MAT 240 - Problem Set 5

Due Thursday November 8th

Questions 1a), 1f), 5a), 6, and 8b) will be marked.

1. Determine whether the function $T : V \to W$ is a linear transformation.
   a) Let $V = W = P(F)$, where F is a field. Define $T(f)(x) = (x^2 - 1)f(x) - f(-1)x^3$, $f \in V$.
   b) Let $V = W = M_{2\times 2}(\mathbb{C})$. Define $T(A) = -i\,A + A^t$, $A \in V$. (Here, A^t is the transpose of the matrix A.)
   c) Let $V = P_2(\mathbb{C})$ and $W = P_4(\mathbb{C})$. Define $T(f)(x) = (f(x))^2$, $f \in V$.
   d) Let $V = P(F)$ (F a field) and $W = F^3$. Define $T(f) = (a_0, -a_2, a_1 - a_4)$, for $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in V$.
   e) Let V be a finite-dimensional vector space over a field F. Let $n = \dim V$ and let $W = F^n$. Let $\{\, x_1, \ldots, x_n \,\}$ be a basis of V. Define $T(x) = (\,-c_1, c_2 - c_1, c_3 - c_2, c_4 - c_3, \ldots, c_j - c_{j-1}, \ldots, c_n - c_{n-1}\,)$ for $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n \in V$ ($c_1, \ldots, c_n \in F$).
   f) Let $V = F^3$ and $W = P_2(F)$, where F is a field. Let $\{\, x_1, x_2, x_3 \,\}$ be a basis of V. Define $T(c_1x_1 + c_2x_2 + c_3x_3) = c_1 + c_2^2 x - c_3 x^2$, c_1, c_2, $c_3 \in F$.

2. Assume that $T : V \to W$ is a linear transformation, where $V = M_{2\times 2}(\mathbb{R})$ and $W = P_2(\mathbb{R})$. Suppose that

$$T\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = x^2 - x,\ T\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 3x,\ T\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = x^2 + 4,\ T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = -x^2.$$

   Determine $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ for all real numbers a, b, c and d.

3. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by:

$$T((x_1, x_2, x_3)) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3).$$

   a) Verify that T is a linear transformation.
   b) If $(a, b, c) \in \mathbb{C}^3$, what are the conditions on a, b, and c that $(a, b, c) \in R(T)$? (Here, $R(T)$ denotes the range of T). Find a basis for $R(T)$.
   c) If $(a, b, c) \in \mathbb{C}^3$, what are the conditions on a, b, and c that $(a, b, c) \in N(T)$? (Here, $N(T)$ denotes the null space of T). Find a basis for $N(T)$.

4. Let V and W be vector spaces over a field F.
   a) Assume that $\dim V \geq \dim W$. Suppose that V_1 is a subspace of V such that $\dim V_1 \geq \dim V - \dim W$. Prove that there exists a linear transformation $T : V \to W$ such that $N(T) = V_1$.
   b) Suppose that V_2 is a subspace of V such that $\dim V_2 < \dim V - \dim W$. Prove that $\dim V_2$ cannot contain the null space of any linear transformation from V to W.

5. Find a basis of $N(T)$ and a basis of $R(T)$ for the given linear transformation T.

   a) Let $V = M_{2\times 3}(\mathbb{C})$ and $W = M_{2\times 2}(\mathbb{C})$. Define

   $$T\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\right) = \begin{pmatrix} a_{13} & -2\,a_{11} \\ (1-i)(a_{12}-a_{21}) & 0 \end{pmatrix}, \quad a_{ij} \in \mathbb{C}, 1 \le i \le 2,\ 1 \le j \le 3.$$

   b) Let $V = W = P(\mathbb{R})$. Define $T(f)(x) = (x^2-1)(f(x) - f(1)\,x)$.

6. Let $T : V \to W$ be a linear transformation. Let V_1 be a subspace of V. Define $W_1 = \{\, T(x) \mid x \in V_1 \,\}$.

   a) Prove that W_1 is a subspace of W.

   b) Assume that V is finite-dimensional. Prove that $\dim W_1 = \dim V_1$ if and only if $V_1 \cap N(T) = \{\mathbf{0}\}$.

7. #14, §2.1.

8. Let V be a vector space of dimension $n \ge 1$.

   a) Suppose that n is odd. Show that if $T : V \to V$ is a linear transformation, then $N(T) \ne R(T)$.

   b) Suppose that $n = 2m$ and $m \ge 1$. Let V_1 be a subspace of V of dimension m. Prove that there exists a linear transformation $T : V \to V$ such that $N(T) = R(T) = V_1$.

   c) Find a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $N(T) = R(T) = \operatorname{span}(\{\, e_1, e_2 \,\})$. Is T unique?

9. Let $T : V \to V$ be a linear transformation. Prove that the following are equivalent:

   (i) $R(T) \cap N(T) = \{\mathbf{0}\}$.

   (ii) For $x \in V$, $T(T(x)) = \mathbf{0}$ implies that $T(x) = \mathbf{0}$.

10. Determine whether or not the linear transformation $T : V \to V$ is one-to-one. Also determine whether or not T is onto.

    a) Let $V = P(\mathbb{R})$. Define $T(f)(x) = x\,f'(x) + f(0)$, $f \in V$.

    b) Let V be the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define $T(f)(x) = (x+1)f(x)$, $f \in V$.

    c) Let $V = P(\mathbb{R})$. Define $T(f)(x) = \int_0^x f(t)\,dt$, $f \in V$.