

A MASLOV MAP FOR COISOTROPIC SUBMANIFOLDS, LEAF-WISE FIXED POINTS AND PRESYMPLECTIC NON-EMBEDDINGS

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Let (M, ω) be a symplectic manifold, $N \subseteq M$ a coisotropic submanifold, and Σ a compact oriented (real) surface. I define a natural Maslov index for each continuous map $u : \Sigma \rightarrow M$ that sends every connected component of $\partial\Sigma$ to some isotropic leaf of N . This index is real valued and generalizes the usual Lagrangian Maslov index. The idea is to use the linear holonomy of the isotropic foliation of N to compensate for the loss of boundary data in the case $\text{codim } N < \dim M/2$. The definition is based on the Salamon-Zehnder (mean) Maslov index of a path of linear symplectic automorphisms. I prove a lower bound on the number of leafwise fixed points of a Hamiltonian diffeomorphism, if (M, ω) is geometrically bounded and N is closed, regular (i.e. "fibering"), and monotone. As an application, we obtain a presymplectic non-embedding result. I also prove a coisotropic version of the Audin conjecture.

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1. Motivation and main results

This article is concerned with the following two problems. Let (M, ω) be a symplectic manifold and $N \subseteq M$ a coisotropic submanifold. A leafwise fixed point of a map $\varphi : M \rightarrow M$ is by definition a point $x \in N$ such that $\varphi(x)$ lies in the isotropic leaf through x . We denote by $\text{Fix}(\varphi, N) := \text{Fix}(\varphi, N, \omega)$ the set of such points.

Problem A: Find conditions on (M, ω, N, φ) under which $\text{Fix}(\varphi, N)$ is non-empty and give a lower bound on $|\text{Fix}(\varphi, N)|$.

Note that in the case $N = M$ the set $\text{Fix}(\varphi, N)$ equals the set $\text{Fix}(\varphi)$ of usual fixed points. In the other extreme case, in which N is Lagrangian, we have $\text{Fix}(\varphi, N) = N \cap \varphi^{-1}(N)$.

To formulate the second problem, let V be a real vector space and ω a skew-symmetric form on V . We denote $\text{corank} \omega := \dim \ker (V \ni v \mapsto \omega(v, \cdot) \in V^*)$. A presymplectic form on a manifold M is a closed two-form ω of constant corank. We say that a presymplectic manifold (M', ω') embeds into another presymplectic manifold (M, ω) iff there exists an embedding $\psi : M' \rightarrow M$ such that $\psi^* \omega = \omega'$. The next problem generalizes the symplectic and Lagrangian non-embedding problems:

Problem B: Find conditions on (M, ω) and (M', ω') under which (M', ω') does not embed into (M, ω) .

In [Zi], I gave some solution to problem A, imposing the conditions that N is regular and the Hofer distance of φ and the identity is small enough. In the present article, the second condition is replaced by the assumption that N is monotone. The paper [Zi] also contains some solution to problem B, assuming that ω is non-degenerate and aspherical. In the present article the latter condition is replaced by monotonicity of ω .

To define monotonicity for a coisotropic submanifold N , I introduce a natural Maslov map for N , which equals the usual Maslov index in the case $\dim N = \dim M/2$, and twice the first Chern class of (M, ω) in the case $N = M$.

1.1. Definition of the Maslov map. Let (M, ω) be a symplectic manifold (without boundary), $N \subseteq M$ a coisotropic submanifold, and X a topological manifold. We denote by $\mathcal{C}(X)$ the set of connected components of X and by N_ω the set of isotropic leaves of N . We define

$$C(X, M; N, \omega) := \{u \in C(X, M) \mid \forall Y \in \mathcal{C}(\partial X) \exists F \in N_\omega : u(Y) \subseteq F\}.$$

Let $u \in C([0, 1] \times X, M)$. We call u an (N, ω) -admissible homotopy iff for every $Y \in \mathcal{C}(\partial X)$ there exists $F \in N_\omega$ such that $u(t, x) \in F$, for every $t \in [0, 1]$, $x \in Y$. We denote by $[X, M; N, \omega]$ the corresponding set of all (N, ω) -admissible homotopy classes of maps from X to M .

We denote by \mathcal{S} the class of all compact oriented (real) topological surfaces (possibly with boundary and disconnected). Let $\Sigma \in \mathcal{S}$. The Maslov map

introduced in this article is a map

$$(1) \quad m_{\Sigma, N} := m_{\Sigma, \omega, N} : [\Sigma, M; N, \omega] \rightarrow \mathbb{R}.$$

Its definition involves the following four steps. A more direct, but less natural definition is given on page 5.

The Salamon-Zehnder Maslov index. Let (V, ω) be a symplectic vector space. We denote by $\text{Aut}\omega$ the group of linear symplectic automorphisms of V . We define the *Salamon-Zehnder Maslov index*

$$(2) \quad m_\omega : C([0, 1], \text{Aut}\omega) \rightarrow \mathbb{R}$$

as follows. We define the winding map $\alpha : C([0, 1], \mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}$ by $\alpha(z) := \tilde{z}(1) - \tilde{z}(0)$, where $\tilde{z} \in C([0, 1], \mathbb{R})$ is any path such that $\tilde{z}(t) + \mathbb{Z} = z(t)$, for every $t \in \mathbb{R}$. We denote by $\rho_\omega : \text{Aut}(\omega) \rightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ the Salamon-Zehnder map (see Proposition 42 below). Let $\Phi \in C([0, 1], \text{Aut}\omega)$. We define $m_\omega(\Phi) := 2\alpha(\rho_\omega \circ \Phi)$.

The Maslov map for pairs of flat transports. Let X be a topological manifold. We denote by ΠX the fundamental groupoid of X . This is a topological groupoid. Its set of objects is X and its set of morphisms consists of all homotopy classes (with fixed end-points) of continuous paths in X .

For two vector spaces V and V' we denote by $\text{Iso}(V, V')$ the set of all isomorphisms from V to V' . Let $E \rightarrow X$ be a vector bundle. We denote by $\text{GL}(E)$ the general linear groupoid of E . This is a topological groupoid. Its set of objects is X and its set of morphisms consists of all triples (x, y, Φ) , where $x, y \in X$ and $\Phi \in \text{Iso}(E_x, E_y)$.

By a flat (linear) transport we mean a (continuous) representation Φ of ΠX on E , i.e. a morphism of topological groupoids from ΠX to $\text{GL}(E)$ that covers the identity on X . Such a Φ associates to every homotopy class of paths $x \in C([0, 1], X)$ an isomorphism $\Phi([x]) \in \text{Iso}(x(0), x(1))$. It is equivariant with respect to concatenation of paths. We denote by $\mathcal{T}(E)$ the set of all flat transports on E .

We call $\Phi \in \mathcal{T}(E)$ *regular* iff $\Phi([x]) = \text{id}$, for every $x \in C([0, 1], X)$ satisfying $x(0) = x(1)$. Note that if X is a smooth manifold and E is a smooth vector bundle then the parallel transport of a smooth flat connection on E is a flat transport.

For symplectic vector spaces (V, ω) and (V', ω') we denote by $\text{Iso}(\omega, \omega')$ the set of linear isomorphisms $\Phi : V \rightarrow V'$ such that $\Phi^*\omega' = \omega$. Let X be a topological manifold and (E, ω) be a symplectic vector bundle over X . We define $\text{GL}(E, \omega)$ to be the subgroupoid of $\text{GL}(E)$ consisting of all (x, y, Φ) such that $\Phi \in \text{Iso}(\omega_x, \omega_y)$. We call a transport $\Phi \in \mathcal{T}(E)$ symplectic iff $\Phi(\Pi X) \subseteq \text{GL}(E, \omega)$, and denote by $\mathcal{T}(E, \omega)$ the set of all such Φ 's.

Let X be an oriented closed curve (i.e. topological real one-manifold), (E, ω) a symplectic vector bundle over X , and $\Phi, \Phi' \in \mathcal{T}(E, \omega)$ be such that Φ or Φ' is regular. We define the number $m_\omega(\Phi, \Phi') \in \mathbb{R}$ as follows.

Namely, we choose a path $z \in C([0, 1], X)$ such that $z(0) = z(1)$ and the map $S^1 \cong [0, 1]/\{0, 1\} \ni [t] \mapsto z(t) \in X$ has degree one. We define $\Psi \in C([0, 1], \text{Aut}(\omega_{z(0)}))$ by $\Psi(t) := \Phi'([z|_{[0,t]})^{-1}\Phi([z|_{[0,t]})$, and

$$(3) \quad m_\omega(\Phi, \Phi') := m_{\omega_{z(0)}}(\Psi).$$

By Lemma 7 below this number is well-defined.

The coisotropic Maslov map for bundles. Let (V, ω) be a symplectic vector space and $W \subseteq V$ be a subspace. We denote by $W^\omega := \{v \in V \mid \omega(v, w) = 0, \forall w \in W\}$ its symplectic complement. Assume that W is coisotropic. We denote by $(W_\omega := W/W^\omega, \omega_W)$ its linear symplectic quotient, and for $\Phi \in \text{Aut}(\omega)$ we define

$$\Phi_W : W_\omega \rightarrow (\Phi W)_\omega, \quad \Phi_W(v + W^\omega) := \Phi v + (\Phi W)^\omega.$$

Let E be a vector bundle over X , $W \subseteq E$ a subbundle and $\Phi \in \mathcal{T}(E)$. We say that Φ leaves W invariant iff $\Phi([z])W_{z(0)} = W_{z(1)}$, for every $[z] \in \Pi X$. Let (E, ω) be a symplectic vector bundle over X . We define $\mathcal{C}^{\text{flat}}(E, \omega)$ to be the set of all pairs (W, Φ) , where $W \subseteq E$ is an ω -coisotropic subbundle, and $\Phi \in \mathcal{T}(W_\omega, \omega_W)$. Let $W \subseteq E$ be a coisotropic subbundle, and $\Phi \in \mathcal{T}(E, \omega)$ be a transport that leaves W invariant. We define $\Phi_W \in \mathcal{T}(W_\omega, \omega_W)$ by $\Phi_W([z]) := \Phi([z])_{W_{z(0)}}$.

Theorem 1. (*Coisotropic Maslov map for bundles*) *Let $\Sigma \in \mathcal{S}$ be connected and such that $\partial\Sigma \neq \emptyset$, and let (E, ω) be a symplectic vector bundle over Σ . Then there exists a unique map $m_{\Sigma, E, \omega} : \mathcal{C}^{\text{flat}}(E, \omega) \rightarrow \mathbb{R}$ with the following properties.*

- (i) (*Boundary*) *For every regular transport $\Phi_0 \in \mathcal{T}(E, \omega)$ and every $\Phi \in \mathcal{T}((E, \omega)|_{\partial\Sigma})$ we have $m_{\Sigma, E, \omega}(E|_{\partial\Sigma}, \Phi) = m_{\partial\Sigma, \omega|_{\partial\Sigma}}(\Phi, \Phi_0)$.*
- (ii) (*Invariant subbundle*) *Let $W \subseteq E|_{\partial\Sigma}$ be an ω -coisotropic subbundle, and $\Psi \in \mathcal{T}((E, \omega)|_{\partial\Sigma})$. If Ψ leaves W invariant then $m_{\Sigma, E, \omega}(E|_{\partial\Sigma}, \Psi) = m_{\Sigma, E, \omega}(W, \Psi_W)$.*

For the proof of this theorem, the idea is to define

$$m_{\Sigma, E, \omega}(W, \Phi) := m_{\partial\Sigma, \omega|_{\partial\Sigma}}(\Psi, \Phi_0),$$

where $\Psi \in \mathcal{T}((E, \omega)|_{\partial\Sigma})$ is a lift of (W, Φ) , and $\Phi_0 \in \mathcal{T}(E, \omega)$ is a regular transport. In order to show that this does not depend on the choice of Ψ , the following result is crucial. Namely, let (V, ω) be a symplectic vector space, $W \subseteq V$ a coisotropic subspace, and $\Psi \in \text{Aut}(\omega)$ be such that $\Psi W = W$. Then $\rho_\omega(\Psi) = \pm \rho_{\omega_W}(\Psi_W)$. (See Proposition 28 below.) The proof of this identity is based on the existence of a path $\Psi \cdot \in C([0, 1], \text{Aut}(\omega))$, such that $\Psi^1 = \Psi$, Ψ^0 leaves three fixed subspaces of V invariant, and the map $[0, 1] \ni t \mapsto \rho_\omega(\Psi^t) \in \mathbb{R}$ is constant.

Let $\Sigma \in \mathcal{S}$ be a connected surface satisfying $\partial\Sigma \neq \emptyset$. We define \mathcal{E}_Σ to be the class of all quadruples (E, ω, W, Φ) , where (E, ω) is a symplectic vector bundle over Σ and $(W, \Phi) \in \mathcal{C}^{\text{flat}}((E, \omega)|_{\partial\Sigma})$. We define

$$m_\Sigma : \mathcal{E}_\Sigma \rightarrow \mathbb{R}, \quad m_\Sigma(E, \omega, W, \Phi) := m_{\Sigma, E, \omega}(W, \Phi),$$

where $m_{\Sigma, E, \omega}$ is the unique map satisfying the conditions of Theorem 1.

