v_2 = (0,1,1) - \frac{2}{5}(1,1,0) - \frac{(-1/5)}{(55/25)}(2/5, -3/5, 1)$$
$$= (-4/11, 6/11, 12/11)$$

and $\{v_1, v_2, v_3\}$ form an orthogonal basis. If $v = (0,2,1)$, we can write

$$v = \frac{< v, v_1 >}{||v_1||^2} v_1 + \frac{< v, v_2 >}{||v_2||^2} v_2 + \frac{< v, v_3 >}{||v_3||^2} v_3 = \frac{4}{5} v_1 - v_2 + \frac{9}{4} v_3$$

Problem 4

One approach would be to apply the Gram-Schmidt procedure to the basis $\{1, x, x^2\}$, which yields an orthonormal basis $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}(x-1), x^2+1\}$. Then, as in problem 3 (or by eyeball), we can write

$$x^2 + (1+i)x + i = (x^2+1) + (1+i)\sqrt{2}(\frac{x-1}{\sqrt{2}}) + 2\sqrt{2}i(\frac{1}{\sqrt{2}})$$

Of course, if you chose a different basis your answers may vary.

Problem 5

Note that $W = span\{(1,0,\sqrt{2},-1),(0,1,0,i)\}$, so applying the Gram-Schmidt procedure to these two vectors (which are clearly independent) and then normalizing yields a basis $\{\frac{1}{2}(1,0,\sqrt{2},-1), \frac{1}{2\sqrt{7}}(i,4,\sqrt{2}i,3i)\}$.

Now $W^\perp = \{(x,y,z,w) \in \mathbb{C}^4 \,|x + \sqrt{2}z - w = 0, y + iw = 0\}$, which is spanned by (say) $(\sqrt{2},0,-1,0)$ and $(0,-\sqrt{2}i,1,\sqrt{2})$. Applying Gram-Schmidt and normalizing gives a basis $\{\frac{1}{\sqrt{3}}(\sqrt{2},0,-1,0), \frac{1}{\sqrt{118}}(\sqrt{2}/5, -\sqrt{2}i, 4/5, \sqrt{2})\}$ for $W^\perp$.

Problem 6

Let (t_{ij}) be the matrix of T with respect to the basis $\beta = \{x_1, \ldots, x_n\}$. For any $v = \sum_{i=1}^n a_i x_i \in V$, we have that

$$< T(v), T(v) > = < T(\sum_{i=1}^n a_i x_i), T(\sum_{j=1}^n a_j x_j) > = \sum_{i,j=1}^n a_i \overline{a_j} < T(x_i), T(x_j) >$$
$$= \sum_{i,j=1}^n a_i \overline{a_j} < T(x_i), \sum_{k=1}^n t_{kj} x_k >$$
$$= \sum_{i,j,k=1}^n a_i \overline{a_j t_{kj}} < T(x_i), x_k > = 0$$

since each $< T(x_i), x_j >= 0$. So for each $v \in V$, $T(v) = 0$; i.e., T is the zero operator.

Problem 7

a) Suppose V is an n-dimensional complex inner product space, and T is an invertible linear operator on V. Let $c_T(x) = det(x \cdot Id - T)$ be the characteristic polynomial of T. This is a polynomial of degree n over the complex numbers, hence it has a zero ($\mathbb{C}$ is an algebraically closed field - Appendix D in the text has one proof of this fact). In other words, there is a $\lambda \in \mathbb{C}$ such that $det(\lambda \cdot Id - T) = 0$, and so there is some non-zero $x \in V$ such that $T(x) = \lambda x$. Applying T^{-1} to this expression, we have $x = \lambda T^{-1}(x)$, which implies $\lambda \neq 0$, since x is non-zero.

