MAT 247S - Problem Set 4

Solution to question 5

5. Let V be a finite-dimensional real inner product space and let m be a positive integer. Suppose that $T \in \mathcal{L}(V)$, T is normal, and $T^m = T_0$ (that is, $T^m(x) = \mathbf{0}$ for all $x \in V$). Prove that $T = T_0$.

   Solution: Because V is a real inner product space and we only assumed that T is normal (not necessarily self-adjoint), we *cannot* assume that there exists an orthonormal basis for V consisting of eigenvectors of T. Let $\beta = \{ x_1, \ldots, x_n \}$ be an orthonormal basis for V. Let $A = [T]_\beta$. Because β is orthonormal, we know that $A^* = [T^*]_\beta$. Because $TT^* = T^*T$, we have that
$$AA^* = [T]_\beta [T^*]_\beta = [TT^*]_\beta = [T^*T]_\beta = [T^*]_\beta [T]_\beta = A^*A.$$

   That is, A is a normal matrix. Because the real numbers are a subset of the complex numbers, we can consider A as a complex matrix. We still have $AA^* = A^*A$ because complex conjugation has no effect on real numbers. So A is a normal complex matrix. Therefore there exists a unitary matrix P and a diagonal matrix D such that $A = P^{-1}DP$. Note that $[T^m]_\beta = ([T]_\beta)^m = A^m = 0$. It follows that
$$0 = A^m = (P^{-1}DP)^m = P^{-1}D^mP.$$

   This implies $D^m = PA^mP^{-1} = 0$. Now D is diagonal, so $D^m = 0$ implies $D = 0$. This tells us that $A = P^{-1}0P = 0$. Since $[T_0]_\beta = 0$, we have shown that $[T]_\beta = [T_0]_\beta$. Because the map $U \mapsto [U]_\beta$ from $\mathcal{L}(V)$ to $M_{n\times n}(\mathbb{R})$ is one-to-one, we conclude that $T = T_0$.