Definition of $m_{\Sigma, \omega, N}$. We now define the map (1) as follows. Assume first that Σ is connected. If $\partial\Sigma = \emptyset$ then we define $m_{\Sigma, \omega, N}(a) := 2\langle c_1(M, \omega), a \rangle$. Assume now that $\partial\Sigma \neq \emptyset$. We denote by $\text{hol}^{N, \omega}$ the linear holonomy of the isotropic foliation of N (see (38) below). We define the map $\tilde{m}_{\Sigma, \omega, N} : C(\Sigma, M; N, \omega) \rightarrow \mathbb{R}$ by

$$(4) \quad \tilde{m}_{\Sigma, \omega, N}(u) := m_\Sigma(u^*(TM, \omega), u|_{\partial\Sigma}^*(TN, \text{hol}^{N, \omega})).$$

It follows from Theorem 24(iii) below that this map is invariant under (N, ω) -admissible homotopies. For a general $\Sigma \in \mathcal{S}$ we define $\tilde{m}_{\Sigma, \omega, N}$ by $\tilde{m}_{\Sigma, \omega, N}(u) := \sum_{\Sigma' \in \mathcal{C}(\Sigma)} \tilde{m}_{\Sigma', \omega, N}(u|_{\Sigma'})$.

Definition 2. Let (M, ω) be a symplectic manifold, $N \subseteq M$ a coisotropic submanifold, and $\Sigma \in \mathcal{S}$. We define the Maslov map $m_{\Sigma, N} : [\Sigma, M; N, \omega] \rightarrow \mathbb{R}$ to be the map induced by $\tilde{m}_{\Sigma, \omega, N}$.

As an example, let $\Sigma := \mathbb{D} \subseteq \mathbb{R}^2$ be the unit disk, $M := \mathbb{R}^{2n}$, ω the standard structure ω_0 , $N := S^{2n-1}$, and $u : \mathbb{D} \rightarrow \mathbb{R}^{2n}$ the inclusion $u(z) := (z, 0, \dots, 0)$. Then $m_{\mathbb{D}, \omega_0, S^{2n-1}}(u) = 2$. For more examples see the subsection on page 9 about the Gaiotto-Salamon Maslov index.

The map $m_{\Sigma, \omega, N}$ may be viewed as a *mean* Maslov index. Analogously to the definition of the Conley-Zehnder index there should also be a natural *integer valued* map with the same domain.

The regular case. Let X be a compact topological manifold. We call a map $u \in C([0, 1] \times X, M)$ a *weakly (N, ω) -admissible homotopy* iff for every $Y \in \mathcal{C}(\partial X)$ and $t \in [0, 1]$ there exists $F \in N_\omega$ such that $u(t, x) \in F$, for every $x \in Y$. We denote by $\langle X, M; N, \omega \rangle$ the corresponding set of homotopy classes. We call N regular iff its isotropic leaf relation is a closed subset and a submanifold of $N \times N$. Assume now that N is regular. Then it follows from Theorem 19(iv,vi) below that the Maslov map $m_{\Sigma, N}$ takes on integer values and is invariant under weak homotopies. If N is also orientable then by Theorem 19(vi) $m_{\Sigma, N}$ takes on even values.

1.2. More elementary description. In more elementary, but less natural terms, the map $\tilde{m}_{\Sigma, \omega, N}$ is given as follows. Let (V, ω) be a symplectic vector space, and $W_0 \subseteq V$ a coisotropic subspace. We define the *framed coisotropic Grassmannian* $G(\omega, W_0)$ to be the manifold consisting of all pairs (W, Φ) , where $W \subseteq V$ is a coisotropic subspace and $\Phi \in \text{Iso}((W_0)_\omega, \omega_{W_0}; W_\omega, \omega_W)$.

Let $(W, \Phi) \in C([0, 1], G(\omega, W_0))$ be a path such that $W(0) = W(1)$. We choose a path $\Psi \in C([0, 1], \text{Aut}\omega)$ satisfying $\Psi(t)W_0 = W(t)$ and $\Psi(t)_{W_0} = \Phi(t)$. (It follows from Lemma 11 below that such a path exists.) We define $m_\omega(W, \Phi) := m_\omega(\Psi)$. (It follows from Theorem 9(ii) below that this number does not depend on the choice of Ψ .)

Let now M, ω, N, Σ and u be as above. For simplicity, assume that $\Sigma = \mathbb{D}$. We denote $2n := \dim M$. We choose a symplectic trivialization $\Psi : \mathbb{D} \times \mathbb{R}^{2n} \rightarrow u^*TM$, and define $W_0 := \Psi_{u(1)}^{-1}T_{u(1)}N \subseteq \mathbb{R}^{2n}$. We define the path $(W, \Phi) : [0, 1] \rightarrow G(W_0, \omega_0)$ as follows. Let $s \in [0, 1]$. We set $W(s) := \Psi_{u(e^{2\pi is})}^{-1}T_{u(e^{2\pi is})}N \subseteq \mathbb{R}^{2n}$. Furthermore, we define $F_s : (T_{u(1)}N)_\omega \rightarrow (T_{u(e^{2\pi is})}N)_\omega$ to be the linear holonomy of the isotropic foliation of N along the path $[0, 1] \ni t \mapsto u(e^{2\pi ist}) \in F$. We set $\Phi(s) := (\Psi_{u(e^{2\pi is})})_{W_0}^{-1}F_s(\Psi_{u(1)})_{W_0}$. The Maslov index of u is now given by

$$\tilde{m}_{\mathbb{D}, \omega, N}(u) = m_{\omega_0}(W, \Phi).$$

1.3. Leaf-wise fixed points, presymplectic embeddings and minimal Maslov numbers.

Leaf-wise fixed points. Assume that N is regular. We define the minimal Maslov number

$$m(N) := m(N, \omega) := \inf(\{m_{\mathbb{D}, N}(a) \mid a \in \langle X, M; N, \omega \rangle\} \cap \mathbb{N}) \in \mathbb{N} \cup \{\infty\}.$$

We call N monotone iff there exists a constant $c > 0$ such that for every $u \in C(\mathbb{D}, M; N, \omega)$ we have $\tilde{m}_{\mathbb{D}, \omega, N, F}(u) = c \int_{\mathbb{D}} u^*\omega$.

We denote by $\text{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms on M . For every $\varphi \in \text{Ham}(M, \omega)$ the pair (N, φ) is called *non-degenerate* iff the following holds. For $x_0 \in N$ we denote by $\text{pr}_{x_0} : T_{x_0}N \rightarrow (T_{x_0}N)_\omega = T_{x_0}N / (T_{x_0}N)^\omega$ the canonical projection. Let $F \subseteq N$ be an isotropic leaf, and $x \in C^\infty([0, 1], F)$ a path. Assume that $\varphi(x(0)) = x(1)$, and let $v \in T_{x(0)}N \cap T_{x(0)}\varphi^{-1}(N)$ be a vector. Then $v \neq 0$ implies that

$$(5) \quad \text{hol}_x^{\omega, N} \text{pr}_{x(0)}v \neq \text{pr}_{x(1)}d\varphi(x(0))v.$$

In the case $N = M$ this condition means that for every $x_0 \in \text{Fix}(\varphi)$, 1 is not an eigenvalue of $d\varphi(x_0)$. Furthermore, in the case that N is Lagrangian the condition means that for every connected component $N' \subseteq N$ we have $N' \pitchfork \varphi(N')$, i.e. N' and $\varphi(N')$ intersect transversely.

For a topological space X and $i \in \mathbb{N} \cup \{0\}$ we denote by $b_i(X, \mathbb{Z}_2)$ the i -th \mathbb{Z}_2 -Betti number of X .

Theorem 3. *Let (M, ω) be a (geometrically) bounded symplectic manifold, $N \subseteq M$ a closed monotone regular coisotropic submanifold and $\varphi \in \text{Ham}(M, \omega)$. If (N, φ) is non-degenerate then*

$$(6) \quad |\text{Fix}(\varphi, N)| \geq \sum_{i=\dim N - m(N) + 2, \dots, m(N) - 2} b_i(N, \mathbb{Z}_2).$$

This theorem generalizes a result for the case $\dim N = \dim M/2$, which is due to P. Albers [AI].

Examples. A big class of examples is given as follows. Let (X, σ) and (X', σ') be closed symplectic manifolds and $L \subseteq X$ a closed Lagrangian submanifold. We define $(M, \omega, N) := (X' \times X, \sigma' \oplus \sigma, X' \times L)$. Then N is a closed regular coisotropic submanifold of M .

Let $\Sigma \in \mathcal{S}$. We define Σ' to be the closed surface obtained from Σ by collapsing each boundary circle to a point. By straight-forward arguments the map $\Phi : [\Sigma', X'] \times \langle \Sigma, X; L, \sigma \rangle \rightarrow \langle \Sigma, M; N, \omega \rangle$, $\Phi([u'], [u]) := [(u', u)]$, is well-defined and a bijection. Furthermore, $m_{\Sigma, \omega, N} \circ \Phi([u'], [u]) = 2\langle c_1(TX', \sigma'), [u'] \rangle + m_{\Sigma, \sigma, L}([u])$. This follows from Theorem 19(ii, viii) below. It follows that $m(N, \omega)$ is the greatest common divisor of twice the minimal Chern number of (TX', σ') and $m(L, \sigma)$.

Assume that there exists $c > 0$ such that $2\langle c_1(TX', \sigma'), [u] \rangle = c \int_{S^2} u^* \sigma'$ for every $u \in C^\infty(S^2, X')$, and $m_{\mathbb{D}, L, \sigma}([u]) = c \int_{\mathbb{D}} u^* \sigma$, for every $u \in C^\infty(\mathbb{D}, X)$ satisfying $u(S^1) \subseteq L$. Then N is monotone, and hence (M, ω, N) satisfies the conditions of Theorem 3.

As a concrete example, let $n \in \mathbb{N}$, $k \in \{1, \dots, 2n\}$, $X' := \mathbb{C}P^n$, σ' be the Fubini-Study form ω_{FS} , X the torus \mathbb{T}^{2k} with the standard form $\sigma := \omega_0$, and $L = \mathbb{T}^k \subseteq \mathbb{T}^{2k}$ the standard Lagrangian subtorus. Let $\varphi \in \text{Ham}(M, \omega)$ be such that (N, φ) is non-degenerate. Then applying Theorem 3 we obtain $|\text{Fix}(\varphi, N)| \geq \sum_{i=k, \dots, 2n, j=0, \dots, 2n} k \text{ choose } i - 2j$.

Idea of proof of Theorem 3. The idea is to find a Lagrangian embedding of N into a suitable symplectic manifold, and then apply the Main Theorem in [AI]. Since N is regular, the set of isotropic leaves N_ω carries canonical smooth and symplectic structures $\mathcal{A}_{N, \omega}$ and ω_N . We define

$$(7) \quad \widetilde{M} := M \times N_\omega, \quad \widetilde{\omega} := \omega \oplus (-\omega_N),$$

$$(8) \quad \iota_N : N \rightarrow \widetilde{M}, \quad \iota_N(x) := (x, N_x), \quad \widetilde{N} := \iota_N(N).$$

Then ι_N is an embedding of N into \widetilde{M} that is Lagrangian with respect to the symplectic form $\widetilde{\omega}$ on \widetilde{M} . In order for the hypotheses of Albers' result to be satisfied, the inequality $m(\widetilde{N}, \widetilde{\omega}) \geq m(N, \omega)$ is crucial. It follows from Theorem 19(x) and Propositions 61 below.

Application: presymplectic non-embeddings. Let (M, ω) be a symplectic manifold. We denote by $c_1^{M, \omega} : [S^2, M] \rightarrow \mathbb{R}$ the contraction with the first Chern class of (M, ω) , and by $c_1(M, \omega) := \inf (c_1^{M, \omega}([S^2, M]) \cap \mathbb{N}) \in \mathbb{Z}$ the (spherical) minimal Chern number. Let (M', ω') be a regular presymplectic manifold. This means that the isotropic leaf relation of ω' is a closed subset and a submanifold of $M' \times M'$. For $x, y \in \mathbb{N} \cup \{\infty\}$ we denote by $\text{gcd}(x, y) \in \mathbb{N} \cup \{\infty\}$ the greatest common divisor of x and y . (Our convention is that $\text{gcd}(x, \infty) = \text{gcd}(\infty, x) = x$, for $x \in \mathbb{N}$, and $\text{gcd}(\infty, \infty) = \infty$.)

We define $\mu := 2 \gcd(c_1(M, \omega), c_1(M'_{\omega'}, \omega'_{M'}))$. The proof of the following result is based on Theorem 3.

Theorem 4. *Assume that (M, ω) is connected and bounded, every compact subset of M is Hamiltonianly displaceable, M' is connected and closed, there exists an index $i \in \{\dim M' - \mu + 2, \dots, \mu - 2\}$ such that $b_i(M', \mathbb{Z}_2) \neq 0$, for some fiber $F \subseteq M'$ every loop $u \in C(S^1, F)$ is contractible in M' , $\dim M' + \text{corank} \omega' = \dim M$, and the following condition is satisfied.*

- (i) *There exists a constant $c > 0$ such that $c_1^{M, \omega} = c[\omega]$ on $[S^2, M]$ and $c_1^{M', \omega'_{M'}} = c[\omega'_{M'}]$ on $[S^2, M'_{\omega'}]$.*

Then (M', ω') does not embed into (M, ω) .

Note that the condition $\dim M' + \text{corank} \omega' = \dim M$ is critical in the sense that in the case $\dim M' + \text{corank} \omega' > \dim M$ there is no presymplectic embedding of any open non-empty subset of M' into M , whereas in the case $\dim M' + \text{corank} \omega' \leq \dim M$ for every point $x' \in M'$ there exists an open neighbourhood that embeds presymplectically into M .

The next result gives a criterion under which condition (i) in Theorem 4 holds and μ becomes simpler.

Proposition 5. *Let (M, ω) be a connected symplectic manifold and (M', ω') a regular presymplectic manifold, such that some isotropic fiber $F \subseteq M'$ is simply-connected, $\dim M' + \text{corank} \omega' = \dim M$, and (M', ω') embeds into (M, ω) . Then $\mu = 2c_1(M, \omega)$. Furthermore, if (M, ω) is spherically monotone then condition (i) of Theorem 4 holds.*

It follows from Theorem 4 and Proposition 5 that (M', ω') does not embed into (M, ω) , provided that $\dim M' + \text{corank} \omega' = \dim M$ and some conditions on (M, ω) and some conditions on (M', ω') are satisfied. (The point here is that there are no further assumptions involving both (M, ω) and (M', ω') .)

