A QUANTUM KIRWAN MAP, I: FREDHOLM THEORY

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Consider a Hamiltonian action of a compact connected Lie group G on an aspherical symplectic manifold (M,ω). Under some assumptions on (M,ω) and the action, D. A. Salamon conjectured that counting gauge equivalence classes of symplectic vortices on the plane $\mathbb{R}^2$ gives rise to a quantum deformation $Q\kappa_G$ of the Kirwan map. This article is the first of three, whose goal is to define $Q\kappa_G$ rigorously. Its main result is that the vertical differential of the vortex equations over $\mathbb{R}^2$ (at the level of gauge equivalence) is a Fredholm operator of a specified index. Potentially, the map $Q\kappa_G$ can be used to compute the quantum cohomology of many symplectic quotients. Conjecturally it also gives rise to quantum generalizations of non-abelian localization and abelianization (see [WZ]).

Contents

1. Main result and motivation

Let (M,ω) be a symplectic manifold without boundary, and G a compact connected Lie group with Lie algebra $\mathfrak{g}$. We fix a Hamiltonian action of G on M and an (equivariant) moment map $\mu : M \to \mathfrak{g}^*$. Assume that the following hypothesis is satisfied:

(H) *G acts freely on $\mu^{-1}(0)$ and the moment map μ is proper.*

Then the symplectic quotient $\big(\bar{M} := \mu^{-1}(0)/G, \bar{\omega}\big)$ is well-defined, smooth and closed. We denote by $H_*^G(M,\mathbb{Z})$ ($H^*_G(M,\mathbb{Z})$) the equivariant (co-)homology of M with integer coefficients (equipped with the cup product $\smile$), $H_*^G(M) := H_*^G(M,\mathbb{Z})/$ torsion etc., by $\kappa_G : H^*_G(M) \to H^*(\bar{M})$ the Kirwan map, and by $[\omega - \mu]^G \in H^2_G(M)$ the class of $\omega - \mu$ in the Cartan model. We define Λ_ω^μ to be the set of maps $\lambda : H_2^G(M) \to \mathbb{Z}$ such that

$$\Big|\big\{B \in H_2^G(M) \,\big|\, \lambda(B) \neq 0, \langle [\omega - \mu]^G, B\rangle \leq C\big\}\Big| < \infty, \quad \forall C \in \mathbb{R},$$

we equip Λ_ω^μ with the convolution product $\cdot$, and call the triple $(\Lambda_\omega^\mu, +, \cdot)$ the *equivariant Novikov ring*. We denote by $(\mathrm{QH}(\bar{M},\bar{\omega}), \bar{*})$ the (small) quantum cohomology of $(\bar{M},\bar{\omega})$ with coefficients in Λ_ω^μ, and by $\mathcal{J}^G(M,\omega)$ the space of G-invariant and ω-compatible almost complex structures on M. For $x \in M$ we denote by $L_x : \mathfrak{g} \to T_xM$ the infinitesimal action. We call (M,ω) *equivariantly convex at* ∞ iff there exists a proper G-invariant $f \in C^\infty(M,[0,\infty))$, $J \in \mathcal{J}^G(M,\omega)$ and $C \in [0,\infty)$ such that

$$\omega(\nabla_v \nabla f(x), Jv) - \omega(\nabla_{Jv}\nabla f(x), v) \geq 0, \quad df(x) J L_x \mu(x) \geq 0,$$

for every $x \in f^{-1}([C,\infty))$ and $0 \neq v \in T_xM$. Here ∇ denotes the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. A symplectic manifold (X,σ) is called semi-positive iff for every $B \in \pi_2(X)$ the conditions $\langle [\omega], B\rangle > 0$ and $c_1(X,\sigma) \geq 3 - \dim X/2$ imply that $c_1(X,\sigma) \geq 0$ (see [**MS2**]). Note that this holds for example if (X,σ) is weakly monotone or $\dim X \leq 6$. We call (X,σ) aspherical iff $\int u^*\sigma = 0$ for every $u \in C^\infty(S^2, X)$.

1. Conjecture. *Assume that (H) holds, (M,ω) is equivariantly convex at ∞ and aspherical, and $(\bar{M},\bar{\omega})$ is semi-positive (see* [**MS2**]*). Then there exists a Λ_ω^μ-algebra homomorphism*

$$\varphi : H^*_G(M) \otimes \Lambda_\omega^\mu \to \mathrm{QH}^*(\bar{M},\bar{\omega}) \tag{1}$$

of the form $\varphi = \kappa_G \otimes \mathrm{id} + \sum_{0 \neq B} \varphi^B \otimes e^B\cdot$.

This conjecture (without specification of the quantum coefficient ring involved) was formulated by D. A. Salamon. The idea of proof outlined below is also due to him. The present article is part of a project whose goal is to make this idea rigorous and hence prove the conjecture. As an example, consider $M := \mathbb{R}^{2n}$ with the standard structure $\omega := \omega_0$, and a linear action of G. Then (M,ω) is equivariantly convex at ∞ (see Example 2.8 in [**CGMS**]) and aspherical. Assume that G is the torus $\mathbb{R}^k/\mathbb{Z}^k$, and let $w^1, \ldots, w^n \in \mathfrak{g}^* \cong (\mathbb{R}^k)^*$ be the weights of the action and $\mu : \mathbb{R}^{2n} \cong \mathbb{C}^n \to \mathbb{R}^k$ be given by $\mu(z) := \tau - \pi\sum_i |z^i|^2 w^i$, for some $\tau \in \mathfrak{g}^*$. Then μ is proper if and only if there exists $\xi \in \mathfrak{g}$ such that $w^i(\xi) > 0$, for $i = 1, \ldots, n := \dim M/2$ (see Proposition 4.14 in [**GGK**]). The action of G on $\mu^{-1}(0)$ is free if and only if for every subset $I \subseteq \{1, \ldots, n\}$ the following holds. If there exist $a_i > 0$, for $i \in I$, such that $\sum_{i\in I} a_i w^i = 0$ then w^i, for $i \in I$, generate the integral lattice in $\mathfrak{g}^*$ (see Lemma 5.20 in [**GGK**]). The quotient $(\bar{M},\bar{\omega})$ is semi-positive if for every $\xi\in\mathfrak{g}$ the conditions $\langle\sum_{i=1}^n w^i,\xi\rangle\geq 3-n+k$, $\langle\tau,\xi\rangle>0$ imply $\langle\sum_{i=1}^n w^i,\xi\rangle\geq 0$. Assume that the hypotheses of Conjecture 1 are satisfied, the action of G on M is monotone (and hence $(\bar{M},\bar{\omega})$ is monotone), and $H_G^*(M)$ is generated by classes of degree less than twice the minimal Maslov number of this action. With these hypotheses R. Gaio and D. A. Salamon [**GS**] proved a version of the conjecture involving the Novikov ring of $(\bar{M},\bar{\omega})$. This was used by K. Cieliebak and D. A. Salamon [**CS**] to compute $\mathrm{QH}^*(\bar{M},\bar{\omega})$ for monotone torus actions on $\mathbb{R}^{2n}$ with Maslov number at least 2. A potential application of Conjecture 1 is to extend these computations to the more general setting of this conjecture.

The idea of proof of Conjecture 1 is to define φ by counting symplectic vortices on the plane. To explain this, we fix a G-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. Let (Σ,ω_Σ,j) be a real surface equipped with an area form and a compatible complex structure, and $\pi:P\to\Sigma$ a principal G-bundle. We denote by $C_G^\infty(P,M)$ the space of smooth equivariant maps from P to M, and by $\mathcal{A}(P)$ the space of smooth connections on P. A (symplectic) vortex is a solution $(u,A)\in C_G^\infty(P,M)\times\mathcal{A}(P)$ of

$$\bar{\partial}_{J,A}(u)=0,\quad F_A+(\mu\circ u)\omega_\Sigma=0. \tag{2}$$

Here $\bar{\partial}_{J,A}(u)$ denotes the complex anti-linear part of $d_Au:=du+L_uA$, which we think of as a one-form on Σ with values in the complex vector bundle $TM^u:=(u^*TM)/G\to\Sigma$. Similarly, we view the curvature F_A of A as a two-form on Σ with values in the adjoint bundle $\mathfrak{g}_P:=(P\times\mathfrak{g})/G\to\Sigma$. Finally, we view $\mu\circ u$ as a section of $\mathfrak{g}_P$. The vortex equations (2) were discovered by K. Cieliebak, A. R. Gaio and D. A. Salamon [**CGS**], and independently by I. Mundet i Riera [**Mu1**, **Mu2**].

The energy density and energy of a $w:=(u,A)\in C_G^\infty(P,M)\times\mathcal{A}(P)$ are given by $e_w:=\frac{1}{2}\big(|d_Au|^2+|F_A|^2+|\mu\circ u|^2\big)$ and $E(w):=\int_\Sigma e_w\omega_\Sigma$. Consider $\Sigma:=\mathbb{R}^2$, equipped with the standard area form $\omega_{\mathbb{R}^2}:=\omega_0$ and complex structure $j:=i$, and $P:=\mathbb{R}^2\times G$. Let $B\in H_2^G(M)$. The group $\mathcal{G}:=\mathcal{G}(P)$ of smooth gauge transformations on P acts on the set of solutions of (2). We denote by $\mathcal{M}_B$ the set of gauge (equivalence) classes of vortices $w:=(u,A)$ on P for which $E(w)<\infty$, $\overline{u(P)}\subseteq M$ is compact and w represent the equivariant homology class B (see [**Zi1**]). We denote by EG a contractible topological space on which G acts continuously and freely. There are natural evaluation maps $\mathrm{ev}_z:\mathcal{M}_B\to(M\times\mathrm{EG})/G$ (at $z\in\mathbb{R}^2$) and $\overline{\mathrm{ev}}_\infty:\mathcal{M}_B\to\bar{M}$ (at $\infty\in\mathbb{R}^2\cup\{\infty\}$) (see again [**Zi1**]). Heuristically, for $\alpha\in H_G^*(M)$ and $\bar{\beta}\in H^*(\bar{M})$, we define

$$\mathrm{Q}\kappa_G^B(\alpha,\bar{\beta}):=\int_{\mathcal{M}_B}\mathrm{ev}_0^*\,\alpha\smile\overline{\mathrm{ev}}_\infty^*\bar{\beta}. \tag{3}$$

We fix dual bases $(\bar{e}_i)_{i=1,\dots,N}$ and $(\bar{e}_i^*)_{i=1,\dots,N}$ of $H^*(\bar{M})$, in the sense that $\int_{\bar{M}} \bar{e}_i \smile \bar{e}_j^* = \delta_{ij}$. The idea is now to define a map $\varphi := \mathrm{Q}\kappa_G$ as in (1) by

$$\mathrm{Q}\kappa_G(\alpha) := \sum_{i=1,\dots,N,\, B\in H_2^G(M)} \mathrm{Q}\kappa_G^B(\alpha,\bar{e}_i)\bar{e}_i^* \otimes e^B.$$

The goal of the present and two subsequent papers [**Zi3**, **Zi4**] is to make this definition rigorous: In the present article I introduce weighted Sobolev spaces $\mathcal{X}_w^{p,\lambda}$ and $\mathcal{Y}_w^{p,\lambda}$ that will serve as model spaces for a Banach manifold $\mathcal{B}_\lambda^p$ of gauge classes of pairs (u,A) and the fiber of a Banach bundle $\mathcal{E}_\lambda^p \to \mathcal{B}_\lambda^p$. Furthermore, I show that the vertical differential of the equations (2) (viewed as a section $\mathcal{S}$ of $\mathcal{E}_\lambda^p$) is a Fredholm map between $\mathcal{X}_w^{p,\lambda}$ and $\mathcal{Y}_w^{p,\lambda}$. In [**Zi3**] the notion of a stable map of vortices on $\mathbb{R}^2$ and (pseudo-)holomorphic spheres in $\bar{M}$ is defined, and a bubbling result for vortices over $\mathbb{R}^2$ is proven. The hypotheses of equivariant convexity and asphericity are needed here. The former forces vortices on $\mathbb{R}^2$ to stay in a fixed compact subset of M, and the latter rules out bubbling of holomorphic spheres in M. The present paper and [**Zi3**] are based on my Ph.D.-thesis [**Zi1**]. In [**Zi4**], I define an atlas for $\mathcal{B}_\lambda^p$ consisting of charts with targets open subsets of the spaces $\mathcal{X}_w^{p,\lambda}$ (and a similar atlas for $\mathcal{E}_\lambda^p$). The charts are based on the choice of a G-invariant Riemannian metric on M whose exponential map along $\mu^{-1}(0)$ is compatible with the moment map μ. (This is needed to make the transition maps well-defined.) Furthermore, I show how to perturb $\mathcal{S}$ in order to make it transverse to the zero section. Combining all these results, it follows that $\mathcal{S}^{-1}(0) \subseteq \mathcal{B}_\lambda^p$ is a smooth finite dimensional submanifold, and the evaluation map $\mathrm{ev}_0 \times \overline{\mathrm{ev}}_\infty$ is a pseudo-cycles. Here the semi-positivity of $(\bar{M},\bar{\omega})$ is needed. The formula (3) is then made rigorous as an intersection number of pseudo-cycles.

In order to show that $\mathrm{Q}\kappa_G$ intertwines $\smile$ with $\bar{*}$, it suffices to prove that

$$\Big\langle \mathrm{Q}\kappa_G^B(\alpha_1 \smile \alpha_2), \bar{a} \Big\rangle = \Big\langle \big(\mathrm{Q}\kappa_G(\alpha_1) \bar{*}\, \mathrm{Q}\kappa_G(\alpha_2)\big)_B, \bar{a} \Big\rangle, \tag{4}$$

for every $B \in H_2^G(M)$, $\alpha_1,\alpha_2 \in H_G^*(M)$ and $\bar{a} \in H_*(\bar{M})$. The idea for proving this is the following. We choose "oriented submanifolds" $X_1, X_2 \subseteq (M \times \mathrm{EG})/G$ that are "Poincaré dual" to α_1 and α_2, and an oriented submanifold $\bar{X} \subseteq \bar{M}$ representing $\bar{a}$. (To make this rigorous one has to pass to some finite dimensional approximation of EG, a compact submanifold of M with boundary and rational multiples of α_1, α_2 and $\bar{a}$.) Consider the marked points $z_\nu^\pm := \pm\nu$, $z_\nu^\infty := \infty$, and a sequence of gauge classes of vortices $W_\nu \in \mathcal{M}_B$, such that $\mathrm{ev}_{z_\nu^+}(W_\nu) \in X_1$, $\mathrm{ev}_{z_\nu^-}(W_\nu) \in X_2$, and $\overline{\mathrm{ev}}_\infty(W_\nu) \in \bar{X}$. By the main result of [**Zi3**] a subsequence of W_ν converges to a stable map of (gauge classes of) vortices on $\mathbb{R}^2$ and holomorphic spheres in $\bar{M}$. In the transverse case, this map consists of two classes W^1 and W^2 of vortices on $\mathbb{R}^2$, each equipped with a marked point $z_i \in \mathbb{C}$, and a holomorphic map

$\bar{u}: S^2 \to \bar{M}$, equipped with a marked point $z_\infty \in S^2$. W^i is attached to $\bar{u}$ at the nodal point $\infty \in \mathbb{R}^2 \cup \{\infty\}$, for $i = 1, 2$. The total homology class of the stable map equals B, $\mathrm{ev}_{z_i}(W^i) \in X_i$, for $i = 1, 2$, and $\bar{u}(z_\infty) \in \bar{X}$. The number of such stable maps equals the right hand side of (4).

Consider now the marked points $z_\nu^\pm := \pm 1/\nu$, $z_\nu^\infty := \infty$, and a sequence $W_\nu \in \mathcal{M}_B$, such that $\mathrm{ev}_{z_\nu^+}(W_\nu) \in X_1$, $\mathrm{ev}_{z_\nu^-}(W_\nu) \in X_2$ and $\overline{\mathrm{ev}}_\infty(W_\nu) \in \bar{X}$. Using [**Zi3**] again, a subsequence of W_ν converges to a stable map of vortices on $\mathbb{R}^2$ and spheres in $\bar{M}$. In the transverse case, this stable map consists of a single class $W \in \mathcal{M}_B$, satisfying $\overline{\mathrm{ev}}_\infty(W) \in \bar{X}$, and a ghost bubble $u \equiv \mathrm{pt} \in X_1 \cap X_2$ that is attached to W at some point in $\mathbb{R}^2$ and contains two marked points. The number of such stable maps equals the left hand side of (4). Equality (4) follows by combining this with the argument of the last paragraph and a gluing argument for vortices on $\mathbb{R}^2$ and spheres in $\bar{M}$. The gluing result is part of my future research.

A similar argument involving an adiabatic limit in the vortex equations over S^2 will show that $\mathrm{Q}\kappa_G$ intertwines the genus 0 symplectic vortex invariants with the Gromov-Witten invariants of $(\bar{M}, \bar{\omega})$. The idea of proof of Conjecture 1 presented here is different from the construction used by Gaio and Salamon in [**GS**]. They use an adiabatic limit argument, which fails in the more general situation considered here, because of bubbling off of vortices on $\mathbb{R}^2$.

Assume now just that (H) holds and (M, ω) is equivariantly convex at ∞. We denote by $\mathrm{GW}_G(M, \omega)$ the G-equivariant Gromov-Witten theory of (M, ω). In joint work with Christopher Woodward [**WZ**] we interpret $\mathrm{Q}\kappa_G$ as a morphism of cohomological field theories between $\mathrm{GW}_G(M, \omega)$ and $\mathrm{GW}(\bar{M}, \bar{\omega})$. We formulate "functoriality" for $\mathrm{GW}_\bullet$ under reduction in stages. We also conjecture quantum generalizations involving $\mathrm{Q}\kappa_G$ of non-abelian localization and abelianization.

For a vector bundle $E \to X$ and $k \in \mathbb{N} \cup \{0\}$ we denote by $\Gamma(E)$ the space of its smooth sections and by $\bigwedge^k(E)$ the bundle of k-forms on X with values in E. For an almost complex manifold X and a complex vector bundle $E \to X$ is we denote by $\bigwedge^{0,1}(E) \to X$ the bundle of anti-linear one-forms on X with values in E. We equip the bundle $TM^u = (u^*TM)/G \to \mathbb{R}^2$ with the complex structure induced by J. We define

$$\widetilde{\mathcal{B}} := C^\infty_G(P, M) \times \mathcal{A}(P),$$
$$\widetilde{\mathcal{E}} := \big\{(w; \zeta') \,\big|\, w \in \widetilde{\mathcal{B}},\, \zeta' \in \Gamma\big(\textstyle\bigwedge^{0,1}(TM^u) \times \bigwedge^2(TM^u)\big)\big\},$$
$$\widetilde{\mathcal{S}}: \widetilde{\mathcal{B}} \to \widetilde{\mathcal{E}}, \quad \widetilde{\mathcal{S}}(u, A) := \big(\bar{\partial}_{J,A}(u), F_A + (\mu \circ u)\omega_\Sigma\big).$$

The group $\mathcal{G}$ acts naturally on $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{E}}$. We denote by $\mathcal{B} := \widetilde{\mathcal{B}}/\mathcal{G}$ and $\mathcal{E} := \widetilde{\mathcal{E}}/\mathcal{G}$ the quotients. The map $\widetilde{\mathcal{S}}$ is $\mathcal{G}$-equivariant, and hence induces a map $\mathcal{S}: \mathcal{B} \to \mathcal{E}$. Note that $\mathcal{S}^{-1}(0) \subseteq \mathcal{B}$ is the set of gauge classes of vortices. Assume that the action of $\mathcal{G}$ on $\widetilde{\mathcal{B}}$ is free. (This is satisfied for

$\Sigma := \mathbb{R}^2$, $\omega_\Sigma := \omega_{\mathbb{R}^2}$ and $j := i$ under hypothesis (H), see Lemma 29 below.) Then heuristically, $\mathcal{B}$ is an infinite dimensional manifold, $\mathcal{E}$ is an infinite dimensional vector bundle over $\mathcal{B}$, and $\mathcal{S}$ is a smooth section of $\mathcal{E}$.

Assume that $W \in \mathcal{S}^{-1}(0)$. Then formally, there is a canonical map $T : T_{(W,0)}\mathcal{E} \to \mathcal{E}_W$, where $\mathcal{E}_W \subseteq \mathcal{E}$ denotes the fiber over W. The vertical differential of $\mathcal{S}$ at W is given by $d^V\mathcal{S}(W) = T\, d\mathcal{S}(W) : T_W\mathcal{B} \to \mathcal{E}_W$. Let $w := (u, A) \in W$ be a representative. We denote by $L_w : \mathrm{Lie}(\mathcal{G}) \to T_w\widetilde{\mathcal{B}}$ the infinitesimal action of $\mathcal{G}$ on $\widetilde{\mathcal{B}}$. Formally, $T_w\widetilde{\mathcal{B}} = \Gamma\big(TM^u \oplus \bigwedge^1(\mathfrak{g}_P)\big)$, $\mathrm{Lie}(\mathcal{G}) = \Gamma(\mathfrak{g}_P)$, and $L_w\xi = (L_u\xi, -d_A\xi)$.

Assume that $\widetilde{\mathcal{B}}$ and $\mathrm{Lie}(\mathcal{G})$ are equipped with a $\mathcal{G}$-invariant Riemannian metric and $\mathcal{G}$-invariant inner product respectively. For $w \in \widetilde{\mathcal{B}}$ we denote by $L_w^* : T_w\widetilde{\mathcal{B}} \to \mathrm{Lie}(\mathcal{G})$ the adjoint of L_w. Then formally, the tangent space $T_W\mathcal{B}$ is the quotient of the disjoint union of $\ker L_w^*$, with w ranging over all representatives of W, by the linearized action of $\mathcal{G}$. We now equip M with the Riemannian metric $\omega(\cdot, J\cdot)$, and TM^u with the induced bundle metric. The bundle $\mathfrak{g}_P$ inherits a bundle metric from $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. We equip $\mathrm{Lie}(\mathcal{G})$ and $T_w\widetilde{\mathcal{B}}$ with the corresponding L^2 inner products. Then L_w^* is given by

$$L_w^*(v, \alpha) = L_u^* v - d_A^*\alpha. \tag{5}$$

Here we view L_u as a map from $\mathfrak{g}_P$ to TM^u, and we denote by $L_u^* : TM^u \to \mathfrak{g}_P$ the adjoint of L_u, and $d_A^* = - * d_A *$, where $*$ denotes the Hodge-star operator on sections of $\mathfrak{g}_P$ with respect to the metric $\omega_\Sigma(\cdot, j\cdot)$.