Thus,

$$< T(x), x >=< \lambda x, x >= \lambda < x, x >= \lambda ||x||^2 \neq 0$$

since both x and λ are non-zero.

b) Now suppose V is real, and has odd dimension. The characteristic polynomial $c_T(x) = det(x \cdot Id - T) = x^n + \dots$ is a (real) polynomial in x of degree n, which is odd, and so it has a (real) zero (note that $\lim_{x \to +\infty} c_T(x) = +\infty$ and $\lim_{x \to -\infty} c_T(x) = -\infty$, so being a continuous function $c_T(x)$ has to hit the x-axis somewhere). So, as above, there is some $\lambda \in \mathbb{R}$ and (non-zero) $x \in V$ such that $T(x) = \lambda x$. Applying T^{-1}, we see that $\lambda \neq 0$, and that $< T(x), x >= \lambda ||x||^2 \neq 0$.

c) Now suppose V is real, and of dimension $n = 2m$. Changing notation slightly, let $\beta = \{x_1, \dots x_m, y_1, \dots y_m\}$ be an orthonormal basis for V, and define $U : V \to V$ on basis elements by $U(x_i) = -y_i$, $U(y_i) = x_i$ (a linear operator is clearly determined uniquely by where it sends basis elements). Let $T : V \to V$ be defined by $T(x_i) = y_i$, $T(y_i) = -x_i$. Then $T \circ U$ and $U \circ T$ are both the identity, so $T = U^{-1}$ i.e. U is invertible. However, for any $x = \sum_{i=1}^m a_i x_i + b_i y_i \in V$, we have

$$\begin{aligned}
< U(x), x > &=< \sum_{i=1}^m a_i U(x_i) + b_i U(y_i), \sum_{j=1}^m a_j x_j + b_j y_j > \\
&=< \sum_{i=1}^m a_i(-y_i) + b_i(x_i), \sum_{j=1}^m a_j x_j + b_j y_j > \\
&= \sum_{i,j=1}^m (-a_i a_j < y_i, x_j > -a_i b_j < y_i, y_j > +b_i a_j < x_i, x_j > +b_i b_j < x_i, y_j >) \\
&= \sum_{i=1}^m (-a_i b_i < y_i, y_i > +b_i a_i < x_i, x_i >) \\
&= \sum_{i=1}^m (-a_i b_i + a_i b_i) = 0
\end{aligned}$$

where the fourth and fifth lines follow from the assumption that our basis was orthonormal.

Problem 8

We have that for any y in W_2, the inner product $< x, y >$ vanishes for all x in W_1, so by the definition of the perp space, $W_2 \subseteq {W_1}^{\perp}$. On the other hand, $dim({W_1}^{\perp}) = dim(V) - dim(W_1)$, since one can take an orthonormal basis of W_1 and extend it to an orthonormal basis for V; convince yourself that the additional vectors form a basis for ${W_1}^{\perp}$. Now ${W_1}^{\perp}$ and W_2 are both subspaces of the same dimension, and $W_2 \subseteq {W_1}^{\perp}$, so $W_2 = {W_1}^{\perp}$.

Problem 9

Let $\beta_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\beta_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\beta_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}$, $\beta_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}$. You can easily check that these vectors form an orthonormal basis β for $M_{2x2}(\mathbb{C})$, with the inner product $< A, B >= Tr(AB^*)$. For example,

$$< \beta_2, \beta_3 >= Tr(\frac{1}{2} \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}) = 0$$

If $T : V \to V$ is a linear operator, then the (i, j)th entry of the matrix $t = [T]_\beta$ is given by $t_{(i,j)} =< T(\beta_j), \beta_i >$ (which is the just coefficient of β_i that appears when we write $T(\beta_j)$ in terms of the basis). So for $T(A) = iA^t - A$, we can compute these inner products and recover the matrix for T (though in this particular case, it might be simpler to just compute $T(\beta_i)$ and read off the coefficients). For example,

$$t_{(3,4)} =< T(\beta_3), \beta_4 >=< i\beta_3^t - \beta_3, \beta_4 >= \frac{1}{2} Tr(\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}) = i$$