As an example, let m and n be positive integers, (X, σ) a closed symplectic manifold and $\pi : M' \rightarrow X$ a closed smooth fiber bundle with simply connected fibers, such that $\dim X/2 + k = m + n$ and there exists $i \in \{2n - k, \dots, 2m\}$ such that $b_i(M', \mathbb{Z}_2) \neq 0$, where k denotes the dimension of the fibers. We define $\omega' := \pi^* \sigma$ and denote by ω_{FS} the Fubini-Study form on $\mathbb{C}P^m$ and by ω_0 the standard symplectic form on \mathbb{R}^{2n} . It follows from Theorem 4 that (M', ω') does not embed into $(\mathbb{C}P^m \times \mathbb{R}^{2n}, \omega_{\text{FS}} \oplus \omega_0)$.

More concretely, let m be a positive integer and $k \in \{2, \dots, 2m\}$. Then $(\mathbb{C}P^m \times S^k, \omega_{\text{FS}} \oplus 0)$ does not embed into $(\mathbb{C}P^m \times \mathbb{R}^{2k}, \omega_{\text{FS}} \oplus \omega_0)$.

Coisotropic Audin conjecture. Recall that a topological space X is called aspherical iff $\pi_k(X) = 0$, for every $k \geq 2$. Furthermore, a manifold is called spin iff it is orientable and its second Stiefel-Whitney number vanishes.

Theorem 6. *Let (M, ω) be a symplectic manifold that is convex at infinity, and $N \subseteq M$ a coisotropic submanifold that is closed, regular, aspherical, spin, and displaceable. Then $m(N, \omega) = 2$.*

In the Lagrangian case this result is due to K. Fukaya [Fu]. It generalizes a conjecture by Audin about the minimal Maslov number of a Lagrangian submanifold of \mathbb{R}^{2n} diffeomorphic to the torus \mathbb{T}^n . The idea of proof of Theorem 6 is to reduce to the Lagrangian case using the construction (7,8), and then to apply Fukaya's result.

1.4. Related work.

Oh's Maslov index. Let J be an ω -compatible almost complex structure, assume that N is gradable and equipped with a grading $[\Delta]$ in the sense of [Oh], and that $\Sigma = \mathbb{D}$. In this situation, Y.-G. Oh defined a Maslov index $\mu_{(N,\Delta)} : \{u \in C^\infty(\mathbb{D}, M) \mid u(S^1) \subseteq N\} \rightarrow \mathbb{Z}$, see Definition 3.3. in [Oh]. If $u \in C(\mathbb{D}, M; N, \omega)$ is a smooth map then $\mu_{(N,\Delta)}(u) = m_{\mathbb{D},\omega,N}(u)$. Note that $\mu_{(N,\Delta)}$ is defined on a larger set of maps than $m_{\mathbb{D},\omega,N}$ (after restriction to $C^\infty(\mathbb{D}, M)$), but requires $[\Delta]$ as an additional datum. Observe also that the definition of $m_{\Sigma,\omega,N}$ does not involve the choice of any ω -compatible almost complex structure on M .

The Gaio-Salamon Maslov index. Let (M, ω, G, ω) be a Hamiltonian G -manifold. This means that (M, ω) is a symplectic manifold, and G is a connected Lie group acting on M in a Hamiltonian way, with moment map μ . Assume that G acts freely on $N := \mu^{-1}(0)$. Let $\Sigma \in \mathcal{S}$. We define the map $m_{\Sigma,\omega,\mu} : [\Sigma, M; N, \omega] \rightarrow \mathbb{Z}$ as follows. Let $a \in [\Sigma, M; N, \omega]$. We choose a representative u of a , a symplectic vector space (V, Ω) of dimension $\dim M$, a trivialization $\Psi \in \text{Iso}(\Sigma \times V, \Omega; u^*(TM, \omega))$, and points $z_X \in X$, for every $X \in \mathcal{C}(\partial\Sigma)$. We define $g : \partial\Sigma \rightarrow G$ by defining $g(z)$ to be the unique solution of $u(z) = g(z)u(z_X)$, for every $z \in X$ and $X \in \mathcal{C}(\partial\Sigma)$.

We define $m_{\Sigma,\omega,\mu}(a) := m_\Omega(S^1 \ni z \mapsto \Psi_z^{-1}g(z) \cdot \Psi_1)$, where for every $g_0 \in G$ we denote by $g_0 \cdot : TM \rightarrow TM$ the differential of the action of g_0 . By a standard homotopy argument, this number does not depend on the choices of u, Ψ and z_X . By Lemma 45 below the maps $m_{\Sigma,\omega,\mu}$ and $m_{\Sigma,\omega,N}$ agree.

For $\Sigma = \mathbb{D}$ the map $m_{\mathbb{D},\omega,\mu}$ was introduced by R. Gaio and D. A. Salamon in [GS]. (More precisely, their definition relies on a choice of an ω -compatible almost complex structure J on M and a unitary trivialization of u^*TM .)

Work by M. Entov and L. Polterovich and by V. L. Ginzburg. Let now (M, ω) be a closed (spherically) monotone symplectic manifold and G a torus acting on M in a Hamiltonian way, with moment map μ . Then by Theorem 1.7 in the article [EP] by M. Entov and L. Polterovich the pre-image N of the special element of \mathfrak{g}^* under μ is strongly (i.e. symplectically) non-displaceable.

Assume that the action of G on N is free. Then by Lemma 46 below $N \subseteq M$ is a closed, monotone regular coisotropic submanifold. Hence if

$b_i(N, \mathbb{Z}_2)$ is non-zero for some $i \in \{\dim N - m(N, \omega) + 2, \dots, m(N, \omega) - 2\}$ then it follows from Theorem 3 that N is not leafwise displaceable (and hence not displaceable). Thus in this case we obtain a stronger statement than in Theorem 1.7 in [EP], provided that also $H^1(M, \mathbb{R}) = 0$.

In his recent paper [Gi] (Theorem 1.5) V. L. Ginzburg proved an upper bound on the minimal Maslov number of a closed, stable, displaceable coisotropic submanifold.

1.5. Organization and Acknowledgments.

Organization of the article. In Section 2 it is shown that the Maslov map for pairs of flat transports is well-defined, and Theorem 1 is proved. Section 3 contains the proofs of the other results of Section 1. They are based on Theorem 19, which summarizes the main properties of the Maslov map. Section 4 is devoted to the proof of this theorem, using a similar result for the coisotropic Maslov index for bundles (Theorem 24). The appendix contains some results about the Salamon-Zehnder map, the Gaiò-Salamon Maslov index, the relation with the mixed action-Maslov index, the linear holonomy of a foliation, and some topological results.

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2. Proof of Theorem 1 (Coisotropic Maslov map for bundles)

The following lemma was used in Section 1.

Lemma 7. *The number $m_\omega(\Phi, \Phi')$ in (3) is well-defined, i.e. it does not depend on the choice of z . Furthermore, if Φ and Φ' are regular then $m_\omega(\Phi, \Phi') \in 2\mathbb{Z}$.*

The next Remark is used in the proof of Lemma 7.

Remark 8. *Let X be a topological space and (E, ω) a symplectic vector bundle over X . Then the map $\text{Aut}(E, \omega) \ni (x, \Phi) \mapsto \rho_{\omega_x}(\Phi) \in S^1$ is continuous. To see this, we choose a symplectic vector space (V, Ω) of dimension $\text{rank} E$. Let (U, Φ) be a pair, where $U \subseteq X$ is an open subset and $\Phi \in \text{Iso}(U \times V, \Omega; (E, \omega)|_U)$. By Proposition 42(i) we have $\rho_{\omega_x}(\Psi) = \rho_\Omega(\Phi_x^{-1} \Psi \Phi_x)$, for every $x \in U$ and $\Phi \in \text{Aut}(E_x, \omega_x)$. Since the map $\rho_\Omega : \text{Aut}(\Omega) \rightarrow S^1$ is continuous, the statement follows.*

Proof of Lemma 7. To prove the first assertion, let z_0 and z_1 be two choices of a path z as above. We choose a map $z \in C([0, 1] \times [0, 1], C)$ such that $z(s, 0) = z(s, 1)$, for every $s \in [0, 1]$, and $z(i, \cdot) = z_i$. We denote $z_s := z(s, \cdot)$, and we define $\Psi_s(t) := \Phi'([z_s|_{[0, t]})^{-1} \Phi([z_s|_{[0, t]})$, for $s, t \in [0, 1]$. We also

define $f : [0, 1] \times [0, 1] \rightarrow S^1$ by $f(s, t) := \rho_{\omega_{z_s(0)}}(\Psi_s(t))$. It follows that $\Psi_s(0) = \text{id}_{E_{z_s(0)}}$ and hence $f(s, 0) = 1$, for every $s \in [0, 1]$. By Remark 8 the map f is continuous.

Claim 1. *The map $[0, 1] \ni s \mapsto f(s, 1) \in S^1$ is constant.*

Proof of Claim 1. Consider the case in which Φ is regular. We choose a path $\tilde{z} \in C([0, 1], C)$ such that $\tilde{z}(i) = z_i(0)$, for $i = 0, 1$. We fix $s \in [0, 1]$. By assumption we have $\Phi([\tilde{z} \# z_s \# \tilde{z}]) = \text{id}_{E_{z_0(0)}}$. Furthermore, the paths z_0 and $\tilde{z} \# z_s \# \tilde{z}$ are homotopic with fixed end-points. It follows that $\Phi'([z_0]) = \Phi'([\tilde{z}])^{-1} \Phi'([z_s]) \Phi'([\tilde{z}]) = \Phi'([\tilde{z}])^{-1} \Psi_s(1) \Phi'([\tilde{z}])$. Hence by Proposition 42(i) we have $f(s, 0) = f(s, 1)$. The case in which Φ' is regular, is treated similarly. This proves Claim 1. \square

Claim 1, the fact $\Psi_s(0) = \text{id}_{E_{z_s(0)}}$ (for every $s \in [0, 1]$) and continuity of f imply that $m_{\omega_{z_0(0)}}(\Psi_0) = m_{\omega_{z_1(0)}}(\Psi_1)$. Hence $m_\omega(\Phi, \Phi')$ is well-defined.

The second assertion of the lemma follows directly from the definition of the Maslov index of a path of automorphisms of a symplectic vector space. This proves Lemma 7. \square

For the proof of Theorem 1 we need the following. Let X be a topological manifold and $\mathcal{X} \subseteq C([0, 1], X)$. We define the equivalence relation $\sim_{\mathcal{X}}$ on \mathcal{X} by $x_0 \sim_{\mathcal{X}} x_1$ iff there exists $x \in C([0, 1] \times [0, 1], X)$ such that $x(s, \cdot) \in \mathcal{X}$, $x(s, i) = x(0, i)$ and $x(i, \cdot) = x_i$, for every $s \in [0, 1]$ and $i = 0, 1$. We equip \mathcal{X} with the compact open topology and $\mathcal{X}/\sim_{\mathcal{X}}$ with the quotient topology. Then $\mathcal{X}/\sim_{\mathcal{X}}$ is a topological groupoid. We call \mathcal{X} *admissible* iff it contains the constant paths, and the following conditions hold. If $x \in \mathcal{X}$ and $f \in C([0, 1], [0, 1])$ then $x \circ f \in \mathcal{X}$. Furthermore, if $x, x' \in \mathcal{X}$ are such that $x(1) = x'(0)$ then the concatenation $x \# x'$ lies in \mathcal{X} . Assume that \mathcal{X} is admissible, and let $E \rightarrow X$ be a topological vector bundle. A *flat transport on E along \mathcal{X}* a morphism of topological groupoids $\Phi : \mathcal{X}/\sim_{\mathcal{X}} \rightarrow \text{GL}(E)$ that descends to the identity on $X \times X$. We denote by $\mathcal{T}(\mathcal{X}, E)$ the set of such Φ 's. Let X' be another topological manifold and $f \in C(X', X)$. Then the pullback $f^*\mathcal{X} := (f \circ)^{-1}(\mathcal{X}) \subseteq C([0, 1], X')$ is again admissible. For $\Phi \in \mathcal{T}(\mathcal{X}, E)$ we define the pullback $f^*\Phi \in \mathcal{T}(f^*\mathcal{X}, E)$ by $(f^*\Phi)_{x'} := \Phi_{f \circ x'}$.

Let X be a topological manifold, (E, ω) a symplectic vector bundle over X , $(W, \Phi) \in \mathcal{C}^{\text{flat}}(E, \omega)$ and $\Psi \in \mathcal{T}(E, \omega)$. We call Ψ a *lift* of (W, Φ) iff for every $z \in C([0, 1], C)$ we have $\Psi([z])W_{z(0)} = W_{z(1)}$ and $\Psi([z])W_{z(0)} = \Phi([z])$. Let X be a closed curve. We denote by $\pi : [0, 1] \times X \rightarrow X$ the canonical projection, and for $s \in [0, 1]$, we define $\iota_s : X \rightarrow [0, 1] \times X$ by $\iota_s(z) := (s, z)$. Furthermore, we define $\mathcal{X} := \{t \mapsto (s, z(t)) \mid s \in [0, 1], z \in C([0, 1], X)\}$. Let (E, ω) be a symplectic vector bundle over X .