The Levi-Civita connection ∇ of $\omega(\cdot, J\cdot)$ and A induce a connection ∇^A on TM^u (see Section 2). For $w \in \widetilde{\mathcal{B}}$ consider the operator $\mathcal{D}_w : \ker L_w^* \to \widetilde{\mathcal{E}}_w$,

$$\mathcal{D}_w(v, \alpha) := \begin{pmatrix} \big(\nabla^A v + L_u\alpha\big)^{0,1} - \frac{1}{2}J(\nabla_v J)(d_A u)^{1,0} \\ d_A\alpha + \omega_\Sigma\, d\mu(u)v \end{pmatrix}, \tag{6}$$

For $W \in \mathcal{S}^{-1}(0)$ the map $d^V\mathcal{S}(W) : T_W\mathcal{B} \to \mathcal{E}_W$ is given by $d^V\mathcal{S}(W)\mathcal{G}^*(v, \alpha) = \mathcal{G}^*\mathcal{D}_w(v, \alpha)$. (This follows for example from [**CGMS**], formula (23), p. 27.)

Consider now the case $\Sigma := \mathbb{R}^2, \omega_{\mathbb{R}^2} := \omega_0$ and $j := i$. The purpose of this article is to find a Banach space setup in which the vertical differential $d^V\mathcal{S}(W)$ is Fredholm. To this end, let $n \in \mathbb{N}$, $p \in [1, \infty]$, $\lambda \in \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We denote $\|f\|_p := \|f\|_{L^p(\mathbb{R}^n)} \in [0, \infty]$, and define $\|f\|_{p,\lambda} := \big\|f(1 + |\cdot|^2)^{\frac{\lambda}{2}}\big\|_p$. Assume now that $p > 2$. We denote by $W^{1,p}_{\mathrm{loc},G}(P, M)$ and $\mathcal{A}^{1,p}_{\mathrm{loc}}(P)$ the spaces of G-equivariant maps from $P \to M$ and connections on P, of class locally $W^{1,p}$. We abbreviate $\widetilde{\mathcal{B}}^p_{\mathrm{loc}} := W^{1,p}_{\mathrm{loc},G}(P, M) \times \mathcal{A}^{1,p}_{\mathrm{loc}}(P)$, and we define

$$\widetilde{\mathcal{B}}^p_\lambda := \big\{(u, A) \in \widetilde{\mathcal{B}}^p_{\mathrm{loc}} \,\big|\, \overline{u(P)} \text{ compact}, \ \|\sqrt{e_{(u,A)}}\|_{p,\lambda} < \infty\big\}. \tag{7}$$

Furthermore, we denote by $\mathcal{G}^{2,p}_{\mathrm{loc}}(P)$ the group of locally $W^{2,p}$ gauge transformations on P, and we define $\mathcal{B}^p_\lambda := \widetilde{\mathcal{B}}^p_\lambda / \mathcal{G}^{2,p}_{\mathrm{loc}}(P)$.

Let $w := (u, A) \in \widetilde{\mathcal{B}}^p_\lambda$ be a smooth pair. ∇, A and the Levi-Civita connection of the standard metric $g_{\mathbb{R}^2}$ on $\mathbb{R}^2$ induce a linear connection ∇^A on $TM^u \oplus \bigwedge^1(\mathfrak{g}_P)$. For $\zeta := (v, \alpha) \in W^{1,p}_{\mathrm{loc}}\big(TM^u \oplus \bigwedge^1(\mathfrak{g}_P)\big)$ we define

$$\|\zeta\|_{w,p,\lambda} := \|\zeta\|_\infty + \big\||\nabla^A\zeta| + |d\mu(u)v| + |\alpha|\big\|_{p,\lambda} \in [0, \infty]. \tag{8}$$

Here the pointwise norms are taken with respect to the metrics $\omega(\cdot, J\cdot)$ and $g_{\mathbb{R}^2}$, and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. We define

$$\mathcal{X}^{p,\lambda}_w := \big\{\zeta \in W^{1,p}_{\mathrm{loc}}\big(TM^u \oplus \bigwedge^1(\mathfrak{g}_P)\big) \,\big|\, L^*_w\zeta = 0,\ \|\zeta\|_{w,p,\lambda} < \infty\big\}, \tag{9}$$

$$\mathcal{Y}^{p,\lambda}_w := \big\{\zeta' \in L^p_{\mathrm{loc}}\big(\bigwedge^{0,1}(T\mathbb{C}, TM^u) \oplus \bigwedge^2(\mathfrak{g}_P)\big) \,\big|\, \|\zeta'\|_{p,\lambda} < \infty\big\}. \tag{10}$$

Here L^*_w is as in (5). In [**Zi4**], the set $\mathcal{B}^p_\lambda$ will be equipped with a Banach manifold structure, such that for every $W \in \mathcal{B}^p_\lambda$ admitting a smooth representative $w \in \widetilde{\mathcal{B}}^p_\lambda$, the tangent space $T_W\mathcal{B}^p_\lambda$ can be identified with $\mathcal{X}^{p,\lambda}_w$. Furthermore, the spaces $\mathcal{Y}^{p,\lambda}_w$ will be identified with the fibers of a Banach bundle $\mathcal{E}^p_\lambda \to \mathcal{B}^p_\lambda$.

From now on throughout this article, we assume that hypothesis (H) is satisfied.

We denote by $m(w)$ the Maslov index of w, see Section 2. The main result of this article is the following.

2. Theorem. *Assume that* $\dim M > 2\dim G$. *Let* $p > 2$, $\lambda > 1 - 2/p$ *and* $w := (u, A) \in \widetilde{\mathcal{B}}^p_\lambda$ *be a smooth pair. Then the following statements hold.*

(i) The normed vector spaces $\mathcal{X}^{p,\lambda}_w$ *and* $\mathcal{Y}^{p,\lambda}_w$ *are complete.*

(ii) If $1 - 2/p < \lambda < 2 - 2/p$ *then the operator* $\mathcal{D}^{p,\lambda}_w : \mathcal{X}^{p,\lambda}_w \to \mathcal{Y}^{p,\lambda}_w$ *given by the formula (6) is well-defined and Fredholm of real index* $\mathrm{ind}\mathcal{D}^{p,\lambda}_w = \dim M - 2\dim G + 2m(w)$.

The condition $1-2/p < \lambda < 2-2/p$ in this theorem captures the geometry of finite energy vortices. More precisely, let $w := (u, A) \in \mathcal{B}^p_{\mathrm{loc}}$ be a finite energy vortex such that $\overline{u(P)} \subseteq M$ is compact. Then for every $\varepsilon > 0$ there exists a constant C such that $e_w(z) \leq C|z|^{-4+\varepsilon}$, for every $z \in \mathbb{R}^2 \setminus B_1$ (see [**Zi2**], Corollary 4). It follows that $w \in \widetilde{\mathcal{B}}^p_\lambda$ if $\lambda < 2 - 2/p$. This bound is sharp. To see this, let $\lambda > 2 - 2/p$, let $M := S^2, \omega := \omega_0, G := \{e\}$ and $J := i$, and consider the inclusion $u : \mathbb{R}^2 \to S^2 \cong \mathbb{R}^2 \cup \{\infty\}$.

On the other hand, every $w \in \widetilde{\mathcal{B}}^p_\lambda$ satisfies $E(w) < \infty$ if $p > 2$ and $\lambda > 1 - 2/p$. The latter condition is sharp. Namely, let $\lambda < 1 - 2/p$, and consider $M := S^2$ with the standard symplectic form ω_0, complex structure $J := i$ and the action of the trivial group $G := \{e\}$. We choose a number

$2 < a < 3 - 2/p - \lambda$ and a smooth map $u : \mathbb{R}^2 \to S^2 \cong \mathbb{C} \cup \{\infty\}$ such that $u(z) = |z|^a$, for $z \in \mathbb{R}^2 \setminus B_1$. Then $(u, 0) \in \widetilde{\mathcal{B}}^p_\lambda$ and $E(u, 0) = \infty$.

The condition $\lambda < 2 - 2/p$ is also needed for $\mathcal{D}^{p,\lambda}_w$ to have the right Fredholm index. Namely, let $\lambda > 1 - 2/p$ be such that $\lambda + 2/p \notin \mathbb{Z}$, and $w \in \widetilde{\mathcal{B}}^p_\lambda$. Then the proof of Theorem 2 shows that $\mathcal{D}^{p,\lambda}_w$ is Fredholm with index equal to $(2 - k)(\dim M - 2 \dim G) + 2m(w)$, where k is the largest integer less than $\lambda + 2/p$. In particular, the index changes when λ passes the value $2 - 2/p$.

The definition of the space $\mathcal{X}^{p,\lambda}_w$ is natural, since it parallels the definition of $\widetilde{\mathcal{B}}^p_\lambda$. Namely, by linearizing with respect to u and A the terms $d_A u, F_A$ and $\mu \circ u$ occurring in the energy density e_w, we obtain the terms $\nabla^A \zeta, d\mu(u)v$ and $L_u \alpha$. These expressions occur in $\|\zeta\|_{w,p,\lambda}$, except for the factor L_u in $L_u \alpha$. (It follows from (H) and Lemma 30 (appendix) that this factor is irrelevant.) The expression $\|\zeta\|_\infty$ is needed in order to make $\| \cdot \|_{w,p,\lambda}$ non-degenerate.

Remark. *Naively, one could define the domain of $\mathcal{D}_w$ to be the kernel of L^*_w defined on the space of usual $W^{1,p}$-sections, and its target to consist of L^p-sections. However, then in general $\mathcal{D}_w$ would not have closed image, and hence is not Fredholm. Note also that the 0-th order terms $\alpha \mapsto (L_u \alpha)^{0,1}$ and $v \mapsto \omega_0\, d\mu(u)v$ in (6) are not compact (neither with $\mathcal{X}^{p,\lambda}_w$ and $\mathcal{Y}^{p,\lambda}_w$ defined as in (9,10) nor the naive choices). The reason is that the Kondrachov compactness theorem fails on $\mathbb{R}^2$. Observe also that because of these terms, $\mathcal{D}_w$ is not well-defined if we choose spaces that look like the standard (weighted) Sobolev spaces in "logarithmic" coordinates $\tau + i\varphi$ (with $e^{\tau + i\varphi} = z \in \mathbb{C} \setminus \{0\}$).*

The proof of Theorem 2 is based on a Fredholm result for the augmented vertical differential (Theorem 5) and surjectivity of L^*_w (Theorem 6). The proof of Theorem 5 has two main ingredients. The first one is a suitable complex trivialization of the bundle $TM^u \oplus \bigwedge^1(\mathfrak{g}_P)$. For R large, $z \in \mathbb{R}^2 \setminus B_R$ and $p \in \pi^{-1}(z) \subseteq P$ this trivialization respects the splitting $T_{u(p)}M = (\mathrm{im} L^{\mathbb{C}}_{u(p)})^\perp \oplus \mathrm{im} L^{\mathbb{C}}_{u(p)}$, where $L^{\mathbb{C}}_x : \mathfrak{g} \otimes \mathbb{C} \to T_x M$ denotes the complexified infinitesimal action, for $x \in M$. The second ingredient are two propositions stating that the standard Cauchy-Riemann operator $\partial_{\bar{z}}$ and a related matrix differential operator are Fredholm maps between suitable weighted Sobolev spaces. These results are based on the analysis of weighted Sobolev spaces carried out by R. B. Lockhart and R. C. McOwen. (See [**Lo3**] and references therein.) Note that for a *compact* Riemann surface Σ without boundary, in [**CGMS**] K. Cieliebak et al. proved that the augmented vertical differential of the vortex equations is Fredholm.

Organization of the article. Section 2 contains some background about the connection ∇^A and the definition of the Maslov index $m(w)$. In Section 3.1 a Fredholm theorem for the augmented vertical differential (Theorem 5), and an existence result for a right inverse for L^*_w (Theorem 6) are stated.

Furthermore, the main result is deduced from these results. Section 3.2 contains the core of the proof of Theorem 5. Here the notion of a good complex trivialization is introduced and an existence result for such a trivialization is stated (Proposition 8). Furthermore, a result is formulated saying that every good trivialization transforms $\mathcal{D}_w$ into a compact perturbation of the direct sum of $\partial_{\bar{z}}$ and a matrix operator (Proposition 9). The results of Section 3.2 are proved in Section 3.3. In Section 3.4 Theorem 6 is proved, using the existence of a right inverse for d_A^* (Proposition 11). Appendix A contains a Hardy-type inequality, which is used in the proof of Proposition 9, some standard embedding and compactness results for weighted Sobolev spaces, and Fredholm results for the Cauchy-Riemann operator and a matrix valued operator on $\mathbb{R}^2$ (Propositions 16 and 19 and Corollary 18). In Appendix B Proposition 11 is proved. Appendix C contains some other auxiliary results.

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2. Background and notation

The connection ∇^A. Let $E \to M$ be a real (smooth) vector bundle. We denote by $\mathcal{C}(E)$ the affine space of (smooth linear) connections on E. Let $\nabla^E \in \mathcal{C}(E)$. Let N be a smooth manifold, and $u : N \to M$ be a smooth map. We denote by $u^*E \to N$ the pullback bundle. The pullback connection $u^*\nabla^E \in \mathcal{C}(u^*E)$ is uniquely determined by $(u^*\nabla^E)_v s \circ u = \nabla^E_{u_*v} s$, for every $v \in TN$ and every $s \in \Gamma(E)$. Let G be a Lie group, $\pi : P \to X$ a (right-)principal G-bundle, and $E \to P$ a G-equivariant vector bundle. Then the quotient E/G has a natural structure of a vector bundle over X. Let now $E \to M$ be a G-equivariant vector bundle. We denote by $\mathcal{C}^G(E)$ the space of G-invariant connections on E. We fix $A \in \mathcal{A}(P)$, $\nabla^E \in \mathcal{C}^G(E)$, and $u \in C^\infty_G(P, M)$. We define $\widetilde{\nabla}^A \in \mathcal{C}^G(u^*E)$ by $\widetilde{\nabla}^A_{\widetilde{v}} \widetilde{s} := (u^*\nabla^E)_{\widetilde{v} - pA\widetilde{v}} \widetilde{s}$, for $\widetilde{s} \in \Gamma(u^*E)$, $p \in P$, and $\widetilde{v} \in T_pP$. We denote $E^u := (u^*E)/G \to X$. The connection $\widetilde{\nabla}^A$ is basic (i.e. G-invariant and horizontal), hence there exists a unique $\nabla^A \in \mathcal{C}(E^u)$ with the following property. Let $s \in \Gamma(E^u)$ and $v \in TX$. We define $\nabla^A_v s := G \cdot (p_0, \widetilde{\nabla}^A_{\widetilde{v}} \widetilde{s})$, where $(p_0, \widetilde{v}) \in TP$ is such that $\pi_* \widetilde{v} = v$, and $\widetilde{s} \in \Gamma(u^*E)$ is a G-invariant section such that $s \circ \pi(p) = G \cdot (p, \widetilde{s}(p))$, for every $p \in P$. Assume now that X is an open subset of $\mathbb{R}^n$, and let $\Psi : X \times V \to E$ be a bundle map (fixing the base). We define $\nabla^A \Psi$ by

$(\nabla_v^A \Psi)w := \nabla_v^A(\Psi w)$, for every $x \in X$, $v \in T_xX$ and $w \in V$. (Here we think of w as a constant section of $X \times V$.)

The Maslov index. Let $M, \omega, G, \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \mu$ and J be as in Section 1, $\Sigma = \mathbb{R}^2, \omega_{\mathbb{R}^2} = \omega_0, j = i$, $P \to \mathbb{R}^2$ a principal G-bundle, $p > 2, \lambda > 1 - 2/p$ and $w = (u, A) \in \widetilde{\mathcal{B}}^p_\lambda$. The definition of the Maslov index of w is based on the following.

3. Proposition. *There exists an extension of P to some smooth G-bundle $\widetilde{P} \to S^2 = \mathbb{R}^2 \cup \{\infty\}$, such that u extends to a continuous map from $\widetilde{P}$ to M. Furthermore, if $\widetilde{P}_1, \widetilde{P}_2$ are two such extensions of P and $\widetilde{u}_1, \widetilde{u}_2$ the corresponding extensions of u, then there exists an isomorphism of continuous G-bundles $\Psi : \widetilde{P}_1 \to \widetilde{P}_2$ such that $\widetilde{u}_1 = \widetilde{u}_2 \circ \Psi$.*

The proof of Proposition 3 is postponed to the appendix (page 37). We choose $\widetilde{P}$ and $\widetilde{u}$ as in Proposition 3. Then ω induces a fiberwise symplectic form $\widetilde{\omega}$ on the continuous bundle $TM^{\widetilde{u}} = (\widetilde{u}^*TM)/G \to S^2$.

4. Definition (Maslov index)**.** *We define the Maslov index $m(w)$ to be the first Chern number of $(TM^{\widetilde{u}}, \widetilde{\omega})$.*

It follows from Proposition 3 that $m(w)$ does not depend on the choice of the extension $\widetilde{P}$. Note that it only depends on the gauge equivalence class of w. The condition $\lambda > 1 - 2/p$ is needed for $m(w)$ to be well-defined for $w \in \widetilde{\mathcal{B}}^p_\lambda$. Consider for example the case $M := \mathbb{R}^2, \omega := \omega_0, J := i$ and $G := \{e\}$. Let $p > 2$ and $\lambda < 1 - 2/p$. We choose $0 < \varepsilon < 1 - 2/p - \lambda$, and a smooth map $u : \mathbb{R}^2 \to \mathbb{R}^2 = \mathbb{C}$ such that $u(z) = \sin(|z|^\varepsilon)$, for $z \in \mathbb{R}^2 \setminus B_1$. Then $(u, 0) \in \widetilde{\mathcal{B}}^p_\lambda$, and $u(re^{i\varphi})$ diverges, as $r \to \infty$, for every $\varphi \in \mathbb{R}$. Therefore, we can not associate any Maslov index with $(u, 0)$.

3. Proof of the main result

Let $M, \omega, G, \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \mu$ and J be as in Section 1 (assuming hypothesis (H)), $\Sigma := \mathbb{R}^2, \omega_{\mathbb{R}^2} := \omega_0, j := i$ and $P \to \mathbb{R}^2$ a principal G-bundle. In the present section we always **assume that** $\bar{n} := (\dim M)/2 - \dim G > 0$.

3.1. Reformulation of the Fredholm theorem.

Let $p > 2$, $\lambda \in \mathbb{R}$, $\widetilde{\mathcal{B}}^p_\lambda$ be defined as in (7), and $w := (u, A) \in \widetilde{\mathcal{B}}^p_\lambda$ be a smooth map. We denote $\mathrm{im}L := \{(x, L_x\xi) \,|\, x \in M, \, \xi \in \mathfrak{g}\}$, and by $\Pr : TM \to TM$ the orthogonal projection onto $\mathrm{im}L$. Pr induces an orthogonal projection $\Pr^u : TM^u \to TM^u$ onto $(u^*\mathrm{im}L)/G$. For $\zeta = (v, \alpha) \in TM^u \oplus \bigwedge^1(\mathfrak{g}_P)$ we write $\Pr^u\zeta := (\Pr^u v, \alpha)$. Note that $\mathrm{im}L$ is in general not a subbundle of TM, since the dimension of $\mathrm{im}L_x$ may vary with $x \in M$. For $\zeta \in W^{1,p}_{\mathrm{loc}}\big(TM^u \oplus \bigwedge^1(\mathfrak{g}_P)\big)$ we define $\widetilde{\|\zeta\|} := \|\zeta\|_{w,p,\lambda} + \|\Pr^u\zeta\|_{p,\lambda}$, where $\|\zeta\|_{w,p,\lambda}$ is as in (8). Recall the definition

(10) of $\mathcal{Y}_w^{p,\lambda}$. We define

$$\widetilde{\mathcal{X}}_w := \widetilde{\mathcal{X}}_w^{p,\lambda} := \big\{\zeta \in W^{1,p}_{\mathrm{loc}}\big(TM^u \oplus \textstyle\bigwedge^1(\mathfrak{g}_P)\big) \,\big|\, \|\zeta\widetilde{\|}_{w,p,\lambda} < \infty\big\}, \tag{11}$$

$$\widetilde{\mathcal{Y}}_w := \widetilde{\mathcal{Y}}_w^{p,\lambda} := \mathcal{Y}_w^{p,\lambda} \oplus L^p_\lambda(\mathfrak{g}_P), \tag{12}$$

$$\widetilde{\mathcal{D}}_w := \widetilde{\mathcal{D}}_w^{p,\lambda} : \widetilde{\mathcal{X}}_w^{p,\lambda} \to \widetilde{\mathcal{Y}}_w^{p,\lambda}, \quad \widetilde{\mathcal{D}}_w\zeta := (\mathcal{D}_w\zeta, L_w^*\zeta). \tag{13}$$

Here $\mathcal{D}_w\zeta$ is defined as in (6). Note that the map $L_w^* : \widetilde{\mathcal{X}}_w :=\to L^p_\lambda(\mathfrak{g}_P)$ given by $L_w^*(v,\alpha) := L_u^*v - d_A^*\alpha$ is well-defined and bounded. This follows from the fact $L_x^* = L_x^*\,\mathrm{Pr}_x$ (for every $x \in M$) and compactness of $\overline{u(P)}$.

5. Theorem. *Let $p > 2$ and $\lambda > -2/p + 1$ be real numbers, and $w := (u,A) \in \widetilde{\mathcal{B}}^p_\lambda$ a smooth pair. Then the following statements hold.*

(i) The normed spaces $(\widetilde{\mathcal{X}}_w^{p,\lambda}, \|\cdot\|_{w,p,\lambda}), \mathcal{Y}_w^{p,\lambda}$ and $L^p_\lambda(\mathfrak{g}_P)$ are complete.

