MAT 247S - Problem Set 1

Due Thursday January 22nd

NOTE: Questions 1 d), 1 e), 2 a), 5 and 7 c) will be marked.

1. Determine whether the indicated function $\langle \cdot, \cdot \rangle$ defines an inner product on the vector space V. If it is an inner product, prove that each of the four conditions in the definition of inner product is satisfied. If it is not an inner product, demonstrate (with an explicit example) how one of the conditions fails to hold.

   a) Let V be the vector space of continuous functions from the interval $[0,1]$ to the real numbers $\mathbb{R}$. Set
   $$\langle f_1, f_2 \rangle = \left(\int_0^1 f_1(t) f_2(t)\, dt \right) + f_1(0) f_2(0), \qquad f_1,\ f_2 \in V.$$

   b) Let $V = \mathbb{C}^2$ and $A = \begin{pmatrix} 5 & \sqrt{2}i \\ -\sqrt{2}i & 3 \end{pmatrix}$. Set $\langle x, y \rangle = xAy^*$, $x,\ y \in \mathbb{C}^2$. (Here, if $y = (y_1, y_2)$, then $y^* = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$.)

   c) Let $V = \mathbb{R}^2$. Set
   $$\langle x, y \rangle = x_1 y_1 - 4 x_2 y_2, \qquad \text{for } x = (x_1, x_2),\ y = (y_1, y_2) \in V.$$

   d) Let $V = P_2(\mathbb{C})$ be the space of polynomials of degree at most two, having complex coefficients. Let
   $$\langle f_1, f_2 \rangle = 3 f_1(0) \overline{f_2(0)} + +2 f_1(i) \overline{f_2(i)}, \qquad f_1,\ f_2 \in V.$$

   e) Let V be an inner product space, with inner product $\langle \cdot, \cdot \rangle'$. Suppose that $T : V \to V$ is a linear transformation. Set
   $$\langle x, y \rangle = \langle\, x,\ y \rangle' + \langle\, T(x),\ T(y)\, \rangle', \qquad x,\ y \in V.$$

   f) Let V be an inner product space, with inner product $\langle \cdot, \cdot \rangle'$. Suppose that $T : V \to V$ is a linear transformation that has the property $\langle\, T(x),\ x\, \rangle' \neq \mathbf{0}$ whenever $x \in V$ and $x \neq \mathbf{0}$. Set
   $$\langle x, y \rangle = \langle\, x,\ T(y) \rangle' + \langle\, T(x),\ y\, \rangle', \qquad x,\ y \in V.$$

2. §6.1, #22.

3. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^3$ defined by
   $$\langle (x, y, z), (x', y', z') \rangle = 3xx' + 2yy' + zz'.$$

   Let $\beta = \{\, (1,1,0),\ (1,0,1),\ (0,1,1)\, \}$.

   a) Use the Gram-Schmidt process to convert β to an orthogonal basis (relative to the above inner product).

   b) Express the vector $(0, 2, -1)$ as a linear combination of the vectors from the orthogonal basis obtained in part a).

4. Let $V = P_2(\mathbb{C}) = \{\, f(x) = ax^2 + bx + c \mid a,\, b,\, c \in \mathbb{C} \,\}$ be the vector space of polynomials of degree at most two, having complex coefficients. Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V that satisfies

$$\begin{array}{lll} \langle 1, 1\rangle = 2 & \langle 1, x\rangle = 2 & \langle 1, x^2\rangle = -2 \\ \langle x, x\rangle = 4 & \langle x, x^2\rangle = -2 & \langle x^2, x^2\rangle = 3 \end{array}$$

   a) Find an orthonormal basis for V.
   b) Express the vector $x^2 + (1+i)x + i$ as a linear combination of the vectors from the orthonormal basis obtained in part b).

5. Let $V = \mathbb{C}^4$ with the standard inner product. Let

$$W = \{\, x = (x_1, x_2, x_3, x_4) \in V \mid \sqrt{2}x_1 - x_3 = 0,\ x_1 - ix_2 + x_4 = 0 \,\}.$$

   a) Find an orthonormal basis for W.
   b) Find an orthonormal basis for $W^\perp$.

6. Let V be an n-dimensional inner product space. Let $\beta = \{\, x_1, \ldots, x_n \,\}$ be an ordered basis of V. (Do not assume that β is orthonormal.) Suppose that $T \in \mathcal{L}(V)$ and $\langle\, T(x_j),\, x_\ell \,\rangle = 0$ for $1 \le j,\, \ell \le n$. Prove that $T = T_0$. (Here, T_0 is the zero operator on V. That is, $T_0(x) = 0$ for all $x \in V$.)

7. Suppose that V is an n-dimensional inner product space. Let $T \in \mathcal{L}(V)$ and assume that T is invertible.
   a) Suppose that V is a complex inner product space. Show that $\langle\, T(x),\, x \,\rangle \neq 0$ for some $x \in V$. (*Hint*: Let x be an eigenvector of T. Explain why T must have an eigenvector, and explain why the corresponding eigenvalue is nonzero.)
   b) Let V be a real inner product space and assume that n is odd. Show that $\langle\, T(x),\, x \,\rangle \neq 0$ for some $x \in V$.
   c) Let V be a real inner product space of dimension $n = 2m$, where $m \geq 1$. Let $\beta = \{\, x_1, \ldots, x_n \,\}$ be an orthonormal basis of V. According to a theorem from Mat 240, there exists a unique $U \in \mathcal{L}(V)$ such that

$$U(x_j) = -x_{j+m}, \quad \text{for } 1 \le j \le m, \quad \text{and} \quad U(x_j) = x_{j-m}, \quad \text{for } m+1 \le j \le 2m = n.$$

   Prove that U is invertible and $\langle\, U(x),\, x \,\rangle = 0$ for all $x \in V$.

8. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space V such that $\dim W_1 + \dim W_2 = \dim V$ and $\langle x, y\rangle = 0$ for all $x \in W_1$ and $y \in W_2$. Prove that $W_1^\perp = W_2$.

9. Let $V = M_{2\times 2}(\mathbb{C})$, with the inner product $\langle A, B\rangle = \text{trace}(AB^*)$.
   a) Verify that the set

$$\beta = \left\{ \sqrt{2}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{2}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sqrt{2}^{-1}\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \sqrt{2}^{-1}\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \right\}$$

   is an orthonormal basis of V.
   b) Let $T : V \to V$ be the linear operator defined by $T(A) = iA^t - A$, $A \in V$. Compute the matrix $[T]_\beta$ of T with respect to the basis β.