Theorem 9. *The following statements hold.*

(i) *For every $(W, \Phi) \in \mathcal{C}^{\text{flat}}(E, \omega)$ there exists a lift $\Psi \in \mathcal{T}(E, \omega)$ of (W, Φ) .*

- (ii) Let $(W, \Phi) \in \mathcal{C}^{\text{flat}}(E, \omega)$, Ψ_0 and Ψ_1 be lifts of (W, Φ) , and $\Psi \in \mathcal{T}(E, \omega)$ a regular transport. Then $m_{X, \omega}(\Psi_0, \Psi) = m_{X, \omega}(\Psi_1, \Psi)$.
- (iii) Let $W \subseteq \pi^*E$ be an $\pi^*\omega$ -coisotropic subbundle and $\Phi \in \mathcal{T}(\Pi([0, 1], X), W_\omega, \omega_W)$. Then there exists $\Psi \in \mathcal{T}(E, \omega)$ such that $\iota_s^*\Psi$ is a lift of $\iota_s^*\Phi$, for every $s \in [0, 1]$.
- (iv) Let $W \subseteq \pi^*E$ be an $\pi^*\omega$ -coisotropic subbundle and $\Phi \in \mathcal{T}(\mathcal{X}, W_\omega, \omega_W)$. Then there exists $\Psi \in \mathcal{T}(\mathcal{X}, E, \omega)$ such that $\iota_s^*\Psi$ is a lift of $\iota_s^*\Phi$, for every $s \in [0, 1]$.

For the proof of Theorem 9 we need the following. Let f be a homeomorphism between two topological manifolds X and X' , and $\pi : E \rightarrow X$ and $\pi' : E' \rightarrow X'$ vector bundles. Assume that there exists $\Psi \in \text{Iso}(E', E)$ that descends to f . We define $\Psi^* : \text{GL}(E) \rightarrow \text{GL}(E')$ by $\Psi^*(x_0, x_1, \Phi) := (f^{-1}(x_0), f^{-1}(x_1), \Psi_{f^{-1}(x_1)}^{-1} \Phi \Psi_{f^{-1}(x_0)})$. For $\Phi \in \mathcal{T}(E)$ we define $\Psi^*\Phi : \Pi X' \rightarrow \text{GL}(E')$ by $(\Psi^*\Phi)(a') := \Psi^*(\Phi f_* a')$.

Lemma 10. *We have $\Psi^*\Phi \in \mathcal{T}(E')$.*

Proof of Lemma 10. It follows from the definitions that $\Psi^*\Phi$ descends to the identity on X' . Furthermore, the map $f_* : \Pi X' \rightarrow \Pi X$ is a morphism of topological groupoids. Since $\Phi \in \mathcal{T}(E)$, the same holds for Φ . Finally, it follows from the definitions that $\Psi^* : \text{GL}(E) \rightarrow \text{GL}(E')$ is a morphism of groupoids. It follows from Lemma 63(iii) that it is continuous. It follows that the map $\Pi X' \ni a' \mapsto \Psi^*(\Phi f_* a') \in \text{GL}(E')$ is a flat transport. This proves Lemma 10. \square

Let (V, ω) be a symplectic vector space and $\ell \in \{\dim V/2, \dots, \dim V\}$. We denote by $G(\omega, \ell)$ the set of ω -coisotropic subspaces of V of dimension ℓ , and equip it with the natural smooth structure. Let $W_0 \subseteq W$ be a coisotropic subspace of dimension ℓ . We define the *framed coisotropic Grassmannian* $G(\omega, W_0)$ to be the set of all pairs (W, Φ) , where $W \subseteq V$ is an coisotropic subspace and $\Phi \in \text{Iso}(\omega_{W_0}, \omega_W)$. This set is naturally equipped with a smooth structure.

Lemma 11. *The maps $\text{Aut}(\omega) \rightarrow G(\omega, \ell)$, $\Psi \mapsto \Psi W$, and $\text{Aut}(\omega) \rightarrow G(\omega, W_0)$, $\Psi \mapsto (\Psi W_0, \Psi_{W_0})$, are smooth (locally trivial) fiber bundles.*

For the proof of Lemma 11 we need the following. The group $\text{Iso}(\omega)$ acts naturally on $G(\omega, \ell)$, and it acts on $G(\omega, W_0)$ by $\Psi(W, \Phi) := (\Psi W, \Psi_W \Phi)$. These actions are smooth.

Lemma 12. *They are transitive.*

Proof of Lemma 12. Transitivity of the first action follows by an elementary argument. Let $(W, \Phi) \in G(\omega, W_0)$. Assume first that $W = W_0$. We choose a maximal symplectic subspace $V_0 \subseteq W$, and define $f : V_0 \rightarrow W_\omega$, $f v_0 := v_0 + W^\omega$. Then $f \in \text{Iso}(\omega|_{V_0}, \omega_W)$, and hence we may define $\Psi : V =$

$V_0 \oplus V_0^\omega \rightarrow V$ by $\Psi(v_0 + v_1) := f^{-1}\Phi f v_0 + v_1$. This map has the required properties.

For a general W we choose $\Psi' \in \text{Iso}(\omega)$ such that $\Psi'W_0 = W$. By what we just proved there exists $\Psi'' \in \text{Iso}(\omega)$ such that $\Psi''W_0 = W_0$ and $\Psi''_{W_0} = \Psi'^{-1}_{W_0}\Phi$. The map $\Psi := \Psi'\Psi''$ has the required properties. This proves Lemma 12. \square

Proof of Lemma 11. If a Lie group G acts smoothly on a manifold X and $x \in X$, then the stabilizer H of x is a closed subgroup, and hence the map $G \rightarrow G/H$, $g \mapsto gH$, is a smooth fiber bundle. If the action is transitive then the map $G/H \rightarrow X$, $gH \mapsto gx$, is a diffeomorphism. Lemma 11 follows from this and Lemma 12. \square

Let f be a homeomorphism between two topological manifolds X and X' , and (π, E, ω) and (π', E', ω') be symplectic vector bundles over X and X' respectively. Assume that there exists $F \in \widetilde{\text{Iso}}(E', \omega'; E, \omega)$.

Lemma 13. *The following statements hold.*

- (i) *The map $F^* : \mathcal{C}^{\text{flat}}(E, \omega) \rightarrow \mathcal{C}^{\text{flat}}(E', \omega')$ defined by $F^*(W, \Phi) := (F^{-1}W, F^*\Phi)$, is a bijection.*
- (ii) *Let $W \subseteq E$ be an ω -coisotropic subbundle and $\Psi \in \mathcal{T}(E, \omega)$ be a transport that leaves W invariant. Then the transport $\Psi' := F^*\Psi \in \mathcal{T}(E', \omega')$ leaves the ω' -coisotropic subbundle $W' := F^{-1}W \subseteq E'$ invariant and $\Psi'_{W'} = F^*(\Psi_W)$.*

Proof of Lemma 13. The statements follow from straight-forward arguments. \square

Remark 14 (Naturality for one-dimensional Maslov map). *Let X and X' be closed oriented curves, (E, ω) and (E', ω') symplectic vector bundles over X and X' , respectively, $\Phi, \Phi_0 \in \mathcal{T}(E, \omega)$, and $\Psi \in \text{Iso}(E', \omega'; E, \omega)$. Assume that Φ_0 is regular. Then $m_{X', \omega'}(\Psi^*\Phi, \Psi^*\Phi_0) = m_{X, \omega}(\Phi, \Phi_0)$. This follows from Proposition 42(i).*

Proof of Theorem 9. Without loss of generality, we may assume that X is connected.

To prove **statement (i)**, assume first that $X = \mathbb{R}/\mathbb{Z}$ and there exists a symplectic vector space (V, Ω) such that $E = \mathbb{R}/\mathbb{Z} \times V$ and ω is constantly equal to Ω . Let $(W, \Phi) \in \mathcal{C}^{\text{flat}}(\mathbb{R}/\mathbb{Z} \times V, \omega)$. It follows from Lemma 11 that there exists a path $\tilde{\Psi} \in C([0, 1], \text{Aut}\Omega)$ such that $\tilde{\Psi}(0) = \text{id}_V$ and $\tilde{\Psi}(s)W_{0+\mathbb{Z}} = W_{s+\mathbb{Z}}$ and $\tilde{\Psi}(s)_{W_{0+\mathbb{Z}}} = \Phi([0, 1] \ni t \mapsto st + \mathbb{Z})$, for every $s \in [0, 1]$. By an elementary argument there exists a unique transport $\Psi \in \mathcal{T}(E, \omega)$ satisfying $\Psi([0, 1] \ni t \mapsto st + \mathbb{Z}) = \tilde{\Psi}(s)$, for every $s \in [0, 1]$. This is a lift of (W, Φ) , as required.

In the general situation, we choose a homeomorphism $f : \mathbb{R}/\mathbb{Z} \rightarrow X$ and a symplectic vector space (V, Ω) of dimension $\text{rank}E$. Since $\text{Aut}\Omega$ is connected,

there exists $F \in \text{Iso}(\mathbb{R}/\mathbb{Z} \times V, \Omega; E, \omega)$ that descends to f . Statement (i) follows now from what we already proved and Lemma 13.

To prove **statement (ii)**, we choose a symplectic vector space (V, Ω) of dimension $\text{rank} E$. Without loss of generality, we may assume that $E = X \times V$, ω is constantly equal to Ω , and $\Psi \equiv \text{id}$. (To see this, we choose $z_0 \in X$ and we define $F \in \text{Iso}(X \times V, \Omega; E, \omega)$ by $F_{z_1} := \Psi([z])$, for $z_1 \in X$, where $z \in C([0, 1], X)$ is a path such that $z(i) = z_i$, for $i = 0, 1$. By regularity of Ψ the map F is well-defined. The claimed equality is a consequence of the equality $m_{X, \omega}(F^* \Psi_0, \text{id}) = m_{X, \omega}(F^* \Psi_1, \text{id})$, the fact $F^* \Psi \equiv \text{id}$, and Remark 14.)

Let $\Psi_0, \Psi_1 \in \mathcal{T}(X \times V, \omega)$ be lifts of (W, Φ) . We choose a path $z \in C([0, 1], X)$ such that $z(0) = z(1)$ and the map $S^1 \cong [0, 1]/\{0, 1\} \ni [t] \mapsto z(t) \in X$ has degree one. We define $\pi : \text{Aut} \omega \rightarrow \text{G}(\omega, W_{z(0)})$ by $\pi(F) := (FW_{z(0)}, FW_{z(0)})$, and $\tilde{\Psi}_i : [0, 1] \rightarrow \text{Aut} \omega$ by $\tilde{\Psi}_i(t) := \Psi_i([z|_{[0, t]})$, for $i = 0, 1$. Then $\pi \circ \tilde{\Psi}_0(t) = (W_{z(t)}, \Phi([z|_{[0, t]})) = \pi \circ \tilde{\Psi}_1(t)$, for every $t \in [0, 1]$, and $\tilde{\Psi}_0(0) = \text{id}_V = \tilde{\Psi}_1(0)$. Therefore, Lemma 11 implies that there exists $\tilde{\Psi} \in C([0, 1] \times [0, 1], \text{Aut} \omega)$ such that $\tilde{\Psi}(i, \cdot) = \tilde{\Psi}_i$, for $i = 0, 1$, $\tilde{\Psi}(s, 0) = \text{id}_V$, and $\pi \circ \tilde{\Psi}(s, t) = (W_{z(t)}, \Phi([z|_{[0, t]}))$, for every $s, t \in [0, 1]$. Therefore, the hypotheses of Proposition 27 are satisfied with $x(s, t) := z(t)$ and $\Psi := \tilde{\Psi}$. By the assertion of that proposition, we have $m_\omega(\tilde{\Psi}_0) = m_\omega(\tilde{\Psi}_1)$. Since $m_{X, \omega}(\Psi_i, \text{id}) = m_\Omega(\tilde{\Psi}_i)$, for $i = 0, 1$, it follows that $m_{X, \omega}(\Psi_0, \text{id}) = m_{X, \omega}(\Psi_1, \text{id})$. This proves statement (ii).

Statements (iii, iv) are proved similarly to statement (i).

This completes the proof of Theorem 9. \square

Lemma 15. *Let Σ be a compact connected oriented surface with non-empty boundary, (E, ω) a symplectic vector bundle over Σ , $\Phi, \Phi' \in \mathcal{T}(E, \omega)$ regular transports, and $\Psi \in \mathcal{T}((E, \omega)|_{\partial \Sigma})$. Then $m_{\partial \Sigma, \omega|_{\partial \Sigma}}(\Psi, \Phi|_{\partial \Sigma}) = m_{\partial \Sigma, \omega|_{\partial \Sigma}}(\Psi, \Phi'|_{\partial \Sigma})$.*

For the proof of Lemma 15 we need the following.