(ii) Assume that $-2/p+1 < \lambda < -2/p+2$. Then the operator $\widetilde{\mathcal{D}}_w^{p,\lambda} : \widetilde{\mathcal{X}}_w^{p,\lambda} \to \widetilde{\mathcal{Y}}_w^{p,\lambda}$ is Fredholm of real index

$$\mathrm{ind}\widetilde{\mathcal{D}}_w^{p,\lambda} = 2\bar{n} + 2m(w). \tag{14}$$

This theorem is proved in Section 3.2. The proof relies on the existence of a suitable trivialization of $TM^u \oplus \bigwedge^1(\mathfrak{g}_P)$ in which the operator $\widetilde{\mathcal{D}}^{p,\lambda}$ becomes standard.

6. Theorem. *Let $p > 2$, $\lambda > 1 - 2/p$, and $w := (u,A) \in \widetilde{\mathcal{B}}^p_\lambda \cap \widetilde{\mathcal{B}}$. Then the map $L_w^* : \widetilde{\mathcal{X}}_w \to L^p_\lambda(\mathfrak{g}_P)$ admits a bounded (linear) right inverse.*

The proof of Theorem 6 is postponed to Section 3.4 (see page 21). It is based on the existence of a bounded right inverse for the operator d_A^* over a compact subset of $\mathbb{R}^n$ diffeomorphic to $\bar{B}_1$ (Proposition 11) and the existence of a neighborhood $U \subseteq M$ of $\mu^{-1}(0)$ such that $\inf\big\{|L_x\xi| \,\big|\, x \in U,\ \xi \in \mathfrak{g} : |\xi| = 1\big\} > 0$. We define

$$M^* := \big\{x \in M \,\big|\, gx = x \Longrightarrow g = \mathbf{1}\big\}. \tag{15}$$

Proof of Theorem 2. Let $p > 2$, $\lambda > 1 - 2/p$, and $w := (u,A) \in \widetilde{\mathcal{B}}^p_\lambda$ be a smooth pair. We **prove (i)**.

1. Claim. *We have $\mathcal{X}_w := \mathcal{X}_w^{p,\lambda} = K := \ker\big(L_w^* : \widetilde{\mathcal{X}}_w \to L^p_\lambda(\mathfrak{g}_P)\big)$, and the restriction of the norm $\|\cdot\widetilde{\|}_{w,p,\lambda}$ to $\mathcal{X}_w$ is equivalent to $\|\cdot\|_{w,p,\lambda}$.*

Proof of Claim 1. It suffices to prove that $\mathcal{X}_w \subseteq K$ and this inclusion is bounded. It follows from hypothesis (H) that there exists $\delta > 0$ such that $\mu^{-1}(\bar{B}_\delta) \subseteq M^*$. We have $c := \min\big\{|L_x\xi| \,\big|\, x \in \mu^{-1}(\bar{B}_\delta),\ \xi \in \mathfrak{g} : |\xi| = 1\big\} > 0$. Lemma 30 below implies that there exists $R > 0$ such that $u(P|_{\mathbb{R}^2\setminus B_R}) \subseteq \mu^{-1}(\bar{B}_\delta)$. Let $\zeta = (v,\alpha) \in \mathcal{X}_w$. Then $L_u^*v = d_A^*\alpha$, and thus, using the last assertion of Remark 27 below,

$$\|\mathrm{Pr}^u v\|_{p,\lambda} \le c^{-1}\|L_u^*v\|_{p,\lambda} \le c^{-1}\|\nabla^A\alpha\|_{p,\lambda} \le c^{-1}\|\zeta\|_{w,p,\lambda} < \infty.$$

Hence $\mathcal{X}_w \subseteq K$, and this inclusion is bounded. This proves Claim 1. $\square$

Part (i) follows from part (i) of Theorem 5 and Claim 1. **Part (ii)** follows from part (ii) of Theorem 5, Theorem 6 and Lemma 24 (appendix) with $X := \widetilde{\mathcal{X}}_w, Y := \mathcal{Y}_w, Z := L^p_\lambda(\mathfrak{g}_P)$, $D' : \widetilde{\mathcal{X}}_w \to \mathcal{Y}_w$ given by (6) and $T := L_w^*$. This proves Theorem 2. $\square$

3.2. Proof of Theorem 5 (augmented vertical differential). We denote by s and t the standard coordinates in $\mathbb{R}^2$. For $v \in \mathbb{R}^n$ we denote $\langle v\rangle := \sqrt{1+|v|^2}$. For $d \in \mathbb{Z}$ we define $p_d : \mathbb{C} \to \mathbb{C}$, $p_d(z) := z^d$. We equip the bundle $\bigwedge^1(\mathfrak{g}_P)$ with the (fiberwise) complex structure J_P defined by $J_P\alpha := -\alpha\, i$. Furthermore, we denote $\mathfrak{g}^\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, $V := \mathbb{C}^{\bar n} \oplus \mathfrak{g}^\mathbb{C} \oplus \mathfrak{g}^\mathbb{C}$, and for $a \in \mathbb{C}$ we denote by $a \cdot \oplus\mathrm{id} : V \to V$ the map $\big(v^1,\ldots,v^{\bar n},\alpha,\beta\big) \mapsto \big(av^1,v^2,\ldots,v^{\bar n},\alpha,\beta\big)$. For $x \in M$ we write $L_x^\mathbb{C} : \mathfrak{g}^\mathbb{C} \to T_xM$ for the complex linear extension of L_x. We define

$$H_x := \ker d\mu(x) \cap (\mathrm{im} L_x)^\perp,\ \forall x \in M.$$

Note that in general, the union H of all the H_x's is not a smooth subbundle of TM, since the dimension of H_x may depend on x. However, there exists an open neighborhood $U \subseteq M$ of $\mu^{-1}(0)$ such that $H|_U$ is a subbundle of $TM|_U$. Let $p > 2$, $\lambda > -2/p+1$ and $w := (u,A) \in \widetilde{\mathcal{B}}^p_\lambda$ be a smooth pair. For $z \in \mathbb{R}^2$ we define $H^u_z := \big\{G\cdot(p,v)\,\big|\, p \in \pi^{-1}(z) \subseteq P,\ v \in H_{u(p)}\big\}$. Consider a complex trivialization (i.e. bundle isomorphism fixing the base $\mathbb{R}^2$)

$$\Psi : \mathbb{R}^2 \times V \to TM^u \oplus \bigwedge^1(\mathfrak{g}_P).$$

7. Definition. *We call* Ψ good*, if the following properties are satisfied.*

(i) **(Splitting)** *For every* $z \in \mathbb{R}^2$ *we have*

$$\Psi_z(\mathbb{C}^{\bar n} \oplus \mathfrak{g}^\mathbb{C} \oplus \{0\}) = TM^u_z \oplus \{0\}, \tag{16}$$

$$\Psi_z(\{0\} \oplus \{0\} \oplus \mathfrak{g}^\mathbb{C}) = \{0\} \oplus \bigwedge^1(\mathfrak{g}_P). \tag{17}$$

Furthermore, there exists a number $R > 0$*, a smooth section* σ *of* $P \to \mathbb{R}^2 \setminus B_1$*, and a point* $x_\infty \in \mu^{-1}(0)$*, such that the following conditions are satisfied. For every* $z \in \mathbb{R}^2 \setminus B_R$ *we have*

$$\Psi_z(\mathbb{C}^{\bar n} \oplus \{0\} \oplus \{0\}) = H^u_z, \tag{18}$$

$u \circ \sigma(re^{i\varphi})$ *converges to* x_∞*, uniformly in* $\varphi \in \mathbb{R}$*, as* $r \to \infty$*,* $\sigma^*A \in L^p_\lambda(\mathbb{R}^2 \setminus B_1, \mathfrak{g})$*, and for every* $z \in \mathbb{R}^2 \setminus B_R$ *and* $\big(\alpha, \beta = \varphi + i\psi\big) \in \mathfrak{g}^\mathbb{C} \oplus \mathfrak{g}^\mathbb{C}$*, we have*

$$\Psi_z(0,\alpha,\beta) = \Big(G\cdot\big(u\circ\sigma(z), L^\mathbb{C}_{u\circ\sigma(z)}(\alpha)\big), G\cdot\big(\sigma(z), \varphi ds + \psi dt\big)\Big). \tag{19}$$

(ii) There exists a number $C > 0$ *such that for every* $(z,\zeta) \in \mathbb{R}^2 \times V$

$$C^{-1}|\zeta| \le \big|\Psi_z(\langle z\rangle^{m(w)} \cdot \oplus\mathrm{id})\zeta\big| \le C|\zeta|. \tag{20}$$

(iii) We have $\left|\nabla^A\big(\Psi(p_{m(w)}\cdot\oplus\mathrm{id})\big)\right|\in L^p_\lambda(\mathbb{R}^2\setminus B_1)$.

8. Proposition. *If* $p>2$, $\lambda>-2/p+1$ *and* $w:=(u,A)\in\widetilde{\mathcal{B}}^p_\lambda$ *is smooth then there exists a good complex trivialization of* $TM^u\oplus\bigwedge^1(\mathfrak{g}_P)$.

The proof of this proposition is postponed to subsection 3.3 (page 14). The next result shows that a good trivialization transforms $\widetilde{\mathcal{D}}_w$ into some standard operator. For $\alpha=(\alpha_1,\ldots,\alpha_n)\in(\mathbb{N}\coprod\{0\})^n$ we denote $|\alpha|:=\sum_{i=1}^n\alpha_i$ and $\partial^\alpha:=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$. Let $1\le p\le\infty$, $n\in\mathbb{N}$, $k\in\mathbb{N}\cup\{0\}$, $\lambda\in\mathbb{R}$, $\Omega\subseteq\mathbb{R}^n$ an open subset, W a real or complex vector space, and $u:\Omega\to W$ a k-times weakly differentiable map. We define

$$\|u\|_{L^{k,p}_\lambda(\Omega,W)} := \sum_{|\alpha|\le k}\big\|\langle\cdot\rangle^{\lambda+|\alpha|}\partial^\alpha u\big\|_{L^p(\Omega,W)}\in[0,\infty],$$

$$\|u\|_{W^{k,p}_\lambda(\Omega,W)} := \sum_{|\alpha|\le k}\|\langle\cdot\rangle^{\lambda}\partial^\alpha u\|_{L^p(\Omega,W)}\in[0,\infty],$$

$$L^{k,p}_\lambda(\Omega,W) := \big\{u\in W^{k,p}_{\mathrm{loc}}(\Omega,W)\,\big|\,\|u\|_{L^{k,p}_\lambda(\Omega,W)}<\infty\big\} \tag{21}$$

$$W^{k,p}_\lambda(\Omega,W) := \big\{u\in W^{k,p}_{\mathrm{loc}}(\Omega,W)\,\big|\,\|u\|_{W^{k,p}_\lambda(\Omega,W)}<\infty\big\}. \tag{22}$$

If $(X_i,\|\cdot\|_i)$, $i=1,\ldots,k$, are normed vector spaces then we endow $X_1\oplus\cdots\oplus X_k$ with the norm $\|(x_1,\ldots,x_k)\|:=\sum_i\|x_i\|_i$. Let $d\in\mathbb{Z}$. If $d<0$ then we choose $\rho_0\in C^\infty(\mathbb{R}^2,[0,1])$ such that $\rho_0(z)=0$ for $|z|\le1/2$ and $\rho_0(z)=1$ for $|z|\ge1$. In the case $d\ge0$ we set $\rho_0:=1$. The isomorphism of Lemma 12 (appendix) induces norms on $\widetilde{\mathcal{X}}'_{p,\lambda,d}:=\mathbb{C}\rho_0p_d+L^{1,p}_{\lambda-1-d}(\mathbb{R}^2,\mathbb{C})$ and $\widetilde{\mathcal{X}}''_{p,\lambda}:=\mathbb{C}^{\bar n-1}+L^{1,p}_{\lambda-1}(\mathbb{R}^2,\mathbb{C}^{\bar n-1})$. We define

$$\widetilde{\mathcal{X}}_d:=\widetilde{\mathcal{X}}^{p,\lambda}_d:=\widetilde{\mathcal{X}}'_{p,\lambda,d}\oplus\widetilde{\mathcal{X}}''_{p,\lambda}\oplus W^{1,p}_\lambda(\mathbb{R}^2,\mathfrak{g}^{\mathbb{C}}\oplus\mathfrak{g}^{\mathbb{C}}),$$

$$\widetilde{\mathcal{Y}}_d:=\widetilde{\mathcal{Y}}^{p,\lambda}_d:=L^p_{\lambda-d}(\mathbb{R}^2,\mathbb{C})\oplus L^p_\lambda\big(\mathbb{R}^2,\mathbb{C}^{\bar n-1}\oplus\mathfrak{g}^{\mathbb{C}}\oplus\mathfrak{g}^{\mathbb{C}}\big).$$

For a complex vector space W we denote by $\partial^W_{\bar z}$ (∂^W_z) the operator $\frac12(\partial_s+i\partial_t)$ ($\frac12(\partial_s-i\partial_t)$) acting on functions from $\mathbb{C}$ to W. We denote by $\langle\cdot,\cdot\rangle^{\mathbb{C}}_{\mathfrak{g}}$ the hermitian inner product on $\mathfrak{g}^{\mathbb{C}}$ (complex anti-linear in its first argument) extending $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. We define

$$F_1:TM^u\to{\textstyle\bigwedge}^{0,1}(T\mathbb{C},TM^u),\,F_2:{\textstyle\bigwedge}^1(\mathfrak{g}_P)\to{\textstyle\bigwedge}^2(\mathfrak{g}_P)\oplus\mathfrak{g}_P, \tag{23}$$

by $F_1(v):=(ds-Jdt)v$ and $F_2(\varphi ds+\psi dt):=(\psi ds\wedge dt,\varphi)$, and $F:=F_1\oplus F_2$.

9. Proposition (Operator in good trivialization)**.** *Let* $2<p<\infty$, $\lambda>-2/p+1$, $w:=(u,A)\in\widetilde{\mathcal{B}}^p_\lambda$ *be smooth and* $\Psi:\mathbb{R}^2\times V\to TM^u\oplus\bigwedge^1(\mathfrak{g}_P)$ *a good trivialization. Then the following statements hold.*

(i) The following maps are well-defined isomorphisms of normed spaces:

$$\widetilde{\mathcal{X}}_{m(w)}\ni\zeta\mapsto\Psi\zeta\in\widetilde{\mathcal{X}}_w,\quad\widetilde{\mathcal{Y}}_{m(w)}\ni\zeta\mapsto F\Psi\zeta\in\widetilde{\mathcal{Y}}_w \tag{24}$$

(ii) There exists a positive $\mathbb{C}$-linear map $S_\infty : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ (i.e. $\langle S_\infty v, v\rangle_{\mathfrak{g}}^{\mathbb{C}} > 0$ for every $0 \neq v \in \mathfrak{g}^{\mathbb{C}}$) such that the following operator is compact:

$$S := (F\Psi)^{-1}\widetilde{\mathcal{D}}_w\Psi - \partial_{\bar{z}}^{\mathbb{C}^{\bar{n}}} \oplus \begin{pmatrix} \partial_{\bar{z}}^{\mathfrak{g}^{\mathbb{C}}} & \mathrm{id}/2 \\ S_\infty & 2\partial_{\bar{z}}^{\mathfrak{g}^{\mathbb{C}}} \end{pmatrix} : \widetilde{\mathcal{X}}_{m(w)} \to \widetilde{\mathcal{Y}}_{m(w)} \tag{25}$$

The proof of Proposition 9 is postponed to subsection 3.3 (page 17). It is based on some inequalities and compactness properties for weighted Sobolev spaces (Proposition 13) and a Hardy-type inequality (Proposition 14).

Proof of Theorem 5. Let $p > 2$, $\lambda > -2/p + 1$, and let $w := (u, A) \in \widetilde{\mathcal{B}}_\lambda^p$ be a smooth pair. The space (21) is complete, see [**Lo1**]. By Proposition 13(ii) (appendix) the same holds for the space (22). Combining this with Propositions 8 and 9(i), **part (i)** follows. **Part (ii)** follows from Propositions 8 and 9(ii), Corollary 18 and Proposition 19 (appendix). This proves Theorem 5. □

10. Remark. *An alternative approach to prove Theorem* 5 *is to switch to "logarithmic" coordinates $\tau + i\varphi$ (defined by $e^{\tau+i\varphi} = s + it \in \mathbb{R}^2 \setminus \{0\}$). In these coordinates and a suitable trivialization the operator $\widetilde{\mathcal{D}}^{p,\lambda}$ is of the form $\partial_\tau + A(\tau)$. Hence one can try to apply the results of* [**RoSa**]. *However, this is not possible, since $A(\tau)$ contains the operator $v \mapsto e^{2\tau} d\mu(u)v\, d\tau \wedge d\varphi$, which diverges for $\tau \to \infty$.*

3.3. Proofs of the results of subsection 3.2.

Proof of Proposition 8 . Let p, λ and w be as in the hypothesis. We choose a section σ of $P|_{\mathbb{R}^2\setminus B_1}$ and a point $x_\infty \in \mu^{-1}(0)$ as in Lemma 30.

1. Claim. *There exists an open G-invariant neighborhood $U \subseteq M$ of x_∞ such that $H|_U$ is a smooth subbundle of TM with the following property. There exists a smooth complex trivialization $\Psi^U : U \times \mathbb{C}^{\bar{n}} \to H|_U$ satisfying $\Psi^U_{gx} v_0 = g\Psi^U_x v_0 := g\Psi^U(x, v_0)$, for every $g \in G$, $x \in U$ and $v_0 \in \mathbb{C}^{\bar{n}}$.*

Proof of Claim 1. By hypothesis (H) we have $x_\infty \in M^*$, where M^* is defined as in (15). We choose a G-invariant neighborhood $U_0 \subseteq M^*$ of x_∞ so small that $\ker d\mu(x)$ and $(\mathrm{im} L_x)^\perp$ intersect transversely, for every $x \in U_0$. Then $H|_{U_0}$ is a smooth subbundle of $TM|_{U_0}$. Furthermore, by the local slice theorem there exists a pair (U, N), where $U \subseteq U_0$ is a G-invariant neighborhood of x_∞ and $N \subseteq U$ is a submanifold of dimension $\dim M - \dim G$ that intersects Gx transversely in exactly one point, for every $x \in U$. We choose a complex trivialization of $H|_N$ and extend it in a G-equivariant way, to obtain a trivialization Ψ^U of $H|_U$. This proves Claim 1. □

We choose U and Ψ^U as in Claim 1. It follows from Lemma 30 that there exists $R > 1$ such that $u(p) \in U$, for $p \in \pi^{-1}(z) \subseteq P$, if $z \in \mathbb{R}^2 \setminus B_R$. We define $\widetilde{\Psi}^\infty : (\mathbb{R}^2 \setminus B_R) \times (\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}}) \to TM^u = (u^*TM)/G$ by

$$\widetilde{\Psi}^\infty_z(v_0,\alpha) = G \cdot \big(u \circ \sigma(z), \Psi^U_{u\circ\sigma(z)}(z^{-m(w)} \cdot \oplus \mathrm{id})v_0 + L^{\mathbb{C}}_{u\circ\sigma(z)}\alpha\big),$$

This is a smooth complex trivialization of $TM^u|_{\mathbb{R}^2\setminus B_R}$.

2. Claim. $\widetilde{\Psi}^\infty|_{\mathbb{C}\setminus B_{R+1}}$ *extends to a smooth complex trivialization of* TM^u*.*

We define $f : \mathbb{C} \setminus \{0\} \to S^1$ by $f(z) := z/|z|$.

Proof of Claim 2. We choose a complex trivialization $\Psi^0 : \bar B_R \times (\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}}) \to TM^u|_{\bar B_R}$. We define $\Phi : S^1_R := \{z \in \mathbb{C} \,|\, |z| = R\} \to \mathrm{Aut}(\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}})$ by

$$\Phi_z(v_0,\alpha) := (\Psi^0_z)^{-1}\big(G \cdot \big(u\circ\sigma(z), \Psi^U_{u\circ\sigma(z)}v_0 + L^{\mathbb{C}}_{u\circ\sigma(z)}\alpha\big)\big).$$

For a continuous map $x : S^1_R \to S^1$ we denote by $\deg(x)$ its degree.