Lemma 16. *Let (V, ω) be a symplectic vector space, and $\Phi, \Psi \in C([0, 1], \text{Aut} \omega)$ be such that $\Phi(0) = \Phi(1) = \text{id}$ and $\Psi(0) = \text{id}$. Then $m_\omega(\Phi\Psi) = m_\omega(\Phi) + m_\omega(\Psi)$.*

Proof of Lemma 16. By an elementary argument, the map $\Phi\Psi$ is homotopic with fixed end-points to the concatenation of Φ with Ψ . The statement follows from this. \square

Lemma 17. *Let Σ be compact connected oriented surface with non-empty boundary, (E, ω) a symplectic vector bundle over Σ , and $\Phi, \Psi \in \mathcal{T}(E, \omega)$ be regular transports. Then $m_{\partial \Sigma, \omega|_{\partial \Sigma}}(\Phi|_{\partial \Sigma}, \Psi|_{\partial \Sigma}) = 0$.*

Proof of Lemma 17. For $z_0 \in \Sigma$ we define $f_{z_0} : \Sigma \rightarrow S^1$ by $f_{z_0}(z_1) := \rho_{\omega_{z_0}}(\Psi([z])^{-1}\Phi([z]))$, for $z_1 \in \Sigma$, where $z \in C([0, 1], \Sigma)$ is a path such that

$z(i) = z_i$, for $i = 0, 1$. By regularity of Φ and Ψ this map is well-defined. Let X be a connected component of $\partial\Sigma$. Then for every $z_0 \in X$ we have $\deg(f_{z_0}|_X) = m_{X,\omega|_X}(\Phi|_X, \Psi|_X)$. Furthermore, for $z_0, z'_0 \in \Sigma$ the maps f_{z_0} and $f_{z'_0}$ are homotopic and hence $\deg(f_{z_0}|_X) = \deg(f_{z'_0}|_X)$. Let $z_0 \in \Sigma$. It follows that $m_{\partial\Sigma,\omega|_{\partial\Sigma}}(\Phi|_{\partial\Sigma}, \Psi|_{\partial\Sigma}) = \deg(f_{z_0}|_{\partial\Sigma}) = 0$. This proves Lemma 17. \square

Proof of Lemma 15. Let X be a connected component of $\partial\Sigma$. We choose a path $z \in C([0, 1], X)$ such that $z(0) = z(1)$ and the map $S^1 \cong [0, 1]/\{0, 1\} \ni [t] \mapsto z(t) \in X$ has degree one. For $s \in [0, 1]$ we define $z_s \in C([0, 1], X)$ by $z_s(t) := z(st)$. Furthermore, we define $F \in C([0, 1], \text{Aut}(\omega_{z(0)}))$ by $F(s) := \Phi'([z_s])^{-1}\Phi([z_s])$. By definition, we have $m_{X,\omega|_X}(\Phi|_X, \Phi'|_X) = m_{\omega_{z(0)}}(F)$. Therefore, using Lemma 16, we obtain $m_{X,\omega|_X}(\Psi, \Phi'|_X) = m_{X,\omega|_X}(\Psi, \Phi|_X) + m_{X,\omega|_X}(\Phi|_X, \Phi'|_X)$. The claimed equality follows now from Lemma 17. This proves Lemma 15. \square

Remark 18. *Let Σ be a compact connected oriented surface with non-empty boundary, and (E, ω) a symplectic vector bundle over Σ . Then there exists a regular transport $\Phi \in \mathcal{T}(E, \omega)$. To see this, we choose a symplectic vector space (V, Ω) of dimension $\text{rank}E$. Since $\text{Aut}(\omega)$ is connected and $\partial\Sigma \neq \emptyset$, there exists $\Psi \in \text{Iso}(\Sigma \times V, \Omega; E, \omega)$. We define $\Phi \in \mathcal{T}(E, \omega)$ by $\Phi([z]) := \Psi_{z(1)}\Psi_{z(0)}^{-1}$.*

Proof of Theorem 1. We show existence of the map $m_{\Sigma, E, \omega}$. Let $(W, \Phi) \in \mathcal{C}^{\text{flat}}(E, \omega)$. By Remark 18 we may choose a regular transport $\Psi_0 \in \mathcal{T}(E, \omega)$. By Theorem 9(i) we may choose a lift $\Psi \in \mathcal{T}((E, \omega)|_{\partial\Sigma})$ of (W, Φ) . We define $m_{\Sigma, E, \omega}(W, \Phi) := m_{\partial\Sigma, \omega|_{\partial\Sigma}}(\Psi, \Psi_0)$. By Theorem 9(ii) and Lemma 15 this number does not depend on the choices of Ψ and Ψ_0 . Furthermore, the conditions (i,ii) follow from the definition of m_{Σ} .

To show uniqueness of the map $m_{\Sigma, E, \omega}$, let $m_{\Sigma, E, \omega} : \mathcal{C}^{\text{flat}}(E, \omega) \rightarrow \mathbb{R}$ be a map satisfying (i,ii). Let $(W, \Phi) \in \mathcal{C}^{\text{flat}}(E, \omega)$. By Theorem 9(i) and Remark 18 we may choose a lift $\Psi \in \mathcal{T}((E, \omega)|_{\partial\Sigma})$ of (W, Φ) and a regular transport $\Psi_0 \in \mathcal{T}(E, \omega)$. By condition (ii) we have $m_{\Sigma, E, \omega}(W, \Phi) = m_{\Sigma, E, \omega}(E|_{\partial\Sigma}, \Psi) = m_{\partial\Sigma, \omega|_{\partial\Sigma}}(\Psi, \Psi_0)$. Uniqueness follows.

This proves Theorem 1. \square

3. Proofs of Theorems 3, 4, 6, and Proposition 5

For the proof of these results, we need the following theorem, which summarizes some properties of the Maslov map. If $\Sigma \in \mathcal{S}$, (M, ω) and (M', ω') are symplectic manifolds, and $N \subseteq M$ and $N' \subseteq M'$ are coisotropic submanifolds, then there is a canonical bijection $\Phi_{M, M', \omega, \omega', N, N'} : [\Sigma, M; N, \omega] \times [\Sigma, M'; N', \omega'] \rightarrow [\Sigma, M \times M'; N \times N', \omega \oplus \omega']$. If X and X' are sets and $f : X \rightarrow \mathbb{R}$ and $f' : X' \rightarrow \mathbb{R}$ are maps then we define $f \oplus f' : X \times X' \rightarrow \mathbb{R}$

by $f \oplus f'(x, x') := f(x) + f'(x')$. Let X be a manifold and $Y \subseteq X \setminus \partial X$ is a submanifold of codimension one. We define X_Y to be the manifold with boundary obtained from X by cutting along Y . (Note that if X is orientable then $\partial X_Y = \partial X \amalg Y \amalg Y$.) There is a canonical map $f_{X,Y} : X_Y \rightarrow X$. If $\Sigma \in \mathcal{S}$, (M, ω) is a symplectic manifold and $L \subseteq M$ is a Lagrangian submanifold then we denote by $m_{\Sigma, \omega}^L$ the Lagrangian Maslov map (see the appendix, (32)). For a manifold X we denote by \sim_X the equivalence relation on X given by $x \sim_X x'$ iff $x = x'$ or x and x' lie in the same connected component of ∂X , and we denote by $\pi_X : X \rightarrow X/\sim_X$ the canonical projection. Let now $\Sigma, \Sigma' \in \mathcal{S}$, $f : \Sigma' \rightarrow \Sigma$ be an embedding (restricting an embedding of $\partial \Sigma'$ into $\partial \Sigma$), M and M' manifolds of the same dimension, ω a symplectic form on M , $N \subseteq M$ a coisotropic submanifold and $\varphi : M' \rightarrow M$ an embedding. We denote $\varphi^*N := \varphi^{-1}(N)$. The map φ induces a map $\varphi_* : [\Sigma', M'; \varphi^*(N, \omega)] \rightarrow [\Sigma, M; N, \omega]$. Recall the definition (7,8). We define the map $\varphi : \langle \mathbb{D}, M; N, \omega \rangle \rightarrow [\mathbb{D}, S^1; \widetilde{M}, \widetilde{N}]$ by $\varphi(a) := [u, u(z_0)]$, where u is an arbitrary representative of a and $z_0 \in S^1$ is any point. This map is well-defined.

Theorem 19 (Properties of the Maslov map). *The following assertions hold.*

- (i) (Naturality) Let $\Sigma, \Sigma', f, M, M', N$ and φ be as above. If f is surjective and orientation preserving then $m_{\Sigma', \varphi^*(\omega, N)} = m_{\Sigma, \omega, N} \circ \varphi_*$.
- (ii) (Product) If $\Sigma \in \mathcal{S}$, (M, ω) and (M', ω') are symplectic manifolds, and $N \subseteq M$ and $N' \subseteq M'$ are coisotropic submanifolds, then $m_{\Sigma, \omega, N} \oplus m_{\Sigma, \omega', N'} = m_{\Sigma, \omega \oplus \omega', N \times N'} \circ \Phi_{M, M', \omega, \omega', N, N'}$.
- (iii) If $u \in C([0, 1] \times \Sigma, M)$ is an admissible homotopy then the map $[0, 1] \ni t \mapsto m_{\Sigma, N}(u(t, \cdot)) \in \mathbb{R}$ is constant.
- (iv) If $u \in C([0, 1] \times \Sigma, M)$ is a weakly (N, ω) -admissible homotopy then the map $[0, 1] \ni t \mapsto m_{\Sigma, N}(u(t, \cdot)) \in \mathbb{R}$ is continuous.
- (v) (Splitting) Let $\Sigma \in \mathcal{S}$, $C \subseteq \Sigma \setminus \partial \Sigma$ a closed curve (possibly disconnected), (M, ω) a symplectic manifold, $N \subseteq M$ a coisotropic submanifold, and $u \in C(\Sigma, M)$ be such that for every $C' \in \mathcal{C}(\partial \Sigma \cup C)$ there exists $F \in N_\omega$ such that $u(C') \subseteq F$. Then $m_{\omega, N}(u) = m_{\omega, N}(u \circ f_{\Sigma, C})$.
- (vi) (Regular case) If N is regular then $\text{im}(m_{\Sigma, N}) \subseteq \mathbb{Z}$. If N is also orientable then $\text{im}(m_{\Sigma, N}) \subseteq 2\mathbb{Z}$.
- (vii) (Lagrangian case) Let $\Sigma \in \mathcal{S}$, (M, ω) be a symplectic manifold and $L \subseteq M$ a Lagrangian submanifold. Then $m_{\Sigma, \omega, L} = m_{\Sigma, \omega}^L$.
- (viii) (Removal of point) Let $\Sigma \in \mathcal{S}$ be such that $\partial \Sigma \neq \emptyset$, $C \in \mathcal{C}(\Sigma)$, (M, ω) be a symplectic manifold, $N \subseteq M$ a coisotropic submanifold, and $u \in C(\Sigma, M; N, \omega)$. Assume that u maps C to a point in N . We define $\tilde{u} : \Sigma/\sim_C \rightarrow M$ by $\tilde{u}([z]) := u(z)$. Then $m_{M, \omega, N}(u) = m_{M, \omega, N}(\tilde{u})$.

- (ix) (Chern class of quotient) Let M, ω and N be as above, with N regular. Let $u \in C(\mathbb{D}, M; N, \omega)$ be such that $u(\mathbb{D}) \subseteq N$. We define $u' : \mathbb{D}/\sim_{\mathbb{D}} \cong S^2 \rightarrow N_\omega$ by $u'([z]) := \pi_N \circ u(z)$. Then $m_{M, \omega, N}(u) = 2c_1^{N_\omega, \omega_N}(u')$.
- (x) Let (M, ω) be a symplectic manifold and $N \subseteq M$ a regular coisotropic submanifold. Then $m_{\mathbb{D}, M, \omega, N} = m_{\mathbb{D}, \widetilde{M}, \widetilde{\omega}, \widetilde{N}} \circ \varphi$.

The proof of Theorem 19 is given on page 22. The proof of Theorem 3 is based on the following result, which is due to P. Albers.

Theorem 20 ([AI], Corollary 2.3). *Let (M, ω) be a bounded symplectic manifold, $L \subseteq M$ a closed monotone Lagrangian submanifold of minimal Maslov number $m(L)$, and $\varphi \in \text{Ham}(M, \omega)$ be such that $L \pitchfork \varphi(L)$. Then $|L \cap \varphi(L)| \geq \sum_{i=\dim L - m(L) + 2}^{m(L) - 2} b_i(L, \mathbb{Z}_2)$.*

Note that in [AI], Corollary 2.3, it is assumed that M is closed. However, the proof of the result carries over to the case in which (M, ω) is bounded.

Proof of Theorem 3. Without loss of generality we may assume that N is connected. Since N is regular, there exists a unique smooth structure $\mathcal{A}_{N, \omega}$ on the set of isotropic leaves N_ω such that the canonical projection $\pi_N : N \rightarrow N_\omega$ is a submersion. (See [Zi], Lemma 15.) We define $\widetilde{M}, \widetilde{\omega}, \iota_N$ and \widetilde{N} as in (7,8), and $\widetilde{\varphi} := \varphi \times \text{id}_{N_\omega} : \widetilde{M} \rightarrow \widetilde{M}$. Then \widetilde{M} is closed, the map $\iota_N : N \rightarrow \widetilde{M}$ is an embedding, and its image \widetilde{N} is a closed Lagrangian submanifold, see [Zi], Lemma 8. By the same lemma, $\widetilde{\varphi}(\widetilde{N}) \pitchfork \widetilde{N}$. We denote by $R^{N, \omega}$ the isotropic leaf relation on N . By Ehresmann's fibration theorem the map π_N is a smooth (locally trivial) fiber bundle. (See [Eh], the proposition on p. 31.) Hence the hypotheses of Proposition 61 with $(X, Y, \sim, \iota, \pi, k) := (M, N, R^{N, \omega}, \iota_N, \pi_N, 2)$ are satisfied. Therefore, by the statement of this result and by Theorem 19(x) the Lagrangian \widetilde{N} is monotone and $m(\widetilde{N}, \widetilde{\omega}) = m(N, \omega)$. Therefore, the hypotheses of Theorem 20 are satisfied with M, ω replaced by $\widetilde{M}, \widetilde{\omega}$, and $L := \widetilde{N}$. Inequality (6) follows from the statement of this theorem and the fact $|\text{Fix}(\varphi, N)| = |\widetilde{N} \cap \widetilde{\varphi}(\widetilde{N})|$, see [Zi], Lemma 8. This proves Theorem 3. \square

For the proof of Theorem 4 we need the following. Let (M, ω) and (M', ω') be presymplectic manifolds such that $\dim M + \text{corank } \omega = \dim M' + \text{corank } \omega'$. Assume that there exists a presymplectic embedding φ of (M', ω') into (M, ω) . Then $N := \varphi(M') \subseteq M$ is a coisotropic submanifold (see [Zi]). Furthermore, (M', ω') is regular if and only if N is regular.

Proposition 21. *Assume that $\text{corank } \omega = 0$, (M', ω') is regular and for every isotropic leaf $F \subseteq M'$ every loop $u \in C(S^1, F)$ is contractible in M' . If there exists a constant $c \in \mathbb{R}$ such that $2c_1^{M, \omega} = c[\omega]$ on $[S^2, M]$ and $2c_1^{M', \omega'} = c[\omega']$ on $[S^2, M']$, then $m_{\mathbb{D}, N, \omega}(u) = c \int_{\mathbb{D}} u^* \omega$, for every*

$u \in C^\infty(\mathbb{D}, M; N, \omega)$. Furthermore, if M is connected then

$$(9) \quad m(N, \omega) = 2 \operatorname{gcd}(c_1(M, \omega), c_1(M'_{\omega'}, \omega'_{M'})).$$

Proof of Proposition 21. To prove the first statement, assume that there exists a constant $c \in \mathbb{R}$ such that $2c_1^{M, \omega} = c[\omega]$ on $[S^2, M]$ and $2c_1^{M'_{\omega'}, \omega'_{M'}} = c[\omega']$ on $[S^2, M']$. Let $a \in [\mathbb{D}, M; N, \omega]$. We choose a smooth representative $u \in a$. Then $\varphi^{-1} \circ u|_{S^1}$ is a continuous loop in $M'_{\varphi^{-1} \circ u(1)}$, and hence by assumption it is contractible in M' . Hence there exists $v \in C(\mathbb{D}, M')$ such that $v|_{S^1} = \varphi^{-1} \circ u|_{S^1}$. Smoothing the map $\varphi \circ v$ out, we obtain a map $w \in C^\infty(\mathbb{D}, N)$ such that $w|_{S^1} = u|_{S^1}$. We denote by $\overline{\mathbb{D}}$ the disk with the reversed orientation and by $u \# w : \mathbb{D} \# \overline{\mathbb{D}} \rightarrow M$ the connected sum of u and w . We have

$$\begin{aligned} m_{M, \omega, N}(a) &= m(u^*(TM, \omega), u|_{S^1}^*(TN, \operatorname{hol}^{N, \omega})) \\ &= 2c_1^{M, \omega}((u \# w)^*(TM, \omega) - m(w^*(TM, \omega), w|_{S^1}^*(TN, \operatorname{hol}^{N, \omega}))) \end{aligned}$$

The first statement follows from this.

To prove the second statement, assume that M is connected. We claim that

$$(10) \quad m([\mathbb{D}, M; N, \omega]) = 2c_1^{M, \omega}([S^2, M]) + 2c_1^{M'_{\omega'}, \omega'_{M'}}([S^2, M'_{\omega'}]).$$

In order to show that the inclusion “ \subseteq ” in (10) holds, let $a \in [\mathbb{D}, M; N, \omega]$. We choose a representative $u \in C(\mathbb{D}, M; N, \omega)$ of a . By assumption the map $\pi_1(N_{u(1)}^\omega) \rightarrow \pi_1(E)$ vanishes. Hence there exists $\tilde{u} \in C(\mathbb{D}, N)$ such that $\tilde{u}|_{S^1} = u|_{S^1}$. We denote by $\overline{\mathbb{D}}$ the disk with the opposite orientation, and define v to be the connected sum $u \# \tilde{u} : \mathbb{D} \# \overline{\mathbb{D}} \cong S^2 \rightarrow M$. It follows from Theorem 19(v) that $m_{M, \omega, N}(a) = 2c_1^{M, \omega}(v) - m_{M, \omega, N}(\tilde{u})$. We define $u' : \mathbb{D}/\sim_{\mathbb{D}} \cong S^2 \rightarrow N_\omega$ by $u'([z]) := \pi_N \circ \tilde{u}(z)$. By Theorem 19(ix) we have $m_{M, \omega, N}(\tilde{u}) = 2c_1^{N_\omega, \omega^N}(u')$. The inclusion “ \subseteq ” in (10) follows.

To prove the inclusion “ \supseteq ”, observe that $c_1^{M, \omega}([S^2, M]) = c_1^{M, \omega}([\mathbb{D}/\sim_{\mathbb{D}}, M])$ and $c_1^{M'_{\omega'}, \omega'_{M'}}([S^2, M'_{\omega'}]) = c_1^{M'_{\omega'}, \omega'_{M'}}([\mathbb{D}/\sim_{\mathbb{D}}, M'_{\omega'}])$, since \mathbb{D}/\sim is homeomorphic to S^2 . Let $a \in [\mathbb{D}/\sim_{\mathbb{D}}, M]$. Since by assumption M is connected, there exists a representative $u \in C(\mathbb{D}/\sim_{\mathbb{D}}, M)$ of a such that $u([1]) \in N$. It follows from Theorem 19(viii) that $m_{M, \omega, N}(u \circ \pi_{\mathbb{D}}) = 2c_1^{M, \omega}(u)$. It follows that $2c_1^{M, \omega}([S^2, M]) \subseteq m_{M, \omega, N}([\mathbb{D}, M; N, \omega])$.

Let now $a' \in [\mathbb{D}/\sim_{\mathbb{D}}, M'_{\omega'}]$. We choose a representative $u' \in C(\mathbb{D}/\sim_{\mathbb{D}}, M'_{\omega'})$ of a' . We claim that there exists a map $v : \mathbb{D} \rightarrow M'$ such that $\pi_{M'} \circ v = u'$. To see this, we define $h' : [0, 1] \times S^1 \rightarrow M'$ by $h'(r, z) := u'(rz)$, and we choose $x_0 \in \pi_N^{-1}(u'(0)) \subseteq N$. By the homotopy lifting property there exists a map $h : [0, 1] \times S^1 \rightarrow M'$ such that $\pi_{M'} \circ h = h'$ and $h(0, z) = x_0$, for every $z \in S^1$. We define $v : \mathbb{D} \rightarrow M'$ by $v(0) := x_0$ and $v(z) := h(|z|, z/|z|)$, for every $z \neq 0$. This map has the required properties. This proves the claim.

We define $\varphi' : M'_{\omega'} \rightarrow N_{\omega}$ to be the unique map satisfying $\pi_N \circ \varphi = \varphi' \circ \pi_{M'}$. Then $\varphi' \in \text{Iso}(\omega'_{M'}, \omega_N)$. Theorem 19(ix) implies that $m_{M,\omega}(\varphi \circ v) = 2c_1^{N_{\omega}, \omega_N}(\varphi' \circ u')$. Furthermore, by Theorem 19(i) we have $c_1^{N_{\omega}, \omega_N}(\varphi' \circ u') = c_1^{M'_{\omega'}, \omega'_{M'}}(u')$. It follows that $2c_1^{M', \omega'}([S^2, M']) \subseteq m_{M,\omega,N}(\mathbb{D}, M; N, \omega)$. The inclusion “ \supseteq ” in (10) follows. This proves (10). Since M is connected, we have $c_1^{M,\omega}([S^2, M]) = c_1(M, \omega)\mathbb{Z}$ and $c_1^{M'_{\omega'}, \omega'_{M'}}([S^2, M'_{\omega'}]) = c_1(M'_{\omega'}, \omega'_{M'})\mathbb{Z}$ (with the convention $\infty\mathbb{Z} = \{0\}$). Combining this with (10), the second statement follows.

This completes the proof of Proposition 21. \square

Proof of Theorem 4. Let (M, ω) be a connected symplectic manifold and (M', ω') a regular connected presymplectic manifold. We define

$$\mu := 2 \gcd(c_1(M, \omega), c_1(M'_{\omega'}, \omega'_{M'})).$$

Assume that $\dim M' + \text{corank} \omega' = \dim M$ and there exists an embedding φ of (M', ω') into (M, ω) . It follows that $N := \varphi(M') \subseteq M$ is a regular coisotropic submanifold (see [Zi]). Furthermore, if there exists a constant $c \in \mathbb{R}$ such that $2c_1^{M,\omega} = c[\omega]$ on $[S^2, M]$ and $2c_1^{M'_{\omega'}, \omega'_{M'}} = c[\omega']$ on $[S^2, M'_{\omega'}]$ then Proposition 21 implies that the coisotropic submanifold $N := \varphi(M') \subseteq M$ is monotone and $m(N, \omega) = \mu$. Hence the statement of Theorem 4 follows from Theorem 3. \square

For the proof of Proposition 5, we need the following remarks.

Remark 22. *Let (M, ω) be a connected symplectic manifold. Then $c_1^{M,\omega}([S^2, M]) = c_1(M, \omega)\mathbb{Z}$, if $c_1(M, \omega) < \infty$, and $c_1^{M,\omega}([S^2, M]) = \{0\}$, otherwise. To see this, we choose a point $x_0 \in M$. Then the composition of the forgetful map $\pi_2(M, x_0) \rightarrow [S^2, M]$ with the map $c_1^{M,\omega} : [S^2, M] \rightarrow \mathbb{Z}$ is a group homomorphism. The statement follows from this.*

Proof of Proposition 5. Let M, ω, M', ω' and F be as in the hypothesis. Using Remark 22, the statement of Proposition 5 is a consequence of the following.

Claim 1. *For every $a' \in [S^2, M'_{\omega'}]$ there exists $a \in [S^2, M]$ such that $\langle [\omega'_{M'}], a' \rangle = \langle [\omega], a \rangle$ and $c_1^{M'_{\omega'}, \omega'_{M'}}(a') = c_1^{M,\omega}(a)$.*

Proof of Claim 1: We choose an isotropic leaf $F \subseteq M'$ and an orientation preserving homeomorphism $f : \mathbb{D}/\sim_{\mathbb{D}} \rightarrow S^2$. Since F is simply-connected, it follows from the long exact homotopy sequence for the fibration $\pi_{M'} : M' \rightarrow M'_{\omega'}$ that there exists $u' \in C(S^2, M')$ such that $[\pi_{M'} \circ u'] = a'$. We define $a := [\varphi \circ u']$. To see that a has the required properties, we denote by $\pi_{\mathbb{D}} : \mathbb{D} \rightarrow \mathbb{D}/\sim_{\mathbb{D}}$ the canonical projection. Then $\langle [\omega], a \rangle = \langle [\omega'], [u'] \rangle = \langle [\omega'_{M'}], a' \rangle$. We define $u := \varphi \circ u' \circ f \circ \pi_{\mathbb{D}}$. Since $N \subseteq M$ is a coisotropic submanifold, it

follows from Theorem 19(viii,i) that $m_{M,\omega,N}(u) = m_{M,\omega,N}(\varphi \circ u' \circ f) = m_{M,\omega,N}(\varphi \circ u') = 2c_1^{M,\omega}(a)$. On the other hand, by Theorem 19(ix) we have $m_{M,\omega,N}(u) = 2c_1^{N\omega,\omega_N}(\pi_N \circ \varphi \circ u' \circ f)$. We denote by $\pi_{M'} : M' \rightarrow M'_\omega$, the canonical projection. The map $M'_\omega \ni \pi_{M'}(x') \mapsto \pi_N \circ \varphi(x') \in N_\omega$ is a well-defined $(\omega'_{M'}, \omega_N)$ -isomorphism. Therefore, Theorem 19(i) implies that $c_1^{N\omega,\omega_N}(\pi_N \circ \varphi \circ u' \circ f) = c_1^{M'_\omega,\omega'_{M'}}(\pi_{M'} \circ u') = c_1^{M'_\omega,\omega'_{M'}}(a')$. It follows that $c_1^{M,\omega}(a) = c_1^{M'_\omega,\omega'_{M'}}(a')$. This proves Claim 1 and completes the proof of Proposition 5. \square

The proof of Theorem 6 is based on the following result, which is due to K. Fukaya.

Theorem 23 ([Fu], Theorem 12.2.). *Let (M, ω) be a symplectic manifold and $L \subseteq M$ a Lagrangian submanifold. Assume that (M, ω) is convex at infinity and L is closed, relatively spin, aspherical and displaceable in a Hamiltonian way. Then there exists $a \in [\mathbb{D}, S^1; M, L]$ such that $m_{\mathbb{D},\omega,L}(a) = 2$.*

Proof of Theorem 6. Since N is regular and orientable, by Theorem 19(vi) we have $\text{im}(m_{\mathbb{D},\omega,N}) \subseteq 2\mathbb{Z}$. Hence the statement follows from Theorem 23 applied with M, ω replaced by $\widetilde{M}, \widetilde{\omega}$ and $L := \widetilde{N}$ (as in (7,8)), Propositions 61 and Theorem 19(x). \square

4. Proof of Theorem 19 (Properties of the Maslov map)

The proof of Theorem 19 is based on the following. Let X be a topological manifold. We define $\mathcal{E}_X^0 \subseteq \mathcal{E}_X$ to be the subclass of all quadruples (E, ω, W, Φ) such that $W = E|_{\partial\Sigma}$ and for every $x \in C([0, 1], \partial X)$ satisfying $x(0) = x(1)$ we have $\Phi([x]) = \text{id}$. Furthermore, we define $\mathcal{E}_X^L \subseteq \mathcal{E}_X$ to be the subclass of all quadruples (E, ω, W, Φ) such that $W \subseteq E|_{\partial\Sigma}$ is Lagrangian.

Let X be a topological manifold and $Y \subseteq X \setminus \partial X$ a hypersurface (i.e. a (real) codimension one submanifold) without boundary. Assume that Y is closed as a subset. Then cutting X along Y we obtain a manifold with boundary X_Y . We denote by $\text{pr}_Y^X : X_Y \rightarrow X$ the natural map, and define $Y^X := (\text{pr}_Y^X)^{-1}(Y) \subseteq X_Y$. (Note that if Y is co-orientable in X then Y^X consists of two copies of Y .) As an example, let Y be a topological manifold. We define $X := \mathbb{R} \times Y$. Then $X_Y = ((-\infty, 0] \times Y) \amalg ([0, \infty) \times Y)$ and $Y^X = (\{0\} \times Y) \amalg (\{0\} \times Y)$. We define $m_\Sigma^L : \mathcal{E}_\Sigma^L \rightarrow \mathbb{Z}$ as in (32) in the appendix.