3. Claim. *The index* $m(w)$ *equals* $\deg(f \circ \det \circ \Phi)$.

Proof of Claim 3. We define $\widetilde{P}$ to be the quotient of $P \coprod \big((S^2 \setminus \{0\}) \times G\big)$ under the equivalence relation generated by $p \sim (z, g)$, where $g \in G$ is determined by $\sigma(z)g = p$, for $p \in \pi^{-1}(z) \subseteq P$, $z \in \mathbb{C} \setminus \{0\}$. Furthermore, we define $\widetilde{u} : \widetilde{P} \to M$ by $\widetilde{u}([p]) := u(p)$, for $p \in P$, and $\widetilde{u}([\infty, g]) := g^{-1}x_\infty$, for $g \in G$. The statement of Lemma 30 implies that this map is continuous and extends u. The (fiberwise linear) complex structure u^*J on u^*TM descends to a complex structure $\widetilde{J}$ on $TM^{\widetilde{u}} = (\widetilde{u}^*TM)/G \to S^2$. By definition, we have $m(w) = c_1\big(TM^{\widetilde{u}}, \widetilde{J}\big)$. We define the local trivialization $\Psi^\infty : (S^2 \setminus B_R) \times (\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}}) \to TM^{\widetilde{u}}$ by

$$\Psi^\infty_z(v_0,\alpha) := \begin{cases} G \cdot \big([u\circ\sigma(z)], \Psi^U_{u\circ\sigma(z)}v_0 + L^{\mathbb{C}}_{u\circ\sigma(z)}\alpha\big), & \text{if } z \in \mathbb{R}^2 \setminus B_R, \\ G \cdot \big([\infty, \mathbf{1}], \Psi^U_{x_\infty}v_0 + L^{\mathbb{C}}_{x_\infty}\alpha\big), & \text{if } z = \infty. \end{cases}$$

Then $\Phi_z = (\Psi^0_z)^{-1}\Psi^\infty_z$, for $z \in S^1_R$, and therefore Φ is the transition map between Ψ^0 and Ψ^∞. Claim 3 follows from this. □

By Claim 3 and Lemma 25 in the appendix the maps Φ and $S^1_R \ni z \mapsto (z^{m(w)} \cdot \oplus \mathrm{id}) \in \mathrm{Aut}(\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}})$ are homotopic. Hence there exists a continuous map $h : \bar B_R \setminus B_1 \to \mathrm{Aut}(\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}})$ such that $h_z := h(z) = (z^{m(w)} \cdot \oplus \mathrm{id})$, if $z \in S^1_1$, and $h_z = \Phi(z)$, if $z \in S^1_R$. We define $\widetilde{\Psi} : \mathbb{R}^2 \times (\mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}}) \to TM^u$ by $\widetilde{\Psi} := \widetilde{\Psi}^\infty$ on $\mathbb{R}^2 \setminus B_R$, and $\widetilde{\Psi}_z(v_0,\alpha) := \Psi^0_z h_z(z^{-m(w)} \cdot \oplus \mathrm{id})(v_0,\alpha)$, for $z \in B_R$, $(v_0,\alpha) \in \mathbb{C}^{\bar n} \oplus \mathfrak{g}^{\mathbb{C}}$. Smoothing $\widetilde{\Psi}$ out on the ball B_{R+1}, we obtain the required extension of $\widetilde{\Psi}^\infty|_{\mathbb{C}\setminus B_{R+1}}$. This proves Claim 2. □

We define $\widehat{\Psi}^\infty : (\mathbb{R}^2 \setminus B_R) \times \mathfrak{g}^{\mathbb{C}} \to \bigwedge^1\big((P|_{\mathbb{R}^2\setminus B_R} \times \mathfrak{g})/G\big)$ by

$$\widehat{\Psi}^\infty_z(\varphi + i\psi) := G \cdot \big(\sigma(z), \varphi ds + \psi dt\big).$$

4. Claim. $\widehat{\Psi}^\infty|_{\mathbb{R}^2\setminus B_{R+1}}$ *extends to a smooth complex trivialization of the bundle* $\bigwedge^1(\mathfrak{g}_P) \to \mathbb{C}$*.*

Proof of Claim 4. We denote by Ad and $\mathrm{Ad}^{\mathbb{C}}$ the adjoint representations of G on $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ respectively. For every $g \in G$ we have $\det(\mathrm{Ad}^{\mathbb{C}}_g) = \det(\mathrm{Ad}_g) \in \mathbb{R}$. We choose a continuous section $\widetilde{\sigma}$ of the restriction $P|_{\bar{B}_R}$. We define $g : S^1_R \to G$ to be the unique map such that $\sigma(z) = \widetilde{\sigma}(z)g(z)$, for every $z \in S^1_R$. It follows that $f \circ \det(\mathrm{Ad}^{\mathbb{C}}_g) \equiv \pm 1$. Therefore, $\deg\big(S^1_R \ni z \mapsto f \circ \det(\mathrm{Ad}^{\mathbb{C}}_{g(z)})\big) = 0$. Hence Lemma 25 (appendix) implies that there exists a continuous map $\Phi : \bar{B}_R \to \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ satisfying $\Phi_z := \Phi(z) = \mathrm{Ad}^{\mathbb{C}}_{g(z)}$, for every $z \in S^1_R$. We define $\widehat{\Psi} : \mathbb{R}^2 \times \mathfrak{g}^{\mathbb{C}} \to \bigwedge^1(\mathfrak{g}_P)$ by $\widehat{\Psi} := \widehat{\Psi}^\infty$ on $\mathbb{R}^2 \setminus B_R$ and by $\widehat{\Psi}_z\alpha := G \cdot \big(\widetilde{\sigma}, \varphi' ds + \psi' dt\big)$, where $\varphi' + i\psi' := \Phi_z\alpha$, for $z \in B_R$, $\alpha \in \mathfrak{g}^{\mathbb{C}}$. Smoothing $\widehat{\Psi}$ out on the ball B_{R+1}, we obtain the required extension of $\widehat{\Psi}^\infty|_{\mathbb{R}^2 \setminus B_{R+1}}$. This proves Claim 4. □

We choose extensions $\widetilde{\Psi}$ and $\widehat{\Psi}$ of $\widetilde{\Psi}^\infty$ and $\widehat{\Psi}^\infty$ as in Claims 2 and 4.

5. Claim. *$\Psi := \widetilde{\Psi} \oplus \widehat{\Psi}$ is a good complex trivialization of $TM^u \oplus \bigwedge^1(\mathfrak{g}_P)$.*

Proof of Claim 5. Condition **(i)** of Definition 7 follows from the construction of Ψ. To prove **(ii)**, note that for $z \in \mathbb{R}^2 \setminus B_{R+1}$ and $(v_0, \alpha, \beta) \in V$, we have

$$|\Psi_z(z^{m(w)} \cdot \oplus \mathrm{id})(v_0, \alpha, \beta)|^2 = |\Psi^U_{u\circ\sigma(z)} v_0|^2 + |L^{\mathbb{C}}_{u\circ\sigma(z)}\alpha|^2 + |\beta|^2. \tag{26}$$

Here we used the fact $H_x = (\mathrm{im} L^{\mathbb{C}}_x)^\perp$, for every $x \in M$. By our choice of U, $H|_U \subseteq TM|_U$ is a smooth subbundle of rank $\dim M - 2\dim G$. It follows that $\mathrm{im} L^{\mathbb{C}}|_U = H^\perp|_U$ is a smooth subbundle of $TM|_U$ of rank $2 \dim G$. Hence $L^{\mathbb{C}}_x : \mathfrak{g}^{\mathbb{C}} \to T_xM$ is injective, for every $x \in U$. Since by assumption $\overline{u(P)} \subseteq M$ is compact, the same holds for the set $\overline{u(P|_{\mathbb{R}^2 \setminus B_{R+1}})} \subseteq \overline{u(P)}$. It follows that there exists a constant $C > 0$ such that

$$C^{-1}|v_0| \leq |\Psi^U_{u\circ\sigma(z)} v_0| \leq C|v_0|, \quad C^{-1}|\alpha| \leq |L^{\mathbb{C}}_{u\circ\sigma(z)}\alpha| \leq C|\alpha|,$$

for every $z \in \mathbb{R}^2 \setminus B_{R+1}$, $v_0 \in \mathbb{C}^{\bar{n}}$ and $\alpha \in \mathfrak{g}^{\mathbb{C}}$. Combining this with equality (26), condition (ii) follows.

We check condition **(iii)**. Let $\zeta := \big(v_0, \alpha, \beta = \varphi + i\psi\big) \in V$, $z \in \mathbb{R}^2 \setminus B_{R+1}$ and $v \in T_z\mathbb{R}^2$. We choose a point $p \in \pi^{-1}(z) \subseteq P$ and a vector $\widetilde{v} \in T_pP$ such that $\pi_*\widetilde{v} = v$. Then

$$\nabla^A_v\big(\widetilde{\Psi}(p_{m(w)} \cdot \oplus \mathrm{id})(v_0, \alpha)\big) = G \cdot \big(u(p), \widetilde{\nabla}^A_{\widetilde{v}}(\Psi^U_u v_0 + L^{\mathbb{C}}_u \alpha)\big). \tag{27}$$

Furthermore, for every smooth vector field X on U we have

$$\widetilde{\nabla}^A_{\widetilde{v}} X = (u^*\nabla)_{\widetilde{v} - pA\widetilde{v}} X = \nabla_{d_A u \cdot \widetilde{v}} X. \tag{28}$$

We define C to be the maximum of $|\nabla_{v'}(\Psi^U_x v'' + L^{\mathbb{C}}_x \alpha)|$ over all $v' \in T_xM$, $x \in \overline{u(P|_{\mathbb{R}^2 \setminus B_{R+1}})}$ and $(v'', \alpha) \in \mathbb{C}^{\bar{n}} \oplus \mathfrak{g}^{\mathbb{C}}$ such that $|v'| \leq 1$, $|(v'', \alpha)| \leq 1$.

Furthermore, we define $C' := \|d_A u\|_{L^p_\lambda(\mathbb{R}^2\setminus B_{R+1})}$. By (27) and (28) with $X(x) := \Psi^U_x v_0 + L^{\mathbb{C}}_x \alpha$, we have

$$\big\|\nabla^A_v\big(\widetilde{\Psi}(p_{m(w)}\cdot\oplus\mathrm{id})(v_0,\alpha)\big)\big\|_{L^p_\lambda(\mathbb{R}^2\setminus B_{R+1})} \leq CC'|v||(v_0,\alpha)|, \tag{29}$$

We now define $\widetilde{\varphi},\widetilde{\psi} : P \to g$ to be the unique equivariant maps such that $\widetilde{\varphi}\circ\sigma \equiv \varphi$, $\widetilde{\psi}\circ\sigma\equiv\psi$. We have $d_A\widetilde{\varphi}\,\sigma_* v = [(\sigma^*A)v,\varphi]$, and similarly for $\widetilde{\psi}$. Since $\widetilde{\nabla}^A_{\sigma_* v}(\widetilde{\varphi}ds + \widetilde{\psi}dt) = (d_A\widetilde{\varphi}\sigma_* v)ds + (d_A\widetilde{\psi}\sigma_* v)dt$, it follows that

$$\big|\nabla^A_v\big(\widehat{\Psi}(\varphi ds + \psi dt)\big)\big| = \big|G\cdot\big(\sigma(z),\widetilde{\nabla}^A_{\sigma_* v}(\widetilde{\varphi}ds+\widetilde{\psi}dt)\big)\big| = \big|\big[(\sigma^*A)v,\beta\big]\big|.$$

Combining this with (29) and the facts $d_A u \in L^p_\lambda(\mathbb{R}^2\setminus B_{R+1})$ and $\|\sigma^*A\|_{p,\lambda} < \infty$, condition (iii) follows. This proves Claim 5 and concludes the proof of Proposition 8. □

Proof of Proposition 9. Let $p,\lambda,w=(u,A)$ and Ψ be as in the hypothesis. We choose $\rho_0\in C^\infty(\mathbb{R}^2,[0,1])$ such that $\rho_0(z)=0$ for $z\in B_{1/2}$ and $\rho_0(z)=1$ for $z\in\mathbb{R}^2\setminus B_1$. We fix $R\geq 1$, σ and x_∞ as in Definition 7(i). We abbreviate $d := m(w)$.

We prove (i). For every $\zeta\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,V)$ Leibnitz' rule implies that

$$\nabla^A(\Psi\zeta) = \big(\nabla^A\big(\Psi(p_d\cdot\oplus\mathrm{id})\big)\big)(p_{-d}\cdot\oplus\mathrm{id})\zeta + \Psi(p_d\cdot\oplus\mathrm{id})D\big((p_{-d}\cdot\oplus\mathrm{id})\zeta\big). \tag{30}$$

1. Claim. *The first map in (24) is well-defined and bounded.*

Proof of Claim 1. Proposition 13(i) below and the fact $\lambda > -2/p+1$ imply that there exists a constant C_1 such that

$$\big\|(\langle\cdot\rangle^{-d}\cdot\oplus\mathrm{id})\zeta\big\|_\infty \leq C_1\|\zeta\|_{\widetilde{\mathcal{X}}_d},\quad \forall\zeta\in\widetilde{\mathcal{X}}_d\,. \tag{31}$$

We choose a constant $C_2 := C$ as in part (ii) of Definition 7. Then by (20) and (31), we have

$$\|\Psi\zeta\|_\infty \leq C_1C_2\|\zeta\|_{\widetilde{\mathcal{X}}_d}\quad\forall\zeta\in\widetilde{\mathcal{X}}_d\,. \tag{32}$$

It follows from (18) and (19), the definition $H_x := \ker d\mu(x)\cap\mathrm{im}L_x^\perp$ and the compactness of $\overline{u(P)}$ that there exists $C_3\in\mathbb{R}$ such that, for every $\zeta\in\widetilde{\mathcal{X}}_d$,

$$\big\|\,|d\mu(u)v'| + |\mathrm{Pr}^u v'| + |\alpha'|\,\big\|_{p,\lambda} \leq C_3\|\zeta\|_{\widetilde{\mathcal{X}}_d}, \tag{33}$$

where $(v',\alpha') := \Psi\zeta$. For $r>0$ we denote $B_r^C := \mathbb{R}^2\setminus B_r$ and $\|\cdot\|_{p,\lambda;r} := \|\cdot\|_{L^p_\lambda(B_r^C)}$. We define $C_4 := \max\big\{\big\|\nabla^A\big(\Psi(p_d\cdot\oplus\mathrm{id})\big)\big\|_{p,\lambda;1}, C_2\big\}$. By condition (iii) of Definition 7 we have $C_4<\infty$. Let $\zeta\in\widetilde{\mathcal{X}}_d$. Then by (30) we have

$$\|\nabla^A(\Psi\zeta)\|_{p,\lambda;1} \leq C_4\big(\|(p_{-d}\cdot\oplus\mathrm{id})\zeta\|_{L^\infty(B_1^C)} + \big\|D\big((p_{-d}\cdot\oplus\mathrm{id})\zeta\big)\big\|_{p,\lambda;1}\big). \tag{34}$$

We define $C_5 := \max\big\{-d2^{(-d+3)/2},2\big\}$. Then $\big\|D\big((p_{-d}\cdot\oplus\mathrm{id})\zeta\big)\big\|_{p,\lambda;1} \leq C_5\|\zeta\|_{\widetilde{\mathcal{X}}_d}$ by Proposition 13(iv). Combining this with (34) and (31), we get

$$\|\nabla^A(\Psi\zeta)\|_{p,\lambda;1} \leq C_4\big(2^{\frac{|d|}{2}}C_1 + C_5\big)\|\zeta\|_{\widetilde{\mathcal{X}}_d}. \tag{35}$$

By a direct calculation there exists a constant C_6 such that $\|\nabla^A(\Psi\zeta)\|_{L^p(B_1)} \leq C_6\|\zeta\|_{\widetilde{\mathcal{X}}_d}$, for every $\zeta \in \widetilde{\mathcal{X}}_d$. Claim 1 follows from this and (32,33,35). □

2. Claim. *The map $\widetilde{\mathcal{X}}_w \ni \zeta' \mapsto \Psi^{-1}\zeta' \in \widetilde{\mathcal{X}}_d$ is well-defined and bounded.*

Proof of Claim 2. We choose a neighborhood $U \subseteq M$ of $\mu^{-1}(0)$ as in Lemma 28 (appendix), and define c as in (65), and $C_1 := \max\{c^{-1}, 1\}$. Since $u \circ \sigma(re^{i\varphi})$ converges to x_∞, uniformly in $\varphi \in \mathbb{R}$, as $r \to \infty$, there exists $R' \geq R$ such that $u(p) \in U$, for every $p \in \pi^{-1}(B^C_{R'}) \subseteq P$. Then (18,19) and (65) imply that

$$\big\|(\alpha, \beta)\big\|_{p,\lambda;R'} \leq C_1\big\|\Psi(0, \alpha, \beta)\big\|_{p,\lambda;R'} \leq C_1\|\zeta'\|_w, \tag{36}$$

where $(v_0, \alpha, \beta) := \Psi^{-1}\zeta'$, for every $\zeta' \in \widetilde{\mathcal{X}}_d$.

3. Claim. *There exists a constant C_2 such that for every $\zeta' \in \widetilde{\mathcal{X}}_w$, we have*

$$\big\|D\big((\rho_0 p_{-d}\cdot \oplus \mathrm{id})\Psi^{-1}\zeta'\big)\big\|_{L^p_\lambda(\mathbb{R}^2)} \leq C_2\|\zeta'\|_w. \tag{37}$$

Proof of Claim 3. It follows from equality (30) and conditions (ii) and (iii) of Definition 7 that there exist constants C and C' such that

$$\begin{aligned}&\big\|D\big((p_{-d}\cdot \oplus \mathrm{id})\Psi^{-1}\zeta'\big)\big\|_{p,\lambda;1} \\ &\leq C\big(\|\nabla^A\zeta'\|_{p,\lambda} + \big\|\nabla^A\big(\Psi(p_d \cdot \oplus \mathrm{id})\big)\big\|_{p,\lambda}\|\zeta'\|_\infty\big) \leq C'\|\zeta'\|_w,\end{aligned} \tag{38}$$

for every $\zeta' \in \widetilde{\mathcal{X}}_w$. On the other hand, Leibnitz' rule implies

$$D(\Psi^{-1}\zeta') = \Psi^{-1}\big(\nabla^A\zeta' - (\nabla^A\Psi)\Psi^{-1}\zeta'\big).$$

Hence by a short calculation, using Leibnitz' rule again, it follows that there exists a constant C'' such that

$$\big\|D\big((\rho_0 p_{-d}\cdot \oplus \mathrm{id})\Psi^{-1}\zeta'\big)\big\|_{L^p(B_1)} \leq C''\|\zeta'\|_w,$$

for every $\zeta' \in \widetilde{\mathcal{X}}_w$. Combining this with (38), Claim 3 follows. □

Let $\zeta' \in \widetilde{\mathcal{X}}_w$. We denote $\widetilde{\zeta} := (\widetilde{v}_0, \widetilde{\alpha}, \widetilde{\beta}) := (\rho_0 p_{-d}\cdot \oplus \mathrm{id})\Psi^{-1}\zeta'$. By inequality (37) the hypotheses of Proposition 14 with $n := 2$ and λ replaced by $\lambda - 1$ are satisfied. It follows that there exists $\zeta_\infty := \big(v_\infty, \alpha_\infty, \beta_\infty\big) \in V = \mathbb{C}^{\bar{n}} \oplus \mathfrak{g}^{\mathbb{C}} \oplus \mathfrak{g}^{\mathbb{C}}$, such that $\widetilde{\zeta}(re^{i\varphi}) \to \zeta_\infty$, uniformly in $\varphi \in \mathbb{R}$, as $r \to \infty$, and

$$\|(\widetilde{\zeta} - \zeta_\infty)|\cdot|^{\lambda-1}\|_{L^p(\mathbb{R}^2)} \leq (\dim M + 2\dim G)p/(\lambda + 2/p)\big\|D\widetilde{\zeta}|\cdot|^\lambda\big\|_{L^p(\mathbb{R}^2)}. \tag{39}$$

Since $\lambda > -2/p + 1$, we have $\int_{B^C_{R'}} \langle\cdot\rangle^{p\lambda} = \infty$. Hence the convergence $(\widetilde{\alpha}, \widetilde{\beta}) \to (\alpha_\infty, \beta_\infty)$ and the estimate (36) imply that $(\alpha_\infty, \beta_\infty) = (0, 0)$. We choose a constant $C > 0$ as in part (ii) of Definition 7. The convergence $\widetilde{v}_0 \to v_\infty$ and the first inequality in (20) imply that

$$|v_\infty| \leq \|\widetilde{v}_0\|_\infty \leq 2^{\frac{|d|}{2}}C\|\zeta'\|_\infty. \tag{40}$$

We define $\big(v^1,\ldots,v^{\bar n},\alpha,\beta\big):=\Psi^{-1}\zeta'-\big(\rho_0p_dv^1_\infty,v^2_\infty,\ldots,v^{\bar n}_\infty,0,0\big)$. Proposition 13(iv) in the appendix and inequalities (39) and (37) imply that there exists a constant C_6 (depending on p,λ,d and Ψ, but not on ζ') such that

$$\|v^1\|_{L^{1,p}_{\lambda-1-d}(B_1^C)}+\big\|\big(v^2,\ldots,v^{\bar n},\alpha,\beta\big)\big\|_{L^{1,p}_{\lambda-1}(B_1^C)}\le C_6\|\zeta'\|_w\,. \tag{41}$$

Finally, by a straight-forward argument, there exists a constant C_7 (independent of ζ') such that $\|\Psi^{-1}\zeta'\|_{W^{1,p}(B_{R'})}\le C_7\|\zeta'\|_w$. Combining this with (36,40,41), Claim 2 follows. □

Claims 1 and 2 imply that the first map in (24) is an isomorphism (of normed vector spaces). It follows from condition (ii) of Definition 7 that the second map in (24) is an isomorphism. This completes the proof of **(i)**.