Let X be topological manifold, $Y \subseteq X$ a closed subset, $E \rightarrow X$ a real vector bundle and $\Phi : Y \times Y \rightarrow \text{GL}(E)$ a morphism of topological groupoids whose composition with the canonical projection $\text{GL}(E) \rightarrow X \times X$ is the identity. We denote by X/Y the topological space obtained by collapsing Y to a point. Furthermore, we define the equivalence relation \sim_Φ on E

by $(x, v) \sim_{\Phi} (x', v')$ iff $(x, v) = (x', v')$ or $(x, x' \in Y$ and $v' = \Phi_x^{x'} v)$. We define $\pi_{\Phi} : E / \sim_{\Phi} \rightarrow X/Y$ by $\pi_{\Phi}([x, v]) := [x]$. Assume that there exists a pair (U, r) , where $U \subseteq X$ is an open neighborhood of Y and $r \in C([0, 1] \times U, U)$ is a strong deformation retraction to Y . Then by Lemma 59 below $(E_{\Phi} := E / \sim_{\Phi}, \pi_{\Phi})$ is a vector bundle. Let $k \in \mathbb{N}$ and $T : E^{\oplus k} \rightarrow \mathbb{R}$ be a tensor, such that $T(\Phi_x^{x'} v_1, \dots, \Phi_x^{x'} v_k) = T(v_1, \dots, v_k)$, for every $x, x' \in Y$ and $v_1, \dots, v_k \in E_x$. We define $T_{\Phi} : E_{\Phi}^{\oplus k} \rightarrow \mathbb{R}$ by $T_{\Phi}([x, v_1], \dots, [x, v_k]) := T_x(v_1, \dots, v_k)$. By Lemma 59 this is a tensor. Let now $(E, \omega, W, \Phi) \in \mathcal{E}_{\Sigma}$ and $C \subseteq \partial\Sigma$ be a connected component. Assume that $W|_C = E_C$ and $\Phi|_C$ is regular. We define $\Psi : C \times C \rightarrow \text{GL}(E)$ by $\Psi_{z_0}^{z_1} := \Phi([z]) : E_{z_0} \rightarrow E_{z_1}$, where $z \in C([0, 1], C)$ is any path such that $z(i) = z_i$ for $i = 0, 1$. By regularity of Φ this map is well-defined. We denote $(E, \omega, W, \Phi)/C := (E_{\Psi}, \omega_{\Psi}, W|_{\partial\Sigma \setminus C}, \Phi|_{\partial\Sigma \setminus C}) \in \mathcal{E}_{\Sigma/C}$. Assume that $\Sigma = [0, 1] \times S^1$, there exists a vector space V such that $E = \Sigma \times V$, and ω is constant. For every point $z_0 \in S^1$ define $\Psi'_{z_0} : \Sigma' \times V \rightarrow E'$ by $\Psi'_{[t, z]} v := [t, z, \Phi_{z_0}^z v]$. These maps induce on E' the structure of a (trivial) vector bundle over Σ' . In the general case we equip E' with the vector bundle structure that restricts to the structure of E on $\Sigma \setminus \partial\Sigma$ and is given as above on collar neighborhoods of the components of the boundary. The form ω induces a fiberwise symplectic form ω' on E' . We denote by $\mathcal{C}^{\text{flat}}(\mathcal{X}, (E, \omega)|_{[0, 1] \times \partial\Sigma})$ the set of all pairs (W, Φ) , where $W \subseteq E$ is an ω -coisotropic subbundle, and $\Phi \in \mathcal{T}(\mathcal{X}, W_{\omega}, \omega_W)$.

Let X be a topological manifold and $\mathcal{X} \subseteq C([0, 1], X)$ be an admissible subset. We call $\Phi \in \mathcal{T}(\mathcal{X}, E)$ *regular* iff $\Phi([x]) = \text{id}$ for every $x \in \mathcal{X}$ satisfying $x(0) = x(1)$. For a symplectic vector bundle (E, ω) over an oriented topological surface Σ we denote by $c_1(E, \omega)$ its first Chern number.

Theorem 24 (Properties of the coisotropic Maslov map for bundles). *The following statements hold.*

- (i) (Naturality) *If $\Sigma, \Sigma' \in \mathcal{S}$, $(E, \omega, W, \Phi) \in \mathcal{E}_{\Sigma}$, $(E', \omega', W', \Phi') \in \mathcal{E}_{\Sigma'}$, and $\Psi \in \widetilde{\text{Iso}}(\omega, \omega')$ is such that $\Psi|_{\partial\Sigma}^*(W', \Phi') = (W, \Phi)$, then $m_{\Sigma}(E, \omega, W, \Phi) = m_{\Sigma'}(E', \omega', W', \Phi')$.*
- (ii) (Direct sum) *For every $\Sigma \in \mathcal{S}$ and $(E, \omega, W, \Phi), (E', \omega', W', \Phi') \in \mathcal{E}_{\Sigma}$ we have*

$$m_{\Sigma}(E \oplus E', \omega \oplus \omega', W \oplus W', \Phi \oplus \Phi') = m_{\Sigma}(E, \omega, W, \Phi) + m_{\Sigma}(E', \omega', W', \Phi').$$

- (iii) (Homotopy) *Let $\Sigma \in \mathcal{S}$, (E, ω) be a symplectic vector bundle over $[0, 1] \times \Sigma$, and $(W, \Phi) \in \mathcal{C}^{\text{flat}}((E, \omega)|_{[0, 1] \times \partial\Sigma})$. Then the map $[0, 1] \ni t \mapsto m_{\Sigma}((E, \omega)|_{\{t\} \times \Sigma}, W|_{\{t\} \times \partial\Sigma}, \Phi|_{\Pi(\{t\} \times \partial\Sigma)})$ is constant.*
- (iv) (Weak homotopy) *Let $\Sigma \in \mathcal{S}$, (E, ω) be a symplectic vector bundle over $[0, 1] \times \Sigma$, $\mathcal{X} := \{t \mapsto (s, z(t)) \mid s \in [0, 1], z \in C([0, 1], \partial\Sigma)\}$, and $(W, \Phi) \in \mathcal{C}^{\text{flat}}(\mathcal{X}, (E, \omega)|_{[0, 1] \times \partial\Sigma})$. Then the map $[0, 1] \ni t \mapsto m_{\Sigma}((E, \omega)|_{\{t\} \times \Sigma}, W|_{\{t\} \times \partial\Sigma}, \Phi|_{\Pi(\{t\} \times \partial\Sigma)})$ is continuous.*

(v) (Splitting) Let $\Sigma \in \mathcal{S}$, $C \subseteq \Sigma \setminus \partial\Sigma$ be a closed curve, $(E, \omega, W, \Phi) \in \mathcal{E}_\Sigma$ and $(W', \Phi') \in \mathcal{C}^{\text{flat}}((E, \omega)_C)$. Then

$$(11) \quad m(E, \omega, W, \Phi) = m(\text{pr}_C^{\Sigma*}(E, \omega, (W, \Phi) \amalg (W', \Phi'))).$$

- (vi) (Lagrangian case) If $\Sigma \in \mathcal{S}$ and $(E, \omega, W) \in \mathcal{E}_\Sigma^L$ then $m_\Sigma(E, \omega, W, 0) = m_\Sigma^L(E, \omega, W)$.
- (vii) (Full case) Let $\Sigma \in \mathcal{S}$ and $(E, \omega, W, \Phi) \in \mathcal{E}_\Sigma$. Assume that there exists $C \in \mathcal{C}(\partial\Sigma)$ such that $W|_C = E|_C$ and $\Phi|_C$ is regular. Then $m(E, \omega, W, \Phi) = m((E, \omega, W, \Phi)/C)$.
- (viii) (Quotient) Let (E, ω) a symplectic vector bundle over \mathbb{D} , $W \subseteq E$ a coisotropic subbundle, and $\Phi \in \mathcal{T}((W_\omega, \omega_W)|_{S^1})$. Then $m(E, \omega, W|_{S^1}, \Phi) = m(W_\omega, \omega_W, W_\omega|_{S^1}, \Phi)$.
- (ix) (Regular case) Let $\Sigma \in \mathcal{S}$ and $(E, \omega, W, \Phi) \in \mathcal{E}_\Sigma$ be such that Φ is regular. Then $m_\Sigma(E, \omega, W, \Phi) \in \mathbb{Z}$. Furthermore, if W is orientable then this integer is even.
- (x) (Lagrangian embedding) Let $(E, \omega, W, \Phi) \in \mathcal{E}_\mathbb{D}$ and (V', ω') a symplectic vector space. Assume that there exists a surjective homomorphism $\Psi : W \rightarrow S^1 \times V'$ such that $\Psi^*\omega' = \omega$ and the following holds. Denoting by $\Psi_W : W_\omega \rightarrow S^1 \times V'$ the map induced by Ψ , we have $(\Psi_W)_{z(1)}\Phi([z]) = (\Psi_W)_{z(0)}$, for every $z \in C([0, 1], S^1)$. Then the following equality holds. We define $\tilde{E} := E \oplus (S^1 \times V')$, $\tilde{\omega} := \omega \oplus (-\omega')$, $\tilde{W} := \{(z, v, \Psi_z v) \mid (z, v) \in W\} \subseteq \tilde{E}$. Then $m_\mathbb{D}(E, \omega, W, \Phi) = m_\mathbb{D}(\tilde{E}, \tilde{\omega}, \tilde{W}, 0)$.

This result is proved in Section 4.1 (page 30). The trickiest part is the proof of property (iii). It is based on the invariance under homotopy of $m_{\partial\Sigma, \omega} : \mathcal{C}^{\text{flat}}(\partial\Sigma \times V, \omega) \rightarrow \mathbb{R}$, where (V, ω) is a symplectic vector space. This follows from Proposition 27.

Proof of Theorem 19. Statements (i,ii,iii,iv,v,vii) follow from Theorem 24(i,ii,iii,iv,v,vi).

To prove **statement (vi)**, assume that N is regular. Let $u \in C(\Sigma, M; N, \omega)$. We choose a symplectic vector space (V, Ω) of dimension $\dim M$, and $\Psi \in \text{Iso}(\Sigma \times V, \Omega; u^*(TM, \omega))$. We define $(W, \Phi) := \Psi \circ u|_{\partial\Sigma}^*(TN, \text{hol}^{N, \omega})$. It follows from regularity of N that $\text{hol}^{N, \omega}$ is regular. (See [Zi], Lemma 15.) Hence (vi) follows from Theorem 24(ix).

We prove **assertion (viii)**. We define

$$\begin{aligned} (E', \omega') &:= u^*(TM, \omega), \quad (W', \Phi') := u|_{\partial\Sigma \setminus C}^*(TN, \text{hol}^{N, \omega}) \amalg (C \times T_{x_0}M, \text{id}_{T_{x_0}M}), \\ (\tilde{E}, \tilde{\omega}) &:= \tilde{u}^*(TM, \omega), \quad (\tilde{W}, \tilde{\Phi}) := \tilde{u}|_{\partial\Sigma \setminus C}^*(TN, \text{hol}^{N, \omega}). \end{aligned}$$

We have $m_{M, \omega, N}(\tilde{u}) = m(\tilde{E}, \tilde{\omega}, \tilde{W}, \tilde{\Phi})$. Furthermore, since $u|_C^*(TN, \text{hol}^{N, \omega}) = (C \times T_{x_0}N, \text{id}_{T_{x_0}N})$, it follows from the definitions that $m_{M, \omega, N}(u) = m(E', \omega', W', \Phi')$.

On the other hand, the map $\tilde{E} \ni ([z], v) \mapsto [z, v] \in E/\Phi'|_C$ is an $(\tilde{\omega}, \omega'/\Phi'|_C)$ -isomorphism that is the identity outside the point $C \in \tilde{\Sigma} := \Sigma/C$, and hence carries $(\tilde{W}, \tilde{\Phi})$ to $(W, \Phi')/C$. Therefore, Theorem 24(i,vii) imply that $m(\tilde{E}, \tilde{\omega}, \tilde{W}, \tilde{\Phi}) = m((E', \omega', W', \Phi')/C) = m(E', \omega', W', \Phi')$. It follows that $m_{M,\omega,N}(\tilde{u}) = m_{M,\omega,N}(u)$. This proves (viii).

We prove **assertion (ix)**. We have

$$\begin{aligned}
m_{M,\omega,N}(u) &= m(u^*(TM, \omega), u|_{S^1}^*(TN, \text{hol}^{N,\omega})) \\
&= m(u^*((TN)_\omega, \omega_{TN}), u|_{S^1}^*((TN)_\omega, \text{hol}^{N,\omega})) \\
&= m(u^*(T(N_\omega), \omega_N), S^1 \times T_{u(1)}N_\omega, \text{id}_{T_{u(1)}N_\omega}) \\
(12) \qquad &= 2c_1(u^*(T(N_\omega), \omega_N)) = 2c_1^{N_\omega, \omega_N}(u').
\end{aligned}$$

Here in the second equality we used Theorem 24(viii), in the third equality we used Theorem 24(i), and in the fourth equality we used Theorem 24(vii). Assertion (ix) follows from this.