We prove statement (ii). Recall that we have chosen $R>0,\sigma$ and x_∞ as in Definition 7(i). We define $S_\infty:\mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$ to be the complex linear extension of $L^*_{x_\infty}L_{x_\infty}:\mathfrak{g}\to\mathfrak{g}$. By our hypothesis (H) the Lie group G acts freely on $\mu^{-1}(0)$. It follows that L_{x_∞} is injective. Therefore S_∞ is positive with respect to $\langle\cdot,\cdot\rangle^{\mathbb{C}}_{\mathfrak{g}}$. By (16) and (17) there exist complex trivializations

$$\Psi_1:\mathbb{R}^2\times(\mathbb{C}^{\bar n}\oplus\mathfrak{g}^{\mathbb{C}})\to TM^u,\qquad \Psi_2:\mathbb{R}^2\times\mathfrak{g}^{\mathbb{C}}\to\textstyle\bigwedge^1(\mathfrak{g}_P),$$

such that $\Psi=\Psi_1\oplus\Psi_2$. We denote by $\iota:\mathfrak{g}^{\mathbb{C}}\to\mathbb{C}^{\bar n}\oplus\mathfrak{g}^{\mathbb{C}}$ ($\mathrm{pr}:\mathbb{C}^{\bar n}\oplus\mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$) the inclusion as (the projection onto) the second factor. We define

$$\widetilde{\mathcal{X}}^1_d:=L^{1,p}_{\lambda-1-d}(\mathbb{R}^2,\mathbb{C})\oplus L^{1,p}_{\lambda-1}(\mathbb{R}^2,\mathbb{C}^{\bar n-1})\oplus W^{1,p}_\lambda(\mathbb{R}^2,\mathfrak{g}^{\mathbb{C}}),\quad \widetilde{\mathcal{X}}^2_d:=W^{1,p}_\lambda(\mathbb{R}^2,\mathfrak{g}^{\mathbb{C}}),$$

$$\widetilde{\mathcal{X}}'_d:=\widetilde{\mathcal{X}}^1_d\oplus\widetilde{\mathcal{X}}^2_d,\quad \widetilde{\mathcal{X}}^0_d:=\mathbb{C}\rho_0p_d\oplus\mathbb{C}^{\bar n-1}\oplus\{(0,0)\}\subseteq\widetilde{\mathcal{X}}_d,$$

$$\widetilde{\mathcal{Y}}^1_d:=L^p_{\lambda-d}(\mathbb{R}^2,\mathbb{C})\oplus L^p_\lambda(\mathbb{R}^2,\mathbb{C}^{\bar n-1}\oplus\mathfrak{g}^{\mathbb{C}}),\quad \widetilde{\mathcal{Y}}^2_d:=L^p_\lambda(\mathbb{R}^2,\mathfrak{g}^{\mathbb{C}}).$$

Note that $\widetilde{\mathcal{X}}_d=\widetilde{\mathcal{X}}^0_d+\widetilde{\mathcal{X}}'_d$ and $\widetilde{\mathcal{Y}}_d=\widetilde{\mathcal{Y}}^1_d\oplus\widetilde{\mathcal{Y}}^2_d$. We define $S:\widetilde{\mathcal{X}}_d\to\widetilde{\mathcal{Y}}_d$ as in (25). Since $\widetilde{\mathcal{X}}^0_d$ is finite dimensional, $S|_{\widetilde{\mathcal{X}}^0_d}$ is compact. Hence it suffices to prove that $S|_{\widetilde{\mathcal{X}}'_d}$ is compact. To see this, we denote

$$Q:=\begin{pmatrix} ds\wedge dt\,d\mu(u)\\ L_u^*\end{pmatrix},\quad T:=\begin{pmatrix} d_A\\ -d_A^*\end{pmatrix},$$

and we define ${S^i}_j:\widetilde{\mathcal{X}}^j_d\to\widetilde{\mathcal{Y}}^i_d$ (for $i,j=1,2$) and ${\widetilde S^1}_1:\widetilde{\mathcal{X}}^1_d\to\widetilde{\mathcal{Y}}^1_d$ by

$${S^1}_1v:=(F_1\Psi_1)^{-1}((\nabla^A\Psi_1)v)^{0,1},\ {\widetilde S^1}_1v:=-(F_1\Psi_1)^{-1}\big(J(\nabla_{\Psi_1v}J)(d_Au)^{1,0}/2\big),$$

$${S^1}_2\alpha:=(F_1\Psi_1)^{-1}(L_u\Psi_2\alpha)^{0,1}-\iota\alpha/2,\ {S^2}_1v:=\big((F_2\Psi_2)^{-1}Q\Psi_1-S_\infty\mathrm{pr}\big)v,$$

$${S^2}_2\alpha:=(F_2\Psi_2)^{-1}(T\Psi_2)\alpha.$$

Here F_1 and F_2 are as in (23) and $(T\Psi_2)\alpha:=T(\Psi_2\alpha)$, for $\alpha\in\mathfrak{g}^{\mathbb{C}}$ (viewed as a constant section of $\mathbb{R}^2\times\mathfrak{g}^{\mathbb{C}}$). A direct calculation shows that $S(v,\alpha)=\big({S^1}_1v+{\widetilde S^1}_1v+{S^1}_2\alpha,{S^2}_1v+{S^2}_2\alpha\big)$. For a subset $X\subseteq\mathbb{R}^2$ we denote by

$\chi_X : \mathbb{R}^2 \to \{0,1\}$ its characteristic function. It follows that $\chi_{B_R} S|_{\widetilde{\mathcal{X}}'_d}$ is of 0-th order. Since it vanishes outside B_R, it follows that this map is compact.

4. Claim. *The operators $\chi_{B_R^C} S^i{}_j$, $i,j = 1,2$, and $\chi_{B_R^C} \widetilde{S}^1_1$ are compact.*

Proof of Claim 4. To see that the map $\chi_{B_R^C} S^1{}_1$ is compact, note that Leibnitz' rule and holomorphicity of p_d imply that

$$(\nabla^A \Psi_1)^{0,1} = \big(\nabla^A\big(\Psi_1(p_d \cdot \oplus \mathrm{id})\big)\big)^{0,1}(p_{-d} \cdot \oplus \mathrm{id}), \quad \text{on } \mathbb{C} \setminus \{0\}.$$

Since $\lambda > 1 - 2/p$, assertions (iv) and (i) of Proposition 13 imply that the map $(\rho_0 p_{-d} \cdot \oplus \mathrm{id}) : \widetilde{\mathcal{X}}^1_d \to C_b(\mathbb{R}^2, \mathbb{C}^{\bar{n}} \oplus \mathfrak{g}^{\mathbb{C}})$ is well-defined and compact. By condition (iii) of Definition 7, the map

$$\chi_{B_R^C}\big(\nabla^A\big(\Psi_1(p_d \cdot \oplus \mathrm{id})\big)\big)^{0,1} : C_b(\mathbb{R}^2, \mathbb{C}^{\bar{n}} \oplus \mathfrak{g}^{\mathbb{C}}) \to L^p_\lambda\big(\bigwedge^{0,1}(\mathbb{R}^2, TM^u)\big)$$

is bounded. Condition (ii) of Definition 7 implies boundedness of the map $(F_1\Psi_1)^{-1} : L^p_\lambda\big(\bigwedge^{0,1}(\mathbb{R}^2, TM^u)\big) \to \widetilde{\mathcal{Y}}^1_d$. Compactness of $\chi_{B_R^C} S^1{}_1$ follows.

By the definition of $\widetilde{\mathcal{B}}^p_\lambda$, we have $|d_A u| \in L^p_\lambda(\mathbb{R}^2)$. This together with Proposition 13(iv) and (i) and Definition 7(ii) implies that the map $\chi_{B_R^C} \widetilde{S}^1{}_1$ is compact. Furthermore, it follows from Definition 7(i) that $\chi_{B_R^C} S^1{}_2 = 0$.

To see that $\chi_{B_R^C} S^2{}_1$ is compact, we define $f : B_R^C \to \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$ by setting $f(z) : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ to be the complex linear extension of the map $L^*_{u \circ \sigma(z)} L_{u \circ \sigma(z)} - L^*_{x_\infty} L_{x_\infty} : \mathfrak{g} \to \mathfrak{g}$. Since $u \circ \sigma(re^{i\varphi})$ converges to x_∞, uniformly in φ, as $r \to \infty$, the map $f(re^{i\varphi})$ converges to 0, uniformly in φ, as $r \to \infty$. Hence by Proposition 13(iii), the map $W^{1,p}_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}}) \ni \alpha \mapsto \chi_{B_R^C} f\alpha \in L^p_\lambda(\mathbb{C}, \mathfrak{g}^{\mathbb{C}})$ is compact. Definition 7(i) implies that $\chi_{B_R^C} S^2{}_1 = \chi_{B_R^C} f \mathrm{pr}$. It follows this map is compact.

Finally, Proposition 13(i) and parts (iii) and (ii) of Definition 7 imply that the map $\chi_{B_R^C} S^2{}_2$ is compact. Claim 4 follows. This completes the proofs of statement **(ii)** and Proposition 9. □

3.4. Proof of Theorem 6 (Right inverse for L^*_w). Let $n \in \mathbb{N}$, $k, \ell \in \mathbb{N} \cup \{0\}$, $p > n$, G a compact Lie group, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g} := \mathrm{Lie}(G)$, $\Omega \subseteq \mathbb{R}^n$ an open subset, $P \to \Omega$ a principal G-bundle and $A \in \mathcal{A}(P)$. Then A and the standard metric on Ω induce a connection ∇^A on $\bigwedge^k(\mathfrak{g}_P)$. For $\alpha \in W^{\ell,p}_{\mathrm{loc}}\big(\bigwedge^k(\mathfrak{g}_P)\big)$ we define

$$\|\alpha\|_{\ell,p,A} := \|\alpha\|_{W^{\ell,p}_{A,\langle \cdot,\cdot \rangle_{\mathfrak{g}}}(\Omega)} := \sum_{i=0,\dots,\ell} \|(\nabla^A)^\ell \alpha\|_{L^p(\Omega)},$$

where the pointwise norms are taken with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. We denote

$$W^{\ell,p}_A\big(\bigwedge^k(\mathfrak{g}_P)\big) := \big\{\alpha \in W^{\ell,p}_{\mathrm{loc}}\big(\bigwedge^k(\mathfrak{g}_P)\big) \,\big|\, \|\alpha\|_{\ell,p,A} < \infty\big\}.$$

For a subset $X \subseteq \mathbb{R}^n$ we denote by $\mathrm{int}X$ its interior. For the proof of Theorem 6 we need the following.

11. Proposition. *Let n, p, G and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be as above and $K \subseteq \mathbb{R}^n$ a compact subset diffeomorphic to $\bar{B}_1$. Then for every G-bundle $P \to \mathrm{int}K$ and $A \in \mathcal{A}(P)$ there exists a bounded right inverse R of the operator*

$$d_A^* : W_A^{1,p}\big(\bigwedge^1(\mathfrak{g}_P)\big) \to L^p(\mathfrak{g}_P). \tag{42}$$

Furthermore, there exist $\varepsilon > 0$ and $C > 0$ such that for every principal G-bundle $P \to \mathrm{int}K$ and every $A \in \mathcal{A}(P)$ satisfying $\|F_A\|_p \le \varepsilon$ the map R can be chosen such that $\|R\| := \sup\big\{\|R\xi\|_{1,p,A} \,\big|\, \xi \in L^p(\mathfrak{g}_P) : \|\xi\|_p \le 1\big\} \le C$.

The proof of Proposition 11 is postponed to the appendix (page 30).

Proof of Theorem 6. Let p, λ and $w = (u, A)$ be as in the hypothesis. It follows from hypothesis (H) that there exists $\delta > 0$ such that $\mu^{-1}(\bar{B}_\delta) \subseteq M^*$ (defined as in (15)). We define $c := \inf\big\{|L_x\xi|/|\xi| \,\big|\, x \in \mu^{-1}(\bar{B}_\delta),\, 0 \ne \xi \in \mathfrak{g}\big\}$. It follows from Lemma 30 that there exists a number $a > 0$ such that $u(p) \in \mu^{-1}(\bar{B}_\delta)$, for every $p \in \pi^{-1}\big(\mathbb{C} \setminus (-a, a)^2\big) \subseteq P$. We choose constants ε_1 and C_1 as in the second assertion of Proposition 11 (corresponding to ε and C, for $n = 2$). Furthermore, we choose constants ε_2 and C_2 as in Lemma 23 (corresponding to ε and C). We define $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. By assumption we have $F_A \in L^p_\lambda(\mathbb{C})$. Hence there exists an integer $N > a$ such that $\|F_A\|_{L^p_\lambda(\mathbb{C}\setminus(-N,N)^2)} < \varepsilon$. We choose a smooth map $\rho : [-1, 1] \to [0, 1]$ such that $\rho = 0$ on $[-1, -3/4] \cup [3/4, 1]$, $\rho = 1$ on $[-1/4, 1/4]$, and $\rho(-t) = \rho(t)$ and $\rho(t) + \rho(t-1) = 1$, for all $t \in [0, 1]$. We choose a bijection $(\varphi, \psi) : \mathbb{Z} \setminus \{0\} \to \mathbb{Z}^2 \setminus \{-N, \ldots, N\}^2$. We define $\widetilde{\rho} : \mathbb{R} \to [0, 1]$ by

$$\widetilde{\rho}(t) := \begin{cases} 1, & \text{if } |t| \le N, \\ \rho(|t| - N), & \text{if } N \le |t| \le N+1, \\ 0, & \text{if } |t| \ge N+1. \end{cases}$$

and $\rho_0 : \mathbb{R}^2 \to [0, 1]$ by $\rho_0(s, t) := \widetilde{\rho}(s)\widetilde{\rho}(t)$. Furthermore, for $i \in \mathbb{Z} \setminus \{0\}$ we define $\rho_i : \mathbb{R}^2 \to [0, 1]$ by $\rho_i(s, t) := \rho(s - \varphi(i))\rho(t - \psi(i))$. We choose a compact subset $K_0 \subseteq [-N-1, N+1]^2$ diffeomorphic to $\bar{B}_1$, such that $[-N-3/4, N+3/4]^2 \subseteq \mathrm{int}K_0$, and we denote $\Omega_0 := \mathrm{int}K_0$. Furthermore, we choose a compact subset $K \subseteq [-1, 1]^2$ diffeomorphic to $\bar{B}_1$, such that $[-3/4, 3/4] \subseteq \mathrm{int}K$. For $i \in \mathbb{Z} \setminus \{0\}$ we define $\Omega_i := \mathrm{int}K + (\varphi(i), \psi(i))$. For $i \in \mathbb{Z}$ we define $T_i := d_A^* : W_A^{1,p}\big(\bigwedge^1((P|_{\Omega_i} \times \mathfrak{g})/G)\big) \to L^p_\lambda\big((P|_{\Omega_i} \times \mathfrak{g})/G\big)$. By the first assertion of Proposition 11 there exists a bounded right inverse R_0 of T_0. We fix $i \in \mathbb{Z} \setminus \{0\}$. Since $\lambda > 1 - 2/p > 0$, by our choice of N we have $\|F_A\|_{L^p(\Omega_i)} \le \|F_A\|_{L^p_\lambda(\Omega_i)} < \varepsilon$. Hence it follows from the statement of Proposition 11 that there exists a right inverse R_i of T_i, satisfying

$$\|R_i\xi\|_{W_A^{1,p}(\Omega_i)} \le C_1\|\xi\|_{L^p(\Omega_i)}, \quad \forall \xi \in W^{1,p}(\mathfrak{g}_P|_{\Omega_i}). \tag{43}$$

We define

$$\hat{R}: L^p_{\mathrm{loc}}(\mathfrak{g}_P) \to W^{1,p}_{\mathrm{loc}}\big(\bigwedge^1(\mathfrak{g}_P)\big), \quad \hat{R}\xi := \sum_{i\in\mathbb{Z}} \rho_i \cdot R_i(\xi|_{\Omega_i}). \tag{44}$$

Each section $\xi : \mathbb{R}^2 \to \mathfrak{g}_P$ induces a section $L_u\xi : \mathbb{R}^2 \to TM^u$. For $p \in \pi^{-1}\big(\mathbb{R}^2 \setminus (-N,N)^2\big) \subseteq P$ we have $u(p) \in \mu^{-1}(\bar{B}_\delta) \subseteq M^*$, and therefore the map $L^*_{u(p)}L_{u(p)} : \mathfrak{g} \to \mathfrak{g}$ is invertible. Hence we may define

$$\widetilde{R}: L^p_{\mathrm{loc}}(\mathfrak{g}_P) \to L^p_{\mathrm{loc}}(TM^u), \quad (\widetilde{R}\xi)(z) := L_u(L_u^*L_u)^{-1}\big(\xi - d_A^*\hat{R}\xi\big)(z),$$

for $z \in \mathbb{R}^2\setminus(-N,N)^2$, and $(\widetilde{R}\xi)(z) := 0$, for $z \in (-N,N)^2$. Furthermore, we define $R : L^p_{\mathrm{loc}}(\mathfrak{g}_P) \to L^p_{\mathrm{loc}}\big(TM^u \oplus \bigwedge^1(\mathfrak{g}_P)\big)$ by $R\xi := (\widetilde{R}\xi, -\hat{R}\xi)$. It follows that $L_w^*R = \mathrm{id}$. Theorem 6 is now a consequence of the following:

1. Claim. *R restricts to a bounded map from $L^p_\lambda(\mathfrak{g}_P)$ to $\widetilde{\mathcal{X}}^{p,\lambda}_w$.*

Proof of Claim 1. We choose a constant C_3 so big that $\sup_{z\in\Omega_i}\langle z\rangle^{p\lambda} \le C_3 \inf_{z\in\Omega_i}\langle z\rangle^{p\lambda}$, for every $i \in \mathbb{Z}$. For a weakly differentiable section $\xi : \mathbb{R}^2 \to \mathfrak{g}_P$ we denote $\|\xi\|_{1,p,\lambda,A} := \|\xi\|_{p,\lambda} + \|d_A\xi\|_{p,\lambda}$.

2. Claim. *There exists a constant C_4 such that $\|\xi - d_A^*\hat{R}\xi\|_{1,p,\lambda,A} \le C_4\|\xi\|_{p,\lambda}$, for every $\xi \in L^p_\lambda(\mathfrak{g}_P)$.*

Proof of Claim 2. Let $\xi \in L^p_\lambda(\mathfrak{g}_P)$. We denote $\alpha_i := R_i(\xi|_{\Omega_i})$ and $\alpha := \hat{R}\xi$. Since $\sum_{i\in\mathbb{Z}}\rho_i = 1$, a straight-forward calculation shows that

$$d_A^*\alpha = \xi - \sum_{i\in\mathbb{Z}} *\big((d\rho_i)\wedge *\alpha_i\big). \tag{45}$$

Fix $z \in \mathbb{R}^2$. Then $\big|\big\{i \in \mathbb{Z} \,|\, \rho_i(z) \ne 0\big\}\big| \le 4$. Hence (45) implies that

$$\big|\big(\xi - d_A^*\alpha\big)(z)\big|^p \le 4^{p-1}\|\rho'\|^p_\infty \sum_{i\in\mathbb{Z}} |\alpha_i(z)|^p. \tag{46}$$

Inequalities (46) and (43) imply that

$$\|\xi - d_A^*\alpha\|^p_{p,\lambda} \le 4^p\|\rho'\|^p_\infty \max\big\{C_1^p, \|R_0\|^p\big\}C_3\sum_{i\in\mathbb{Z}}\|\xi\|^p_{L^p_\lambda(\Omega_i)}.$$

Equality (45) implies that

$$\big|d_A\big(\xi - d_A^*\alpha\big)(z)\big|^p \le 8^{p-1}\max\big\{\|\rho''\|^p_\infty, \|\rho'\|^p_\infty\big\}\sum_{i\in\mathbb{Z}}\big(|\alpha_i|^p + |\nabla^A\alpha_i|^p\big)(z).$$

Combining this with (43), Claim 2 follows. □

We choose C_4 as in Claim 2. Let $\xi \in L^p_\lambda(\mathfrak{g}_P)$. We abbreviate $\widetilde{\xi} := \xi - d_A^*\hat{R}\xi$. By the fact $\widetilde{\xi}|_{(-N,N)^2} = 0$, Lemma 23, the fact $\lambda > 1 - 2/p > 0$ and Claim 2, we have

$$\|\widetilde{\xi}\|_\infty \le C_2\|\widetilde{\xi}\|_{1,p,\lambda,A} \le C_2C_4\|\xi\|_{p,\lambda}. \tag{47}$$

Recall that $\mathrm{Pr} : TM \to TM$ and $\mathrm{Pr}^u : TM^u \to TM^u$ denote the orthogonal projections onto $\mathrm{im}L$ and $(u^*\mathrm{im}L)/G$. Claim 1 is now a consequence of the following three claims.

3. Claim. *We have*

$$\sup\big\{\big\|R\xi\big\|_\infty + \big\||d\mu(u)\widetilde{R}\xi| + |\mathrm{Pr}^u\widetilde{R}\xi| + |\hat{R}\xi|\big\|_{p,\lambda}\,\big|\,\xi \in L^p_\lambda(\mathfrak{g}_P) : \|\xi\|_{p,\lambda} \le 1\big\} < \infty.$$

Proof of Claim 3. Let $\xi \in L^p_\lambda(\mathfrak{g}_P)$ be such that $\|\xi\|_{p,\lambda} \le 1$. For $i \in \mathbb{Z}$ we denote $\alpha_i := R_i(\xi|_{\Omega_i})$. Since $\big|\big\{i \in \mathbb{Z}\,\big|\,\rho_i(z) \ne 0\big\}\big| \le 4$, for every $z \in \mathbb{R}^2$, we have $\|\hat{R}\xi\|^p_{p,\lambda} \le 4^p \sum_i \|\alpha_i\|^p_{L^p_\lambda(\Omega_i)}$. Using (43), it follows that $\|\hat{R}\xi\|^p_{p,\lambda} \le 4^p \max\{C_1^p, \|R_0\|^p\}C_3$. We define $C := \max\big\{|d\mu(x)| + |\mathrm{Pr}_x|\,\big|\,x \in \overline{u(P)}\big\}$. By Remark 27 below, the statement of Claim 2 and the fact $\widetilde{R}\xi|_{(-N,N)^2} = 0$, we have

$$\big\||d\mu(u)\widetilde{R}\xi| + |\mathrm{Pr}^u\widetilde{R}\xi|\big\|_{p,\lambda} \le Cc^{-1}\|\xi - d_A^*\hat{R}\xi\|_{L^p_\lambda(\mathbb{R}^2\setminus(-N,N)^2)} \le Cc^{-1}C_4.$$

Inequality (47) and Remark 27 imply that $\|\widetilde{R}\xi\|_\infty \le c^{-1}\|\widetilde{\xi}\|_{L^\infty(\mathbb{R}^2\setminus(-N,N)^2)} \le c^{-1}C_2C_3$. We fix $i \in \mathbb{Z}\setminus\{0\}$. Lemma 23, (43) and the fact $\lambda > 1 - 2/p > 0$ imply that $\|\alpha_i\|_\infty \le C_2\|\alpha_i\|_{W^{1,p}_A(\Omega_i)} \le C_2C_1\|\xi\|_{L^p(\Omega_i)} \le C_2C_1$. Furthermore,

$$\|\alpha_0\|_\infty \le C' := \sup\big\{\|\alpha\|_\infty\,\big|\,\alpha \in W^{1,p}_A\big(\bigwedge^1((P|_{\Omega_0}\times\mathfrak{g})/G\big) : \|\alpha\|_{1,p,A} \le 1\big\}\|R_0\|.$$

It follows that $\|\hat{R}\xi\|_\infty \le \sup_i \|\alpha_i\|_\infty \le \max\{C_2C_1, C'\} < \infty$. Claim 3 follows. □

4. Claim. *We have* $\sup\big\{\|\nabla^A(\widetilde{R}\xi)\|_{p,\lambda}\,\big|\,\xi \in L^p_\lambda(\mathfrak{g}_P) : \|\xi\|_{p,\lambda} \le 1\big\} < \infty.$

Proof of Claim 4. Let $\xi \in L^p_\lambda(\mathfrak{g}_P)$. We define $\widetilde{\xi} := \xi - d_A^*\hat{R}\xi$, $\eta := (L_u^*L_u)^{-1}\widetilde{\xi}$ and $\rho \in \Omega^2(M, \mathfrak{g})$ as in (63). By Lemma 26 below, we have

$$\nabla^A(L_u\eta) = L_u d_A\eta + \nabla_{d_Au}X_\eta. \tag{48}$$

Using the second part of Lemma 26 (with $v := L_u\eta$), it follows that

$$L_u^*L_u d_A\eta = d_A\widetilde{\xi} - \rho(d_Au, L_u\eta) - L_u^*\nabla_{d_Au}X_\eta. \tag{49}$$

We choose a constant C so big that $|\rho(v, v')| \le C|v||v'|$ and $|\nabla_v X_{\xi_0}| \le C|v||\xi_0|$, for every $x \in \mu^{-1}(\bar{B}_\delta)$, $v, v' \in T_xM$ and $\xi_0 \in \mathfrak{g}$. We define $C := \max\big\{c^{-1}, 3Cc^{-2}\big\}$. Since $\widetilde{R}\xi = L_u\eta$, equalities (48,49) imply that

$$\|\nabla^A(\widetilde{R}\xi)\|_{p,\lambda} \le C\big(\big\|d_A\widetilde{\xi}\big\|_{p,\lambda} + \|d_Au\|_{p,\lambda}\big\|\widetilde{\xi}\big\|_\infty\big).$$

Here we used Remark 27. Since $\|d_Au\|_{p,\lambda} < \infty$, Claim 2 and (47) now imply Claim 4. □

5. Claim. *We have* $\sup\big\{\|\nabla^A(\hat{R}\xi)\|_{p,\lambda}\,\big|\,\xi \in L^p_\lambda(\mathfrak{g}_P) : \|\xi\|_{p,\lambda} \le 1\big\} < \infty.$

Proof of Claim 5. Let $\xi \in L^p_\lambda(\mathfrak{g}_P)$ be such that $\|\xi\|^p_{p,\lambda} \leq 1$. We write $\alpha_i := R_i(\xi|_{\Omega_i})$. Then $\nabla^A(\hat{R}\xi) = \sum_i \big(\rho_i \nabla^A \alpha_i + d\rho_i \otimes \alpha_i\big)$. Setting $C := 8^p\|\rho'\|^p_\infty C_3 \max\{C_1^p, \|R_0\|^p\}$, it follows that

$$\|\nabla^A(\hat{R}\xi)\|^p_{p,\lambda} \leq 8^{p-1}\sum_i \Big(\|\nabla^A\alpha_i\|^p_{p,\lambda} + \|\rho'\|^p_\infty \|\alpha_i\|^p_{p,\lambda}\Big) \leq C.$$

Here in the second inequality we used the fact $\|\rho'\|_\infty \geq 1$, and (43). This proves Claim 5, and completes the proofs of Claim 1 and Theorem 6. □

Appendix A. Weighted spaces and a Hardy-type inequality

Let $d \in \mathbb{Z}$. The following lemma is used in section 3.2 in order to define a norm on $\widetilde{\mathcal{X}}_d$. If $d < 0$ then let $\rho_0 \in C^\infty(\mathbb{R}^2, [0,1])$ be such that $\rho_0(z) = 0$ for $|z| \leq 1/2$ and $\rho_0(z) = 1$ for $|z| \geq 1$. In the case $d \geq 0$ we set $\rho_0 := 1$. Recall that $p_d : \mathbb{C} \to \mathbb{C}$, $p_d(z) := z^d$.

12. Lemma. *For every $1 < p < \infty$ and $\lambda > -2/p$ the map*

$$\mathbb{C} \oplus L^{1,p}_{\lambda-d}(\mathbb{R}^2, \mathbb{C}) \to \mathbb{C}\cdot \rho_0 p_d + L^{1,p}_{\lambda-d}(\mathbb{R}^2,\mathbb{C}), \quad (v_\infty, v) \mapsto v_\infty \rho_0 p_d + v$$

is an isomorphism of vector spaces.

Proof of Lemma 12. This follows from a straight-forward argument. □

The following proposition was used in the proof of Theorem 5 and Proposition 9. For every normed vector space V we denote by $C_b(\mathbb{R}^n, V)$ the space of bounded continuous maps from $\mathbb{R}^n$ to V. We denote $B_1^C := \mathbb{R}^n \setminus B_1$.

13. Proposition (Weighted Sobolev spaces)**.** *Let $n \in \mathbb{N}$. Then the following statements hold.*

(i) Let $n < p < \infty$. Then for every $\lambda \in \mathbb{R}$ there exists $C > 0$ such that

$$\|u\langle\cdot\rangle^{\lambda+\frac{n}{p}}\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{L^{1,p}_\lambda(\mathbb{R}^n)}, \quad \forall u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n). \tag{50}$$

If $\lambda > -n/p$ then $L^{1,p}_\lambda(\mathbb{R}^n)$ is compactly contained in $C_b(\mathbb{R}^n)$.

(ii) For every $k \in \mathbb{N}\cup\{0\}$, $1 < p < \infty$ and $\lambda \in \mathbb{R}$ the map $W^{k,p}_\lambda(\mathbb{R}^n) \ni u \mapsto \langle\cdot\rangle^\lambda u \in W^{k,p}(\mathbb{R}^n)$ is a well-defined isomorphism (of normed spaces).

(iii) Let $p > 1$, $\lambda \in \mathbb{R}$, and $f \in L^\infty(\mathbb{R}^n)$ be such that $\|f\|_{L^\infty(\mathbb{R}^n\setminus B_i)} \to 0$, for $i \to \infty$. Then the map $W^{1,p}_\lambda(\mathbb{R}^n) \ni u \mapsto fu \in L^p_\lambda(\mathbb{R}^n)$ is compact.

(iv) For every $1 < p < \infty$, $\lambda \in \mathbb{R}$, $d \in \mathbb{Z}$ and $u \in L^{1,p}_\lambda(B_1^C)$ we have

$$\|p_d u\|_{L^{1,p}_{\lambda-d}(B_1^C)} \leq \max\big\{-d2^{(-d+3)/2}, 2\big\}\|u\|_{L^{1,p}_\lambda(B_1^C)}.$$

Proof of Proposition 13. **Proof of (i):** Inequality (50) follows from inequality (1.11) in Theorem 1.2 in the paper by R. Bartnik [**Ba**]. Assume now that $\lambda > -n/p$. Then it follows from Morrey's embedding theorem that there exists a canonical bounded inclusion $L^{1,p}_\lambda(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$. In order to show that this inclusion is compact, let $u_\nu \in L^{1,p}_\lambda(\mathbb{R}^n)$ be a sequence such that $C := \sup_\nu \|u_\nu\|_{L^{1,p}_\lambda(\mathbb{R}^n)} < \infty$. By Kondrachov's compactness theorem on $\bar{B}_j$ (for $j \in \mathbb{N}$), and a diagonal subsequence argument there exists a subsequence u_{ν_j} of u_ν that converges to some map $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, weakly in $W^{1,p}(B_j)$, and strongly in $C(\bar{B}_j)$, for every $j \in \mathbb{N}$.

1. Claim. *We have $u \in C_b(\mathbb{R}^n)$ and u_{ν_j} converges to u in $C_b(\mathbb{R}^n)$.*

Proof of Claim 1. We choose a constant C' as in the first part of (i). For every $R > 0$ we have $\|u\|_{L^{1,p}_\lambda(B_R)} \leq \limsup_j \|u_{\nu_j}\|_{L^{1,p}_\lambda(B_R)} \leq C$. Hence $u \in L^{1,p}_\lambda(\mathbb{R}^n)$. Since $\lambda > -n/p$, by inequality (50), this implies $u \in C_b(\mathbb{R}^n)$. To see the second statement, we choose a smooth function $\rho : \mathbb{R}^n \to [0,1]$ such that $\rho(x) = 0$ for $x \in B_1$, $\rho(x) = 1$ for $x \in B_3^C$, and $|D\rho| \leq 1$. Let $R \geq 1$. We define $\rho_R := \rho(\cdot/R) : \mathbb{R}^n \to [0,1]$. Let $j \in \mathbb{N}$. Abbreviating $v_j := u_{\nu_j} - u$, we have

$$\|v_j\|_\infty \leq \big\|v_j(1-\rho_R)\big\|_\infty + \big\|v_j\rho_R\big\|_\infty. \tag{51}$$

Inequality (50) implies that

$$\big\|v_j\rho_R\big\|_\infty \leq C' R^{-\lambda-\frac{n}{p}} \|v_j\rho_R\|_{1,p,\lambda}. \tag{52}$$

Furthermore, $\|v_j\rho_R\|_{1,p,\lambda} \leq 2\|v_j\|_{1,p,\lambda} \leq 4C$. Combining this with (51) and (52), and the fact $\lim_{j\to\infty} \|v_j\|_{L^\infty(B_{3R})} = 0$, it follows that $\limsup_{j\to\infty} \|v_j\|_\infty \leq 4CC'R^{-\lambda-\frac{n}{p}}$. Since $\lambda > -n/p$ and $R \geq 1$ is arbitrary, it follows that u_{ν_j} converges to u in $C_b(\mathbb{R}^n)$. This proves Claim 1 and completes the proof of statement **(i)**.

Statement (ii) follows from a straight-forward calculation.

Proof of (iii): Let $f \in L^\infty(\mathbb{R}^n)$ be as in the hypothesis. Let $u_\nu \in W^{1,p}_\lambda(\mathbb{R}^n)$ be a sequence such that $C := \sup_\nu \|u_\nu\|_{W^{1,p}_\lambda(\mathbb{R}^n)} < \infty$. By Kondrachov's theorem on $\bar{B}_j$ (for $j \in \mathbb{N}$) and a diagonal subsequence argument there exists a subsequence (ν_j) and a map $v \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, such that fu_{ν_j} converges to v, strongly in $L^p(K)$, as $j \to \infty$, for every compact subset $K \subseteq \mathbb{R}^n$. Standard arguments show that $v \in L^p_\lambda(\mathbb{R}^n)$ and fu_{ν_j} converges to v in $L^p_\lambda(\mathbb{R}^n)$. This proves **(iii)**.

Statement (iv) follows from a straight-forward calculation. This completes the proof of Proposition 13. □

The next result was used in the proof of Proposition 9 and will be used in the proof of Lemma 30.

14. Proposition (Hardy-type inequality)**.** *Let $n \in \mathbb{N}$, $p > n$, $\lambda > -n/p$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$ be such that $\|Du| \cdot |^{\lambda+1}\|_{L^p(\mathbb{R}^n)} < \infty$. Then $u(rx)$ converges to some $y_\infty \in \mathbb{R}$, uniformly in $x \in S^{n-1}$, as $r \to \infty$, and*

$$\|(u - y_\infty)| \cdot |^\lambda\|_{L^p(\mathbb{R}^n)} \leq p/(\lambda + n/p)\|Du| \cdot |^{\lambda+1}\|_{L^p(\mathbb{R}^n)}. \tag{53}$$

For the proof of Proposition 14, we need the following.

15. Lemma (Hardy's inequality). *Let $n \in \mathbb{N}$, $1 < p < \infty$, $\lambda > -n/p$ and $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$. If there exists $R > 0$ such that $u|_{B_R^C} = 0$ then* $\| |u| \cdot |\cdot|^\lambda \|_{L^p(\mathbb{R}^n)} \le p/(\lambda + n/p) \| |Du| \cdot |\cdot|^{\lambda+1} \|_{L^p(\mathbb{R}^n)}$ $(\in [0,\infty])$.

Proof of Lemma 15. If u is smooth then the stated inequality follows from Exercise 21, Chapter 6, in the book by O. Kavian [**Ka**]. The general case can be reduced to this case by mollifying the function u. This proves the lemma. □

Proof of Proposition 14. Let n, p, λ as in the hypothesis. We set $\varepsilon := \lambda + \frac{n}{p}$.

1. Claim. *There exists a constant C_1 such that for every weakly differentiable map $u : \mathbb{R}^n \to \mathbb{R}$ and $x, y \in \mathbb{R}^n$ satisfying $0 < |x| \le |y|$, we have* $|u(x) - u(y)| \le C_1 |x|^{-\varepsilon} \big\| |Du| \cdot |\cdot|^{\lambda+1} \big\|_{L^p(B^C_{|x|})}$.

Proof of Claim 1. By Morrey's theorem there is a constant C such that $|u(0) - u(x)| \le C r^{1-n/p} \|Du\|_{L^p(B_r)}$, for every $r > 0$, weakly differentiable $u : B_r \to \mathbb{R}$ and $x \in B_r$. Let u, x and y be as in the hypothesis of the claim. Let $N \in \mathbb{N}$ be such that $2^{N-1}|x| \le |y| \le 2^N |x|$. For $i = 0, \ldots, N$ we define $x_i := 2^i x \in \mathbb{R}^n$. Furthermore, we set $x_{N+7} := y|x_{N+7}|/|y|$ and $x_{N+8} := y$, and we choose points $x_i \in S^{n-1}_{2^N|x|}$, $i = N+1, \ldots, N+6$, such that $|x_i - x_{i-1}| \le 2^{N-1}|x|$, for $i = N+1, \ldots, N+7$. For $i = 0, \ldots, N-1$ we have $x_i \in \bar{B}_{2^i|x|}(x_{i+1})$. Hence it follows from the statement of Morrey's theorem that $|u(x_{i+1}) - u(x_i)| \le C(2^i|x|)^{-\varepsilon} \big\| |Du| \cdot |\cdot|^{\lambda+1} \big\|_{L^p(B^C_{|x|})}$. Moreover, for $i = N, \ldots, N+7$ we have $x_{i+1} \in \bar{B}_{2^{N-1}|x|}(x_i)$, and hence analogously, $|u(x_{i+1}) - u(x_i)| \le C(2^{N-1}|x|)^{-\varepsilon} \| |Du| \cdot |\cdot|^{\lambda+1} \|_{L^p(B^C_{|x|})}$. Using the inequality $|u(y) - u(x)| \le \sum_{i=0,\ldots,N+7} |u(x_{i+1}) - u(x_i)|$, Claim 1 follows. □

Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R})$ be such that $\| |Du| \cdot |\cdot|^{\lambda+1} \|_{L^p(\mathbb{R}^n)} < \infty$. It follows from Claim 1 that there exists $y_\infty \in \mathbb{R}$ such that $u(rx)$ converges to y_∞, as $r \to \infty$, uniformly in $x \in S^{n-1}$. To prove inequality (53), we choose a smooth map $\rho : [0,\infty) \to [0,1]$ such that $\rho(t) = 1$ for $0 \le t \le 1$, $\rho(t) = 0$ for $t \ge 2$ and $|\rho'(t)| \le 2$. We fix a number $R > 0$ and define $\rho_R : \mathbb{R} \to [0,1]$ by $\rho_R(x) := \rho(|x|/R)$. We abbreviate $v := u - y_\infty$. Using Lemma 15 with u replaced by $\rho_R v$, we have

$$\| |v| \cdot |\cdot|^\lambda \|_{L^p(B_R)} \le \big\| |\rho_R v| \cdot |\cdot|^\lambda \big\|_{L^p(\mathbb{R}^n)} \le p/(\lambda + n/p) \big\| |D(\rho_R v)| \cdot |\cdot|^{\lambda+1} \big\|_{L^p(\mathbb{R}^n)}.$$

Combining this with a calculation using Leibnitz' rule, it follows that

$$(54)\quad \| |v| \cdot |\cdot|^\lambda \|_{L^p(B_R)} \le p/(\lambda + n/p) \big(4 \| |v| \cdot |\cdot|^\lambda \|_{L^p(B_{2R} \setminus B_R)} + \big\| |Du| \cdot |\cdot|^{\lambda+1} \big\|_{L^p(\mathbb{R}^n)} \big).$$

Claim 1 implies that $|v(x)| \leq C_1|x|^{-\varepsilon}\|Du|\cdot|^{\lambda+1}\|_{L^p(B_R^C)}$, for $x \in B_R^C$. Using the equalities $\int_{B_{2R}\setminus B_R}|x|^{-n}dx = \log 2|S^{n-1}|$ and $\varepsilon = \lambda + n/p$, it follows that

$$\|v|\cdot|^\lambda\|^p_{L^p(B_{2R}\setminus B_R)} \leq C_1^p \log 2|S^{n-1}|\|Du|\cdot|^{\lambda+1}\|^p_{L^p(B_R^C)}.$$

Inequality (53) follows by inserting this into (54) and sending R to ∞. This proves Proposition 14. □

The next result will be used to prove Corollary 18 below, which in turn is used in the proof of Theorem 5. For every $d \in \mathbb{Z}$ we define P_d and $\bar{P}_d$ to be the spaces of polynomials in $z \in \mathbb{C}$ and $\bar{z}$ of degree *less than* d. (Note that if $d \leq 0$ then $P_d = \{0\}$.) We abbreviate $L^{1,p}_\lambda := L^{1,p}_\lambda(\mathbb{C},\mathbb{C})$, $L^p_\lambda := L^p_\lambda(\mathbb{C},\mathbb{C})$, $\partial_{\bar{z}} := \partial^{\mathbb{C}}_{\bar{z}}$ and $\partial_z := \partial^{\mathbb{C}}_z$. Let X be a normed vector space and $Y \subseteq X$ be a closed subspace. We denote by X^* the dual space of X and equip X/Y with the quotient norm.

16. Proposition (Fredholm property for $\partial_{\bar{z}}$)**.** *For every $d \in \mathbb{Z}$, $1 < p < \infty$ and $-2/p + 1 < \lambda < -2/p + 2$ the following conditions hold.*

(i) The operator $T := \partial_{\bar{z}} : L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}$ is Fredholm.

(ii) We have $\ker T = P_d$.

(iii) The map $\bar{P}_{-d} \to \big(L^p_{\lambda-d}/\mathrm{im}T\big)^*$, $u \mapsto \big(v + \mathrm{im}T \mapsto \int_{\mathbb{C}} uv\, ds\, dt\big)$ is well-defined and an isometric isomorphism. Here we equip $L^p_{\lambda-d}/\mathrm{im}T$ with the quotient norm.

17. Remark. *For every normed vector space X and every closed subspace $Y \subseteq X$ the map $\big\{\varphi \in X^* \,|\, \varphi(x) = 0,\ \forall x \in Y\big\} \to (X/Y)^*$, $\varphi \mapsto \big(x + Y \mapsto \varphi(x)\big)$, is well-defined and an isometric isomorphism. This follows from a straight-forward argument.*

We denote by $\mathcal{S}$ the space of Schwartz functions on $\mathbb{C}$ and by $\mathcal{S}'$ the space of temperate distributions. By $\widehat{\ }: \mathcal{S}' \to \mathcal{S}'$ we denote the Fourier transform, and by $^\vee : \mathcal{S}' \to \mathcal{S}'$ the inverse transformation.

Proof of Proposition 16*.* Let d, p, λ and T be as in the hypothesis. **We start by proving (ii).** A calculation in polar coordinates shows that for every polynomial u in z we have

$$u \in L^{1,p}_{\lambda-1-d} \iff \deg u < d - \lambda + 1 - 2/p. \tag{55}$$

Hence our assumption $\lambda < -2/p + 2$ implies that $\ker T \supseteq P_d$.

We prove that $\ker T \subseteq P_d$**.** Let $u \in \ker T$. Then $0 = \widehat{\partial_{\bar{z}} u}(\zeta) = \frac{i}{2}\zeta\widehat{u}$ (as temperate distributions). It follows that the support of $\widehat{u}$ is either empty or consists of the point $0 \in \mathbb{C}$. Hence the Paley-Wiener theorem implies that u is real analytic in the variables s and t, where $z = s + it$, and there exists $N \in \mathbb{N}$ such that $\sup_{z\in\mathbb{C}} |u(z)|\langle z\rangle^N < \infty$. (See for example Theorem IX.12 in Vol. I of the book [**ReSi**].) Therefore, by Liouville's Theorem u is a polynomial in the variable z. Since by our assumption $\lambda > -2/p+1$, it follows from (55) that $u \in P_d$. This proves that $\ker T \subseteq P_d$ and completes the proof of (ii).

We prove (i) and (iii). We define $p' := p/(p-1)$. We identify the spaces $L^{p'}_{-\lambda+d}$ and $(L^p_{\lambda-d})^*$ via the isometric isomorphism $u \mapsto \big(v \mapsto \int_{\mathbb{C}} uv\big)$. Then the adjoint operator T^* is given by $T^* = \partial_z : L^{p'}_{-\lambda+d} \cong (L^p_{\lambda-d})^* \to (L^{1,p}_{\lambda-1-d})^*$, where the derivatives are taken in the sense of distributions.

1. Claim. $\ker T^* = \bar{P}_{-d}$.

Proof of Claim 1. For every polynomial u in $\bar{z}$ we have

$$u \in L^{p'}_{-\lambda+d} \iff \deg u < -d + \lambda - 2/p' = -d + \lambda - 2 + 2/p. \tag{56}$$

Our assumption $\lambda > -2/p+1$ and (56) imply that $\ker T^* \supseteq \bar{P}_{-d}$. Furthermore, the inclusion $\ker T^* \subseteq \bar{P}_{-d}$ is proved analogously to the inclusion $\ker T \subseteq P_d$, using $\lambda < -2/p+2$ and (56). This proves Claim 1. □

We apply now Theorem 4.3 in the paper by R. B. Lockhart [**Lo2**] with T (case $d \leq 0$) or T^* (case $d > 0$). The hypotheses of that theorem are satisfied, since by our assumption $-2/p+1 < \lambda < -2/p+2$, and since the operator $T = \partial_{\bar{z}}$ (T^*) has constant coefficients and is elliptic, in the sense that its principal symbol $\sigma_T : \mathbb{C} \to \mathbb{C}$, $\sigma_T(\zeta) = \frac{1}{2}(\zeta_1 + i\zeta_2)$ ($\sigma_{T^*}(\zeta) = \frac{1}{2}(\zeta_1 - i\zeta_2)$) does not vanish on $S^1 \subseteq \mathbb{C}$. Hence that theorem implies that in the case $d \leq 0$ the map T is Fredholm, and in the case $d > 0$ the operator T^* is Fredholm. It follows that $\mathrm{im} T$ is closed if $d \leq 0$. On the other hand, if $d > 0$ then $\mathrm{im} T^*$ is closed, hence the same holds for $\mathrm{im} T$. Statements (iii) and (i) follow now from statement (ii), Remark 17 applied with $X := L^p_{\lambda-d}$ and $Y := \mathrm{im} T$, and Claim 1. This proves Proposition 16. □

Let $d \in \mathbb{Z}$, $1 < p < \infty$, $-2/p+1 < \lambda < -2/p+2$, and $\rho_0 : \mathbb{R}^2 \to [0,1]$ be a smooth function that vanishes on $B_{1/2}$ and equals 1 on B_1^C. We equip $\mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d}$ with the norm induced by the isomorphism of Lemma 12. This norm is complete. (See e.g. [**Lo1**].)

18. Corollary. *The following map is Fredholm, with real index* $2+2d$*:*

$$\partial_{\bar{z}} : \mathbb{C}\rho_0 p_d + L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}\,. \tag{57}$$

Proof of Corollary 18. The composition of the isomorphism of Lemma 12 with (57) is given by $T + S : \mathbb{C} \oplus L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}$, where $T(x_\infty, u) := \partial_{\bar{z}} u$ and $S(x_\infty, u) := x_\infty (\partial_{\bar{z}} \rho_0) p_d$. The map T is the composition of the canonical projection $\mathrm{pr} : \mathbb{C} \oplus L^{1,p}_{\lambda-1-d} \to L^{1,p}_{\lambda-1-d}$ with the operator $\partial_{\bar{z}} : L^{1,p}_{\lambda-1-d} \to L^p_{\lambda-d}$. Using Proposition 16, it follows that T is Fredholm of real index $2+2d$. Furthermore, S is compact, since it equals the composition of the canonical projection $\mathbb{C} \oplus L^{1,p}_{\lambda-1-d} \to \mathbb{C}$ (which is compact) with a bounded operator. Corollary 18 follows. □

The next result is used in the proof of Theorem 5. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional hermitian vector space, $A, B : V \to V$ positive linear maps, $\lambda \in \mathbb{R}$ and $1 < p < \infty$. We define

$$T_\lambda := \begin{pmatrix} \partial_{\bar z} & A \\ B & \partial_z \end{pmatrix} : W^{1,p}_\lambda(\mathbb{C}, V \oplus V) \to L^p_\lambda(\mathbb{C}, V \oplus V).$$

19. Proposition. *The operator T_λ is Fredholm of index 0.*

For the proof of Proposition 19 we need the following result.

20. Proposition. *Let $(V, \langle\cdot,\cdot\rangle), p$ and A be as above, and $n \in \mathbb{N}$. Then the map $-\Delta + A : W^{2,p}(\mathbb{R}^n, V) \to L^p(\mathbb{R}^n, V)$ is an isomorphism (of Banach spaces).*

Proof of Proposition 20. Consider first the case $\dim_{\mathbb{C}} V = 1$ and $A = 1$. We define $G := (2\pi)^{\frac{n}{2}}\big(\langle\cdot\rangle^{-2}\big)^\vee \in \mathcal{S}'$. The map $\mathcal{S} \ni u \mapsto G*u \in \mathcal{S}$ is well-defined. By Calderón's Theorem this map extends uniquely to an isomorphism

$$L^p(\mathbb{R}^n, \mathbb{C}) \ni u \mapsto G * u \in W^{2,p}(\mathbb{R}^n, \mathbb{C}). \tag{58}$$

(See Theorem 1.2.3. in the book [**Ad**].) Note that $(-\Delta + 1)(G * u) = \big(\langle\cdot\rangle^2 (G * u)\widehat{\ }\big)^\vee = u$, for every $u \in \mathcal{S}$. It follows that the inverse of (58) is given by $-\Delta + 1 : W^{2,p}(\mathbb{R}^n, \mathbb{C}) \to L^p(\mathbb{R}^n, \mathbb{C})$. Hence this is an isomorphism.

The general case can be reduced to the above case by diagonalizing the map A. This proves Proposition 20. □

Proof of Proposition 19. We abbreviate $L^p := L^p(\mathbb{C}, V \oplus V)$, etc.

Assume first that $\lambda = 0$. We denote by $A^{1/2}, B^{1/2} : V \to V$ the unique positive linear maps satisfying $(A^{\frac12})^2 = A$, $(B^{\frac12})^2 = B$. We define

$$L := \begin{pmatrix} \partial_{\bar z} & A^{\frac12} B^{\frac12} \\ B^{\frac12} A^{\frac12} & \partial_z \end{pmatrix} : W^{1,p} \to L^p.$$

A short calculation shows that

$$T_0 = \big(A^{\frac12} \oplus B^{\frac12}\big) L \big(A^{-\frac12} \oplus B^{-\frac12}\big). \tag{59}$$

1. Claim. *The operator L is an isomorphism.*

Proof of Claim 1. We define

$$L' := \begin{pmatrix} -\partial_z & A^{\frac12} B^{\frac12} \\ B^{\frac12} A^{\frac12} & -\partial_{\bar z} \end{pmatrix} : W^{2,p} \to W^{1,p}.$$

By a short calculation we have $LL' = \big(-\Delta/4 + A^{\frac12} B A^{\frac12}\big) \oplus \big(-\Delta/4 + B^{\frac12} A B^{\frac12}\big) : W^{2,p} \to L^p$. Since the linear maps $A^{\frac12} B A^{\frac12}, B^{\frac12} A B^{\frac12} : V \to V$ are positive, Proposition 20 implies that LL' is an isomorphism. We denote by $(LL')^{-1} : L^p \to W^{2,p}$ its inverse and define $R := L'(LL')^{-1} : L^p \to W^{1,p}$. Then R is bounded and $LR = \mathrm{id}_{L^p}$.

By a short calculation, we have $LL'(u,v) = L'L(u,v)$, for every Schwartz function $(u,v) \in \mathcal{S}$. This implies that $(LL')^{-1}L|_{\mathcal{S}} = L(LL')^{-1}|_{\mathcal{S}}$, and therefore $RL|_{\mathcal{S}} = \mathrm{id}_{\mathcal{S}}$. Since $RL : W^{1,p} \to W^{1,p}$ is continuous and $\mathcal{S} \subseteq W^{1,p}$ is dense, it follows that $RL = \mathrm{id}_{W^{1,p}}$. Claim 1 follows. □

The maps $A^{\frac{1}{2}} \oplus B^{\frac{1}{2}} : L^p \to L^p$ and $A^{-\frac{1}{2}} \oplus B^{-\frac{1}{2}} : W^{1,p} \to W^{1,p}$ are automorphisms. Therefore, (59) and Claim 1 imply that T is an isomorphism.

Consider now the **general case** $\lambda \in \mathbb{R}$. The map $L^p \ni (u,v) \mapsto \langle\cdot\rangle^{-\lambda}(u,v) \in L^p_\lambda$ is an isometric isomorphism. Furthermore, by Proposition 13(ii) the map $W^{1,p}_\lambda \ni (u,v) \mapsto \langle\cdot\rangle^{\lambda}(u,v) \in W^{1,p}$ is well-defined and an isomorphism. We define $S := \langle\cdot\rangle^{\lambda}(\partial_{\bar z}\langle\cdot\rangle^{-\lambda}) \oplus \langle\cdot\rangle^{\lambda}(\partial_z\langle\cdot\rangle^{-\lambda}) : W^{1,p} \to L^p$. Direct calculations show that $T_\lambda = \langle\cdot\rangle^{-\lambda}(T_0 + S)\langle\cdot\rangle^{\lambda}$, $|\partial_{\bar z}\langle\cdot\rangle^{-\lambda}| \le |\lambda|\langle\cdot\rangle^{-\lambda-1}/2$ and $|\partial_z\langle\cdot\rangle^{-\lambda}| \le |\lambda|\langle\cdot\rangle^{-\lambda-1}/2$. Therefore, Proposition 13(iii) implies that the operator S is compact. We proved that T_0 is an isomorphism. It follows that T_λ is a Fredholm map of index 0. This proves Proposition 19 in the general case. □

Appendix B. Proof of Proposition 11 (Right inverse for d_A^*)

For the proof of Proposition 11 we need the following three results. Let $n \in \mathbb{N}$, $1 \le p \le \infty$, G a compact Lie group with Lie algebra $\mathfrak{g}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g}$. For a Riemannian manifold X and a G-bundle $P \to X$ we denote by $\mathcal{A}^{1,p}(P)$ the space of $W^{1,p}$-connections on P.

21. Proposition. *Let $K \subseteq \mathbb{R}^n$ be a compact subset diffeomorphic to $\bar B_1$. If $n/2 < p < \infty$ then there exist constants $\varepsilon > 0$ and C such that for every principal G-bundle $P \to \mathrm{int}K$, $A_0 \in \mathcal{A}(P)$, and $A \in \mathcal{A}^{1,p}(P)$ the following holds. If A_0 is flat and $\|F_A\|_p \le \varepsilon$ then there exists a gauge transformation $g \in \mathcal{G}^{2,p}(P)$ such that $\|g^*A - A_0\|_{1,p,A_0} \le C\|F_A\|_p$.*

Proof of Proposition 21. Let $n, p, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and K be as in the hypothesis. We denote by $\widetilde A_0$ the trivial connection on $\mathrm{int}K \times G$. By Uhlenbeck's gauge theorem there exist constants $\varepsilon > 0$ and C such that for every connection $\widetilde A \in \mathcal{A}^{1,p}(\mathrm{int}K \times G)$ satisfying $\|F_{\widetilde A}\|_p \le \varepsilon$ there exists $\widetilde g \in \mathcal{G}^{2,p}(\mathrm{int}K \times G)$ such that $\|\widetilde g^*\widetilde A - \widetilde A_0\|_{1,p,\widetilde A_0} \le C\|F_{\widetilde A}\|_p$. (This follows for example from Theorem 6.3 in [**Weh**].) Let $P \to \mathrm{int}K$ be a G-bundle, $A_0 \in \mathcal{A}(P)$ be flat, and $A \in \mathcal{A}^{1,p}(P)$ be such that $\|F_A\|_p \le \varepsilon$. Since A_0 and $\widetilde A_0$ are flat, there exists a smooth isomorphism of G-bundles $\Psi : \mathrm{int}K \times G \to P$ (with fixed base) such that $\Psi^*A_0 = \widetilde A_0$. We choose $\widetilde g \in \mathcal{G}^{2,p}(\mathrm{int}K \times G)$ as in the conclusion of Uhlenbeck's theorem with $\widetilde A := \Psi^*A$. We define $g := \widetilde g \circ \Psi^{-1} \in \mathcal{G}^{2,p}(P)$. A straight-forward calculation shows that $\|g^*A - A_0\|_{1,p,A_0} = \|\widetilde g^*\widetilde A - \widetilde A_0\|_{1,p,\widetilde A_0}$. The statement of Proposition 21 follows from this. □

22. Proposition. *Let $n \in \mathbb{N}$, $1 < p < \infty$, G a compact Lie group with Lie algebra $\mathfrak{g}$, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g}$, and $K \subseteq \mathbb{R}^n$ a compact subset diffeomorphic to $\bar{B}_1$. Then there exists a constant C such that for every principal G-bundle $P \to \mathrm{int}K$ and every smooth flat connection A on P there exists right inverse R of $d_A^* d_A : W_A^{2,p}(\mathfrak{g}_P) \to L^p(\mathfrak{g}_P)$ satisfying $\|R\| := \big\{ \|R\xi\|_{2,p,A} \,\big|\, \xi \in L^p(\mathfrak{g}_P) : \|\xi\|_p \leq 1 \big\} \leq C$.*

Proof of Proposition 22. Let $n \in \mathbb{N}$ and $1 < p < \infty$. For an open subset $U \subseteq \mathbb{R}^n$ we denote by $C_0^\infty(U)$ the compactly supported smooth functions on U. We define the map $\widetilde{T} : C_0^\infty(B_1) \to C^\infty(B_1)$ as follows. We denote by $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ the fundamental solution of the Laplace equation (see e.g. [**Ev**], p. 22). Let $f \in C_0^\infty(B_1)$. We define $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ to be the extension of f by 0 outside B_1. We denote by $*$ convolution in $\mathbb{R}^n$ and define $\widetilde{T}f := (\Phi * \widetilde{f})|_{B_1}$. Note that Φ is locally integrable, hence the convolution is well-defined. Furthermore, $\widetilde{T}f$ is smooth, and $\Delta \widetilde{T} f = f$. (The first assertion follows from differentiation under the integral, and for the second see for example Theorem 1 on p. 23 in the book [**Ev**].)

1. Claim. *There exists a constant C such that $\|\widetilde{T}f\|_{W^{2,p}(B_1)} \leq C\|f\|_{L^p(B_1)}$, for every $f \in C_0^\infty(B_1)$.*

Proof of Claim 1. Young's inequality states that $\|\widetilde{T}f\|_{L^p(B_1)} \leq \|\Phi\|_{L^1(B_2)}\|f\|_{L^p(B_1)}$, for every $f \in C_0^\infty(B_1)$. Furthermore, the Calderón-Zygmund inequality states that there exists a constant C such that for every $f \in C_0^\infty(\mathbb{R}^n)$ we have $\|D^2(\Phi * f)\|_p \leq C\|f\|_p$. (See for example Theorem B.2.7 in [**MS2**]. Note that $\Phi_j * f = (\partial_j \Phi) * f = \partial_j(\Phi * f)$.) Claim 1 follows from this. □

We fix a constant C as in Claim 1. By this claim the map $\widetilde{T}$ uniquely extends to a bounded linear map $T : L^p(B_1) \to W^{2,p}(B_1)$. Since $\Delta\widetilde{T}f = f$, for every $f \in C_0^\infty(B_R)$, a density argument shows that $\Delta T f = f$, for every $f \in W^{2,p}(B_1)$, i.e. T is a right inverse for $\Delta : W^{2,p}(B_R) \to L^p(B_R)$.

Let now $G, \langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and K be as in the hypothesis. Without loss of generality we may assume that $K = \bar{B}_1$. Let $\pi : P \to B_1$ be a G-bundle, and $A \in \mathcal{A}(P)$ be flat. We fix a point $p_0 \in P_0$, and denote by $\sigma : B_1 \to P$ the A-horizontal section through p_0. This is the unique smooth section of P satisfying $A\, d\sigma = 0$ and $\sigma(0) = p_0$. For $k \geq 0$ we define the map $\Psi_k : W^{k,p}(B_1, \mathfrak{g}) \to W_A^{k,p}(B_1, \mathfrak{g}_P)$ by $\Psi_k \xi := G \cdot (\sigma, \xi)$. This is an isometric isomorphism. We define $R := -\Psi_2 T \Psi_0^{-1}$. It follows that $\big\|R\big\| \leq \|\Psi_2\|\|T\|\|\Psi_0^{-1}\| = \|T\| \leq C$, where we use various operator norms. A straight-forward calculation shows that $d_A^* d_A \Psi_2 = \Psi_0 d^* d = -\Psi_0 \Delta : W^{2,p}(B_1, \mathfrak{g}) \to L^p(B_1, \mathfrak{g}_P)$. It follows that R is a right inverse for $d_A^* d_A$. This proves Proposition 22. □

23. Lemma. *Let $n \in \mathbb{N}$, $p > n$, G a compact Lie group with Lie algebra $\mathfrak{g}$, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g}$ and $K \subseteq \mathbb{R}^n$ a compact subset* *diffeomorphic to* $\bar{B}_1$*. Then there exist constants* C *and* $\varepsilon > 0$ *such that for every principal* G*-bundle* $\pi : P \to \mathrm{int}K$, $A \in \mathcal{A}^{1,p}(P)$, $k = 0,1$ *and* $\alpha \in W^{1,p}_A\big(\bigwedge^k(\mathfrak{g}_P)\big)$ *the following holds. If* $\|F_A\|_{L^p(\mathrm{int}K)} \leq \varepsilon$ *then* $\|\alpha\|_{L^\infty(\mathrm{int}K)} \leq C\|\alpha\|_{1,p,A}$.

Proof of Lemma 23. Let $n, p, G, \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and K be as in the hypothesis. We choose constants $\varepsilon > 0$ and $C_1 := C$ as in Proposition 21. Since by assumption $p > n$, it follows from Morrey's theorem that there exists a constant C_2 with the following property. If $P \to \mathrm{int}K$ is a G-bundle, $A_0 \in \mathcal{A}(P)$ is flat, $k = 0,1$ and $\alpha \in W^{1,p}_{A_0}\big(\bigwedge^k(\mathfrak{g}_P)\big)$ then $\|\alpha\|_\infty \leq C_2\|\alpha\|_{W^{1,p}_{A_0}}$. Let $\pi : P \to \mathrm{int}K$ be a G-bundle and $A \in \mathcal{A}^{1,p}(P)$ be such that $\|F_A\|_{L^p(\mathrm{int}K)} \leq \varepsilon$. We choose $g \in \mathcal{G}^{2,p}(P)$ as in Proposition 21. Let $\alpha \in W^{1,p}_A\big(\bigwedge^k(\mathfrak{g}_P)\big)$. We set $A' := g^*A$, $\alpha' := g^*\alpha$ and $C_3 := \max\big\{[\xi,\eta]\,|\,\xi,\eta \in \mathfrak{g} : |\xi| \leq 1, |\eta| \leq 1\big\}$. A direct calculation shows that $(\nabla^{A_0} - \nabla^{A'})\alpha' = [(A' - A_0)\otimes\alpha']$. It follows that

$$\|\alpha\|_\infty \leq \|\alpha'\|_\infty \leq C_2\|\alpha'\|_{1,p,A_0} \leq C_2\big(\|\alpha'\|_{1,p,A'} + C_3\|A' - A_0\|_\infty\|\alpha'\|_p\big). \tag{60}$$

By the statement of Proposition 21, we have $\|A' - A_0\|_{1,p,A_0} \leq C_1\varepsilon$. Combining this with Morrey's theorem and (60), Lemma 23 follows. □

Proof of Proposition 11. Let $n, p, G, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, K, P$ and A be as in the hypothesis. We show that the operator (42) admits a bounded right inverse.

1. Claim. *There exists a bounded linear map* $L : W^{1,p}_A(\mathrm{int}K, \mathfrak{g}_P) \to W^{1,p}_A\big(\bigwedge^1(\mathfrak{g}_P)\big)$ *such that* $d_A^*L = \mathrm{id}$.

Proof of Claim 1. We may assume without loss of generality that $K = \bar{B}_1$. We define $\Omega \subseteq \mathbb{R}\times P$ to be the subset consisting of all (t,p) such that $|t + x_1|^2 + x_2^2 + \cdots + x_n^2 < 1$, where $x := \pi(p) \in B_1$. Furthermore, we denote by $\Psi : \Omega \to P$ the A-parallel transport in x_1-direction. Let $(t_0, p_0) \in \Omega$. We denote $x_0 := \pi(p_0) \in \bar{B}_1$. Then $\Psi(t_0,p_0) = p(t_0)$, where $p : \{t \in \mathbb{R}\,|\,(t,x_0) \in \Omega\} \to P$ is the unique path satisfying $\mathrm{pr}\circ p(t) = x_0 + (t, 0, \ldots 0)$, for every $t \in \mathbb{R}$, $A\dot{p} = 0$ and $p(0) = p_0$. Let $\xi \in W^{1,p}(B_1, \mathfrak{g}_P)$. We define $\widetilde{\xi} : P \to \mathfrak{g}$ by the condition $[p, \widetilde{\xi}(p)] = \xi\circ\pi(p)$, for $p \in P$. Let $p \in B_1$ and $(x_1, \ldots, x_n) := \pi(p)$. We define $\widetilde{\eta} := \int_{-x_1}^0 \widetilde{\xi}\circ\Psi(t,p)dt \in \mathfrak{g}$. Furthermore, we define the section $\eta : B_1 \to \mathfrak{g}_P$ by the condition $\eta\circ\pi(p) = [p, \widetilde{\eta}(p)]$, for every $p \in P$, and $L\xi := \eta\, dx^1$. Then L has the required properties. This proves Claim 1. □

We choose a map L as in Claim 1. Furthermore, we choose a smooth flat connection A_0 on P. By Proposition 22 there exists a bounded right inverse R_0 of $d_{A_0}^*d_{A_0} : W^{2,p}_{A_0}(\mathfrak{g}_P) \to L^p(\mathfrak{g}_P)$. We define the map

$$R := d_AR_0 + L\big(\mathrm{id} - d_A^*d_AR_0\big) : L^p(\mathfrak{g}_P) \to W^{1,p}_A\Big(\bigwedge^1(\mathfrak{g}_P)\Big). \tag{61}$$

The first assertion of Proposition 11 is now a consequence of the following.

2. Claim. *The map* (61) *is well-defined and bounded, and* $d_A^* R = \mathrm{id}$.

Proof of Claim 2. A short calculation shows that $S := d_A^* d_A - d_{A_0}^* d_{A_0}$ is of first or zeroth order. Hence it defines a bounded linear map from $W_A^{2,p}(\mathrm{int}K, \mathfrak{g}_P)$ to $W_A^{1,p}(\mathrm{int}K, \mathfrak{g}_P)$. Furthermore $\mathrm{id} - d_A^* d_A R_0 = -SR_0$. This implies that R is well-defined and bounded. A short calculation shows that $d_A^* R = \mathrm{id}$. This proves Claim 2. □

To prove the second statement of Proposition 11, we choose ε_1, C_1 as in Lemma 23 (corresponding to ε, C) and ε_2, C_2 as in Proposition 21 (corresponding to ε, C). We also fix a constant $C_3 := C$ as in Proposition 22, and we define $\varepsilon := \min\big\{\varepsilon_1, \varepsilon_2, 1/(2C_1C_2C_3)\big\}$. Let $P \to \mathrm{int}K$ be a G-bundle and $A \in \mathcal{A}(P)$ be such that $\|F_A\|_p \leq \varepsilon$. We choose a flat smooth connection A_0 on P. By the assertion of Proposition 21 there exists $g \in \mathcal{G}^{2,p}(P)$ such that

$$\|g^*A - A_0\|_{1,p,A_0} \leq C_2 \|F_A\|_p. \tag{62}$$

Furthermore, by the assertion of Proposition 22 there exists a right inverse R_0 of $d_{A_0}^* d_{A_0} : W_{A_0}^{2,p}(\mathfrak{g}_P) \to L^p(\mathfrak{g}_P)$ satisfying $\|R_0\| \leq C_3$, where $\|R_0\|$ denotes the operator norm of R_0. We define the map $S : W_{A_0}^{1,p}\big(\bigwedge^1(\mathfrak{g}_P)\big) \to L^p(\mathfrak{g}_P)$ by $S\alpha := -*\big[(g^*A - A_0) \wedge *\alpha\big]$. Since A_0 is flat, the hypotheses of Lemma 23 are satisfied. Hence by this lemma and (62), we have

$$\|S\alpha\|_p \leq \|g^*A - A_0\|_p \|\alpha\|_\infty \leq C_2 C_1 \varepsilon \|\alpha\|_{1,p,A_0}(\mathfrak{g}_P),$$

for every $\alpha \in W_{A_0}^{1,p}\big(\bigwedge^1(\mathfrak{g}_P)\big)$. Hence S is well-defined, and

$$\|S d_{A_0} R_0\| \leq \|S\| \|d_{A_0}\| \|R_0\| \leq C_2 C_1 \varepsilon C_3 \leq 1/2,$$

Hence $\mathrm{id} + S d_{A_0} R_0 : L^p(\mathfrak{g}_P) \to L^p(\mathfrak{g}_P)$ is invertible, and the von Neumann series $\sum_{i=0}^\infty (-S d_{A_0} R_0)^i$ converges in the operator norm and equals $(\mathrm{id} + S d_{A_0} R_0)^{-1}$. Furthermore, $\big\|\big(\mathrm{id} + S d_{A_0} R_0\big)^{-1}\big\| \leq \sum_{i=0}^\infty 2^{-i} = 2$. We define $R := g_* d_{A_0} R_0 \big(\mathrm{id} + S d_{A_0} R_0\big)^{-1} g^* : L^p(\mathfrak{g}_P) \to W_{A_0}^{1,p}\big(\bigwedge^1(\mathfrak{g}_P)\big)$. Since $S = d_{g^*A}^* - d_{A_0}^*$, we have $d_A^* R = g_* d_{g^*A}^* g^* R = \mathrm{id}$. Furthermore, for every $\xi \in L^p(\mathfrak{g}_P)$, we have

$$\begin{aligned}\|\nabla^A(R\xi)\|_p = \|\nabla^{g^*A}(g^*R\xi)\|_p &\leq \|\nabla^{A_0}(g^*R\xi)\|_p + \big\|\big[(g^*A - A_0) \otimes g^*R\xi\big]\big\|_p \\ &\leq (2 + 2C_4C_2\varepsilon)C_3 \|\xi\|_p\,,\end{aligned}$$

where $C_4 := \max\big\{|[\alpha \otimes \beta]| \,\big|\, \alpha, \beta \in \bigwedge^1(\mathfrak{g}_P) : |\alpha|, |\beta| \leq 1\big\}$. Here in the last step we used (62) and the fact $\|F_A\|_p \leq \varepsilon$. This proves the second statement and concludes the proof of Proposition 11. □

Appendix C. Other auxiliary results

The next lemma was used in the proof of Theorem 2. Here for a linear map $D : X \to Y$ we denote $\operatorname{coker} D := Y/\mathrm{im}D$.

24. Lemma. *Let X, Y, Z be vector spaces and $D' : X \to Y$ and $T : X \to Z$ be linear maps. We define $D := D'|_{\ker T}$. Then the following holds.*

(i) $\ker D = \ker(D', T)$.

(ii) The map $\Phi : \operatorname{coker} D \to \operatorname{coker}(D', T)$, $\Phi(y+\mathrm{im}D) := (y, 0)+\mathrm{im}(D', T)$, is well-defined and injective. If $T : X \to Z$ is surjective then Φ is also surjective.

(iii) Let $\|\cdot\|_Y$, $\|\cdot\|_Z$ be norms on Y and Z and assume that $\mathrm{im}(D', T)$ is closed in $Y \oplus Z$. Then $\mathrm{im}D$ is closed in Y.

The proof of Lemma 24 is straight-forward and left to the reader. The following result was used in the proof of Proposition 8. We define the map $f : \mathbb{C} \setminus \{0\} \to S^1$ by $f(z) := z/|z|$. For two topological spaces X and Y we denote by $C(X, Y)$ the set of all continuous maps from X to Y, and by $[X, Y]$ the set of all (free) homotopy classes of such maps. Let V be a finite dimensional complex vector space. We denote by $\mathrm{End}(V)$ the space of its (complex) endomorphisms of V, by $\det : \mathrm{End}(V) \to \mathbb{C}$ the determinant map, and by $\mathrm{Aut}(V) \subseteq \mathrm{End}(V)$ the group of automorphisms of V.

25. Lemma. *The map $C(S^1, \mathrm{Aut}(V)) \to \mathbb{Z}$ given by $\Phi \mapsto \deg\big(f \circ \det \circ \Phi\big)$ descends to a bijection $\big[S^1, \mathrm{Aut}(V)\big] \to \mathbb{Z}$.*

Proof of Lemma 25. We choose a hermitian inner product V and denote by $\mathrm{U}(V)$ the corresponding group of unitary automorphisms of V. The map $\det : \mathrm{U}(V) \to S^1$ induces an isomorphism of fundamental groups, see e.g. Proposition 2.23 in the book by D. McDuff and D. A. Salamon [**MS1**]. Furthermore, the space $\mathrm{Aut}(V)$ strongly deformation retracts onto $\mathrm{U}(V)$. (This follows from the Gram-Schmidt orthonormalization procedure.) This implies that the map $f \circ \det : \mathrm{Aut}(V) \to S^1$ induces an isomorphism of fundamental groups. Let $\Phi_0 \in \mathrm{Aut}(V)$. It follows that the map $C(S^1, \mathrm{Aut}(V)) \to \mathbb{Z}$ given by $\Phi \mapsto \deg\big(f \circ \det \circ \Phi\big)$ descends to an isomorphism between $\pi_1(\mathrm{Aut}(V), \Phi_0)$ and $\mathbb{Z}$. Since this group is abelian, the map $\pi_1(\mathrm{Aut}(V), \Phi_0) \to \big[S^1, \mathrm{Aut}(V)\big]$ that forgets the base point Φ_0, is a bijection. The statement of Lemma 25 follows from this. □

The next lemma was used in the proof of Theorem 6. Let X and M be manifolds, G a Lie group with Lie algebra $\mathfrak{g}$, $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g}$, $\langle\cdot,\cdot\rangle_M$ a G-invariant Riemannian metric on M, and ∇ its Levi-Civita connection. For $\xi \in \mathfrak{g}$ we denote by X_ξ the vector field on M generated by ξ. We define the tensor $\rho : TM \oplus TM \to \mathfrak{g}$ by

$$\langle \xi, \rho(v, v')\rangle_{\mathfrak{g}} := \langle \nabla_v X_\xi, v'\rangle_M. \tag{63}$$

A short calculation shows that ρ is skew-symmetric. This two-form was introduced in [**Ga**] (page 181). The next lemma corresponds to Proposition 7.1.3(a,b) in [**Ga**]. Let $P \to X$ be a principal bundle, $A \in \mathcal{A}(P)$, $u \in C^\infty\big(X, (P\times M)/G\big)$, $v \in \Gamma(TM^u)$, and $\xi \in \Gamma(\mathfrak{g}_P)$. We define the connection ∇^A on $TM^u \to X$ as in Section 2.

26. Lemma. $\nabla^A L_u\xi - L_u d_A\xi = \nabla_{d_Au}X_\xi, \quad d_AL_u^*v - L_u^*\nabla^A v = \rho(d_Au, v).$

Proof of Lemma 26. This follows from short calculations. $\square$

Let $M, \omega, G, \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \mu$ and J be as in Section 1, and $\langle\cdot,\cdot\rangle_M := \omega(\cdot, J\cdot)$. The following remark was used in the proofs of Theorems 2 and 6. Recall the definition (15) of $M^* \subseteq M$, and that $\mathrm{Pr} : TM \to TM$ denotes the orthogonal projection onto $\mathrm{im}L$.

27. Remark. *Let $K \subseteq M^*$ be compact. We define $c := \inf\big\{|L_x\xi|/|\xi| \,\big|\, x \in K,\, 0 \neq \xi \in \mathfrak{g}\big\}$. Then $c > 0$. Let $x \in K$. Then $L_x^*L_x$ is invertible, and*

$$(64) \qquad |(L_x^*L_x)^{-1}| \leq c^{-2},\ \big|L_x(L_x^*L_x)^{-1}\big| \leq c^{-1},\ L_x(L_x^*L_x)^{-1}L_x^* = \mathrm{Pr}_x,$$

where the $|\cdot|$'s denote operator norms. Furthermore, $|\mathrm{Pr}_x v| \leq c^{-1}|L_x^*v|$, for every $v \in T_xM$. These assertions follow from short calculations.

Assume that hypothesis (H) holds. The following lemma was used in the proof of Proposition 9.

28. Lemma. *There exists a neighborhood $U \subseteq M$ of $\mu^{-1}(0)$, such that*

$$(65) \qquad c := \inf\big\{\big|d\mu(x)L_x^{\mathbb{C}}\alpha\big| + |\mathrm{Pr}\,L_x^{\mathbb{C}}\alpha| \,\big|\, x \in U,\, \alpha \in \mathfrak{g}^{\mathbb{C}} :\, |\alpha| = 1\big\} > 0.$$

Proof of Lemma 28. It follows from hypothesis (H) that there exists $\delta_0 > 0$ such that $\mu^{-1}(\bar{B}_{\delta_0}) \subseteq M^*$. We define

$$C := \sup\big\{|[\xi,\eta]| \,\big|\, \xi,\eta \in \mathfrak{g} :\, |\xi| \leq 1,\, |\eta| \leq 1\big\},$$
$$c_0 := \inf\big\{|L_x\xi|/|\xi| \,\big|\, x \in \mu^{-1}(\bar{B}_{\delta_0}),\, 0 \neq \xi \in \mathfrak{g}\big\}.$$

Since the action of G on M^* is free, it follows that $L_x : \mathfrak{g} \to T_xM$ is injective, for $x \in M^*$. Furthermore, by hypothesis (H) the set $\mu^{-1}(\bar{B}_{\delta_0})$ is compact. It follows that $c_0 > 0$. We choose a positive number $\delta < \min\{\delta_0, c_0/C, c_0^3/C\}$, and we define $U := \mu^{-1}(B_\delta)$.

1. Claim. *Inequality (*65*) holds.*

Proof of Claim 1. Let $x \in U$ and $\alpha = \xi + i\eta \in \mathfrak{g}^{\mathbb{C}}$. Then

$$(66) \qquad d\mu(x)L_x^{\mathbb{C}}\alpha = [\mu(x), \xi] + L_x^*L_x\eta.$$

Using the last assertion in (64), we have

$$(67) \qquad \mathrm{Pr}_xL_x^{\mathbb{C}}\alpha = L_x\xi - L_x(L_x^*L_x)^{-1}[\mu(x), \eta].$$

By the first assertion in (64), we have $|L_x^* L_x \eta| \geq c_0^2|\eta|$. Combining this with (66,67) and the second assertion in (64), we obtain

$$\big|d\mu(x)L_x^{\mathbb{C}}\alpha\big| + |\Pr L_x^{\mathbb{C}}\alpha| \geq -C\delta|\xi| + c_0^2|\eta| + c_0|\xi| - c_0^{-1}C\delta|\eta|.$$

Inequality (65) follows now from our choice of δ. This proves Claim 1 and completes the proof of Lemma 28. □

The following lemma was mentioned in Section 1. Recall the definition (7) of $\widetilde{\mathcal{B}}^{p,\lambda}$, and that $\mathcal{G}^{2,p}_{\mathrm{loc}}(P)$ denotes the group of gauge transformations on P of class $W^{2,p}_{\mathrm{loc}}$.

29. Lemma. *For $p > 2$ and $\lambda > 1 - 2/p$ $\mathcal{G}^{2,p}_{\mathrm{loc}}(P)$ acts freely on $\widetilde{\mathcal{B}}^{p,\lambda}$.*

Proof of Lemma 29. Let $w := (u, A) \in \widetilde{\mathcal{B}}^{p,\lambda}$ and $g \in \mathcal{G}^{2,p}_{\mathrm{loc}}(P)$ be such that $g_* w = w$. It follows from hypothesis (H) that there exists $\delta > 0$ such that $\mu^{-1}(\bar{B}_\delta) \subseteq M^*$ (defined as in (15)). Furthermore, Lemma 30 below implies that there exists $R > 0$ such that $|\mu \circ u(p)| < \delta$, for $p \in \pi^{-1}(\mathbb{R}^2 \setminus B_R)$. We choose $p_0 \in \pi^{-1}(R) \subseteq P$. Since $g(p_0)u(p_0) = u(p_0)$ and $u(p_0) \in M^*$, it follows that $g(p_0) = \mathbf{1}$. Let $p_1 \in P$. We choose a smooth path $p : [0, 1] \to P$ such that $p(i) = p_i$, for $i = 0, 1$. Then the map $g_p := g \circ p : [0, 1] \to G$ solves the ordinary differential equation $\dot{g}_p = g_p A\dot{p} - (A\dot{p})g_p$, $g_p(0) = \mathbf{1}$. It follows that $g_p \equiv \mathbf{1}$, and hence $g(p_1) = \mathbf{1}$. This proves Lemma 29. □

We now prove Proposition 3. Let $M, \omega, G, , \mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}}, \mu$ and J be as in Section 1, $\Sigma := \mathbb{R}^2, \omega_\Sigma := \omega_0, j := i, P \to \mathbb{R}^2$ a principal G-bundle, $p > 2, \lambda > 1-2/p$ and $w \in \widetilde{\mathcal{B}}^p_\lambda$. Assume that hypothesis (H) holds.

30. Lemma. *There exists a smooth section σ of the restriction of the bundle P to B_1^C and a point $x_\infty \in \mu^{-1}(0)$, such that $u \circ \sigma(re^{i\varphi})$ converges to x_∞, uniformly in $\varphi \in \mathbb{R}$, as $r \to \infty$, and $\sigma^* A \in L^p_\lambda(B_1^C)$.*

Proof of Lemma 30.

1. Claim. *The expression $|\mu \circ u|(re^{i\varphi})$ converges to 0, uniformly in $\varphi \in \mathbb{R}$, as $r \to \infty$.*

Proof of Claim 1. We define the function $f := |\mu \circ u|^2 : M \to \mathbb{R}$. It follows from the ad-invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ that

$$df = 2\big\langle d_A(\mu \circ u), \mu \circ u\big\rangle_{\mathfrak{g}} = 2\big\langle d\mu(u)d_A u, \mu \circ u\big\rangle_{\mathfrak{g}}. \tag{68}$$

Since $\overline{u(P)} \subseteq M$ is compact, we have $\sup_{\mathbb{R}^2}|d\mu(u)| < \infty, \sup_{\mathbb{R}^2}|\mu \circ u| < \infty$. Furthermore, $|d_A u| \leq \sqrt{e_w} \in L^p_\lambda$. Combining this with (68), it follows that $df \in L^p_\lambda$. Therefore, by Proposition 14 (Hardy-type inequality, applied with u replaced by f) the expression $f(re^{i\varphi})$ converges to some number $y_\infty \in \mathbb{R}$, as $r \to \infty$, uniformly in $\varphi \in \mathbb{R}$. Since $|\mu \circ u| \leq \sqrt{e_w} \in L^p_\lambda$, it follows that $y_\infty = 0$. This proves Claim 1. □

It follows from hypothesis (H) that there exists a $\delta > 0$ such that $\mu^{-1}(\bar{B}_\delta) \subseteq M^*$ (defined as in (15)). We choose $R > 0$ so big that $|\mu \circ u|(z) \leq \delta$ if $z \in B_{R-1}^C = \mathbb{R}^2 \setminus B_{R-1}$. Since G is compact, the action of it on M is proper. Hence the local slice theorem implies that M^*/G carries a unique manifold structure such that the canonical projection $\pi^{M^*}: M^* \to M^*/G$ is a submersion. Consider the map $\bar{u}: B_{R-1}^C \to M^*/G$ defined by $\bar{u}(z) := Gu(p)$, where $p \in \pi^{-1}(z) \subseteq P$ is arbitrary.

2. Claim. *The point $\bar{u}(re^{i\varphi})$ converges to some point $\bar{x}_\infty \in \mu^{-1}(0)/G$, uniformly in $\varphi \in \mathbb{R}$, as $r \to \infty$.*

Proof of Claim 2. We choose $n \in \mathbb{N}$ and an embedding $\iota: M^*/G \to \mathbb{R}^n$. Furthermore, we choose a smooth function $\rho: \mathbb{R}^2 \to \mathbb{R}$ that vanishes on B_{R-1} and equals 1 on B_R^C. We define $f: \mathbb{R}^2 \to \mathbb{R}^n$ to be the map given by $\rho \cdot \iota \circ \bar{u}$ on B_{R-1}^C and by 0 on B_{R-1}. It follows that $\|df\|_{L^p_\lambda(B_R^C)} \leq \big\|d\iota(\bar{u})d\bar{u}\big\|_{L^p_\lambda(B_R^C)} + \|(d\rho)\iota\circ\bar{u}\|_{L^p_\lambda(B_R \setminus B_{R-1})}$. Furthermore,

$$\big\|d\iota(\bar{u})d\bar{u}\big\|_{L^p_\lambda(B_R^C)} \leq \|d\iota(\bar{u})\|_{L^\infty(B_R^C)} \|d_A u\|_{L^p_\lambda(B_R^C)}.$$

Our assumption $w = (u, A) \in \widetilde{\mathcal{B}}^p_\lambda$ implies that $\|d_A u\|_{L^p_\lambda(B_R^C)} < \infty$. Furthermore, μ is proper by the hypothesis (H), hence the set $\mu^{-1}(\bar{B}_\delta)$ is compact. Thus the same holds for the set $\pi^{M^*}(\mu^{-1}(\bar{B}_\delta))$. This set contains the image of $\bar{u}$. It follows that $\|d\iota(\bar{u})\|_{L^\infty(B_R^C)} < \infty$, and therefore $\|df\|_{L^p_\lambda(\mathbb{R}^2)} \leq \|df\|_{L^p_\lambda(B_R)} + \|df\|_{L^p_\lambda(B_R^C)} < \infty$. Hence the hypotheses of Proposition 14 are satisfied. It follows that the point $f(re^{i\varphi})$ converges to some point $y_\infty \in \mathbb{R}^n$, uniformly in $\varphi \in \mathbb{R}$, as $r \to \infty$. Claim 2 follows. □

Let $\bar{x}_\infty$ be as in Claim 2. We choose a local slice around $\bar{x}_\infty$, i.e. a pair $(\bar{U}, \widetilde{\sigma})$, where $\bar{U} \subseteq M^*/G$ is an open neighborhood of $\bar{x}_\infty$, and $\widetilde{\sigma}: \bar{U} \to M^*$ is a smooth map satisfying $\pi^{M^*} \circ \widetilde{\sigma} = \mathrm{id}_{\bar{U}}$. Then there exists a unique section σ' of $P|_{B_R^C}$ such that $\widetilde{\sigma} \circ \bar{u} = u \circ \sigma'$. By the homotopy lifting property of P we may extend this to a continuous section σ'' of $P|_{B_1^C}$. Smoothing out σ'' on $B_{R+1} \setminus B_1$, we obtain a smooth section σ of $P|_{B_1^C}$. We define $x_\infty := \widetilde{\sigma}(\bar{x}_\infty)$. It follows from Claim 2 that $u \circ \sigma(re^{i\varphi})$ converges to x_∞, uniformly in φ, for $r \to \infty$. Furthermore,

$$\|L_{u\circ\sigma}\sigma^* A\|_{L^p_\lambda(B_{R+1}^C)} \leq \|du\, d\sigma'\|_{L^p_\lambda(B_{R+1}^C)} + \|d_{A\, d\sigma'} u\|_{L^p_\lambda(B_{R+1}^C)},$$

$$du\, d\sigma' = d(u \circ \sigma') = d\widetilde{\sigma}\, d\bar{u}, \quad |d\bar{u}| \leq |d_A u|.$$

Since $\inf\big\{|L_u(p)\xi| \,\big|\, p \in P|_{B_{R+1}^C},\ \xi \in \mathfrak{g}: |\xi| = 1\big\} > 0$ and $\|d_A u\|_{p,\lambda} < \infty$, it follows that $\sigma^* A \in L^p_\lambda(B_1^C)$. This proves Lemma 30. □

Proof of Proposition 3. We choose σ and x_∞ as in Lemma 30, and we define $\widetilde{P}$ to be the quotient of $P \coprod (S^2 \setminus \{0\}) \times G$ under the equivalence relation generated by $p \sim (\pi(p), g)$, where $g \in G$ is determined by $\sigma(z)g = p$, for $p \in P$. We define $\widetilde{u} : \widetilde{P} \to M$ by $\widetilde{u}([p]) := u(p)$, for $p \in P$, and $\widetilde{u}([\infty, g]) := g^{-1}x_\infty$, for $g \in G$. If follows from the statement of Lemma 30 that this map is continuous. Let now $\widetilde{P}_1$ and $\widetilde{P}_2$ be two extensions of P, for which the map u extends to continuous maps $\widetilde{u}_i : \widetilde{P}_i \to M$. We define $\Psi : \widetilde{P}_1 \to \widetilde{P}_2$ to be the identity on $P \subseteq \widetilde{P}_1$, and for p in the fiber of $\widetilde{P}_1$ over ∞ we define $\Psi(p)$ to be the unique point $p' \in \widetilde{P}_2$ such that $\widetilde{u}_1(p) = \widetilde{u}_2(p')$. (It follows from Lemma 30 that this point is unique.) This map has the required properties. This proves Proposition 3. □

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