We prove **assertion (x)**. Let $a \in [\mathbb{D}, S^1; M, N_\omega]$. We choose a representative $u \in C(\mathbb{D}, M; N, \omega)$ of a . The claimed equality follows from Theorem 24(x) with $(E, \omega, W, \Phi) := u^*(TM, \omega, TN, \text{hol}^{N,\omega})$ and $(V', \omega') := (T_{N_{u(1)}}(N_\omega), (\omega_N)_{N_{u(1)}})$, using the map $\Psi : u|_{S^1}^*TN \rightarrow S^1 \times V'$ given by $\Psi(z, v) := (z, (\pi_N)_*v)$.

This proves assertion (x) and completes the proof of Theorem 19. \square

For the proof of Theorem 24(ii) we need the following.

Remark 25. Let X be a compact oriented curve, (E, ω) and (E', ω') symplectic vector bundles over X , $\Phi, \Psi \in \mathcal{T}(E, \omega)$ and $\Phi', \Psi' \in \mathcal{T}(E', \omega')$, with Ψ and Ψ' regular. Then $m_{X,\omega \oplus \omega'}(\Phi \oplus \Phi', \Psi \oplus \Psi') = m_{X,\omega}(\Phi, \Psi) + m_{X,\omega'}(\Phi', \Psi')$. This follows from Proposition 42(ii).

For the proof of Theorem 24(iii,iv) we need the following. Let X be a closed oriented curve. We denote by $\pi : [0, 1] \times X \rightarrow X$ the canonical projection. For $s \in [0, 1]$ we denote by $\iota_s : \{s\} \times X \rightarrow [0, 1] \times X$ the inclusion. We define $\mathcal{X} := \{t \mapsto (s, z(t)) \mid s \in [0, 1], z \in C([0, 1], X)\}$. Let (E, ω) be a symplectic vector bundle over X .

Lemma 26. Let $W \subseteq \pi^*E$ be an $\pi^*\omega$ -coisotropic subbundle, and $\Phi_0 \in \mathcal{T}(\pi^*(E, \omega))$ a regular transport. The following assertions hold.

- (i) Let $\Phi \in \mathcal{T}(\Pi([0, 1] \times X), W_\omega, \omega_W)$. Assume that $\Psi \in \mathcal{T}(\Pi([0, 1] \times X), E, \omega)$ is such that $\iota_s^*\Psi$ is a lift of $\iota_s^*\Phi$, for every $s \in [0, 1]$. Then the map $[0, 1] \ni s \mapsto m_{X,\omega}(\iota_s^*\Psi, \iota_s^*\Phi_0) \in \mathbb{R}$ is constant.
- (ii) Let $\Phi \in \mathcal{T}(\mathcal{X}, W_\omega, \omega_W)$. Assume that $\Psi \in \mathcal{T}(\mathcal{X}, E, \omega)$ is such that $\iota_s^*\Psi$ is a lift of $\iota_s^*\Phi$, for every $s \in [0, 1]$. Then the map $[0, 1] \ni s \mapsto m_{X,\omega}(\iota_s^*\Psi, \iota_s^*\Phi_0) \in \mathbb{R}$ is continuous.

For the proof of Lemma 26 we need the following.

Proposition 27. *Let (V, ω) be a symplectic vector space, X a topological manifold, $(W, \Phi) \in \mathcal{C}^{\text{flat}}(X \times V, \omega)$, $x \in C([0, 1] \times [0, 1], X)$ and $\Psi \in C([0, 1] \times [0, 1], \text{Aut}(\omega))$ be such that $x(s, 0) = x(s, 1)$, $\Psi(s, t)W_{x(s,0)} = W_{x(s,t)}$ and $\Psi(s, t)W_{x(s,0)} = \Phi([x(s, \cdot)]|_{[0,t]})$, for $s, t \in [0, 1]$. Then the map $[0, 1] \ni s \mapsto m_\omega(\Psi(s, \cdot)) \in \mathbb{R}$ is constant.*

For the proof of this result, we need the following.

Proposition 28. *Let $W \subseteq V$ be a coisotropic subspace and $\Psi \in \text{Iso}(\omega)$ be such that $\Psi W = W$. Then*

$$(13) \quad \rho_\omega(\Psi) = \pm \rho_{\omega_W}(\Psi_W).$$

Furthermore, if $\det(\Psi|_W) > 0$ then $\rho_\omega(\Psi) = \rho_{\omega_W}(\Psi_W)$.

For the proof of Proposition 28 we need the following.

Lemma 29. *Let (V, ω) be a symplectic vector space, $W, W' \subseteq V$ Lagrangian subspaces and $\Psi \in \text{Iso}(\omega)$. Assume that $W + W' = V$, $\Psi W = W$ and $\Psi W' = W'$. Then $\rho_\omega(\Psi) = \pm 1$. If also $\det(\Psi|_W) > 0$ then $\rho_\omega(\Psi) = 1$.*

Let V and W be real vector spaces and $\Psi \in \text{Hom}(V, W)$. We denote by $\Psi^{\mathbb{C}} : V^{\mathbb{C}} \rightarrow W^{\mathbb{C}}$ the complex linear extension. If $V = W$ and this space has dimension n then for every $\lambda \in \mathbb{C}$ we denote $E_\Psi^\lambda := \ker((\lambda - \Psi^{\mathbb{C}})^n) \subseteq V^{\mathbb{C}}$. Let now (V, ω) be a symplectic vector space, $\Psi \in \text{Iso}(\omega)$ and $\lambda \in S^1 \setminus \{\pm 1\}$. We define

$$m_+(\omega, \Psi, \lambda) := \max \{ \dim_{\mathbb{C}} W \mid W \subseteq E_\Psi^\lambda \text{ complex subspace, } \Im \omega(\bar{v}, v) > 0, \forall v \in W \}.$$

The following remarks are used in the proof of Lemma 29.

Remark 30. *We have $m_+(-\omega, \Psi, \bar{\lambda}) = m_+(\omega, \Psi, \lambda)$. This follows, since the map $E_\Psi^\lambda \rightarrow E_{\bar{\Psi}}^{\bar{\lambda}}, v \mapsto \bar{v}$, is a real isomorphism.*

Remark 31. *If V' is another vector space and $\Phi \in \text{Iso}(V', V)$ then $m_+(\Phi^* \omega, \Phi^{-1} \Psi \Phi, \lambda) = m_+(\omega, \Psi, \lambda)$. This follows from the fact $\Phi E_{\Phi^{-1} \Psi \Phi}^\lambda = E_\Psi^\lambda$.*

We define ω^* to be the symplectic form on V^* defined by $\omega^*(\varphi, \psi) := \varphi(w)$, where $w \in V$ is determined by $\omega(w, \cdot) = \psi$.

Remark 32. *The map $\Psi^{-*} := (\Psi^*)^{-1}$ is ω^* -symplectic, and the map $\omega_\# : V \rightarrow V^*$ defined by $\omega_\# v := \omega(v, \cdot)$ satisfies $\omega_\#^*(\omega^*, \Psi^{-*}) = (\omega, \Psi)$.*

Let now W be a finite dimensional vector space. We define the canonical symplectic form ω^W on $V := W \oplus W^*$ by $\omega^W((v, \varphi), (v', \varphi')) := \varphi'(v) - \varphi(v')$. Furthermore, we denote by $\iota^W : W \rightarrow W^{**}$ the canonical isomorphism, and define the map $\Phi_W : V \rightarrow V^*$ by $\Phi_W(v, \varphi) := (\varphi, \iota^W v)$.

Remark 33. We have $\Phi_W^* \omega^{W^*} = \omega^W$. Furthermore, if $\Psi \in \text{Aut}(\omega^W)$ is such that $\Psi W = W$ and $\Psi W^* = W^*$ then $\Phi_W^{-1} \Psi^{-*} \Phi_W = \Psi$. This follows from the fact $\Psi|_{W^*} = \Psi|_{\bar{W}^*}$.

Remark 34. Let V be a real vector space and $\Phi \in \text{End}(V)$ be such that $\det \Phi > 0$. For $\lambda \in \mathbb{C}$ we denote by $m(\Phi, \lambda) \in \mathbb{N} \cup \{0\}$ the algebraic multiplicity over \mathbb{C} of λ as an eigenvalue of Φ . Then $\sum_{\lambda \in (-\infty, 0)} m(\Phi, \lambda)$ is even.

Proof of Lemma 29. We define the map $\Phi : V = W \oplus W' \rightarrow W \oplus W^*$ by $\Phi(w, w') := (w, -(\omega^\# w')|_W)$. Then the tuple $(\tilde{V}, \tilde{W}, \tilde{W}', \tilde{\omega}, \tilde{\Psi}) := (W \oplus W^*, W, W^*, \omega^W, \Phi \Psi \Phi^{-1})$ satisfies $\tilde{W} + \tilde{W}' = \tilde{V}$, $\tilde{\Psi} \tilde{W} = W$, $\tilde{\Psi} \tilde{W}' = \tilde{W}'$ and $\det \Psi|_W = \det \tilde{\Psi}|_{\tilde{W}}$. Hence by (Naturality) for ρ , we may assume without loss of generality that $V = W \oplus W^*$, $W' = W^*$ and $\omega = \omega^W$.

Let $\lambda \in S^1 \setminus \{\pm 1\}$. Remarks 33 and 31 imply that $m_+(\omega, \Psi, \lambda) = m_+(\omega^{W^*}, \Psi^{-*}, \lambda)$. On the other hand, $\omega^{W^*} = -(\omega^W)^*$, hence by Remarks 30, 31 and 32, we obtain $m_+(\omega^{W^*}, \Psi^{-*}, \lambda) = m_+(\omega^W, \Psi, \bar{\lambda}) = m_+(\omega, \Psi, \bar{\lambda})$. It follows that $m_+(\omega, \Psi, \lambda) = m_+(\omega, \Psi, \bar{\lambda})$, and therefore by Lemma 43 $\rho_\omega(\Psi) = \pm 1$.

Assume now also that $\det(\Psi|_W) > 0$. We have $\Psi|_{W^*} = \Psi|_{\bar{W}^*}$ and $\det \Psi|_{\bar{W}^*} = (\det \Psi|_W)^{-1} > 0$. Hence by Remark 32,

$$\sum_{\lambda \in (-\infty, 0)} m(\Psi, \lambda) = \sum_{\lambda \in (-\infty, 0)} m(\Psi|_W, \lambda) + \sum_{\lambda \in (-\infty, 0)} m(\Psi|_{W^*}, \lambda) \in 4\mathbb{Z}.$$

It follows now from Remark 44 that $\rho_\omega(\Psi) = 1$. This proves Lemma 29. \square

Proof of Proposition 28. Assume first that there exists a coisotropic subspace $W' \subseteq V$ such that

$$(14) \quad \dim W' = \dim W, \quad W + W'^\omega = V, \quad \Psi W' = W'.$$

We define $U := W \cap W'$. Since $\Psi W = W$, we have $\Psi W^\omega = W^\omega$, and since $\Psi W' = W'$, we have $\Psi W'^\omega = W'^\omega$. Furthermore, by an elementary argument, we have

$$(15) \quad U^\omega = W^\omega + W'^\omega.$$

It follows that $\Psi U^\omega = U^\omega$ and hence $\Psi U = U$.

Claim 1. *The map*

$$(16) \quad U \rightarrow W_\omega = W/W^\omega, \quad v \mapsto [v]$$

is bijective.

Proof of Claim 1. By an elementary argument, we have

$$W' \cap W^\omega = (W'^\omega + W)^\omega = \{0\}.$$

Here in the second equality we used the facts $W'^{\omega\omega} = W'$ and $W'^{\omega} + W = V$. It follows that

$$(17) \quad U \cap W^\omega = W' \cap W^\omega = \{0\}$$

Claim 2. *We have*

$$(18) \quad U + W^\omega = W.$$

Proof of Claim 2. By (17) and the facts $U \subseteq W$ and $W^\omega \subseteq W$, it suffices to show that

$$(19) \quad \dim U + \dim W^\omega \geq \dim W.$$

To see this inequality, observe that (15) implies

$$\begin{aligned} \dim V - \dim U &= \dim U^\omega \\ &\leq \dim W^\omega + \dim W'^{\omega} \\ &= \dim W^\omega + \dim V - \dim W'. \end{aligned}$$

Since by (14) we have $\dim W = \dim W'$, inequality (19) follows. This proves Claim 2. \square

Claim 1 follows from (17) and Claim 2. \square

Claim 1 implies that the map (16) is a linear symplectic isomorphism. It follows that U and hence U^ω are symplectic subspaces of V . Since they are invariant under Ψ , the (Product) property in Proposition 42 implies that

$$(20) \quad \rho_\omega(\Psi) = \rho_{\omega|_U}(\Psi|_U) \rho_{\omega|_{U^\omega}}(\Psi|_{U^\omega}).$$

Furthermore, the (Naturality) property in Proposition 42 implies that

$$(21) \quad \rho_{\omega|_U}(\Psi|_U) = \rho_{\omega_W}(\Psi_W).$$

Since W^ω and W'^{ω} are complementary Lagrangian subspaces of U^ω that are invariant under Ψ , it follows from Lemma 29 that $\rho_{\omega|_{U^\omega}}(\Psi|_{U^\omega}) = \pm 1$. Combining this with (20) and (21), equality (13) follows.

Assume now that $\det \Psi|_W > 0$. Since $\Psi|_W = \Psi|_U \oplus \Psi|_{W^\omega}$ and $\Psi|_U \in \text{Iso}(\omega|_U)$, it follows that $\det \Psi|_{W^\omega} > 0$. Hence Lemma 29 implies that $\rho_{\omega|_{U^\omega}}(\Psi|_{U^\omega}) = 1$. Combining this with (20) and (21), we obtain $\rho_\omega(\Psi) = 1$.

Consider now the general case, in which we do not assume that a subspace $W' \subseteq V$ satisfying (14) exists. We choose a coisotropic subspace $W' \subseteq V$ such that $\dim W' = \dim W$ and $W + W'^{\omega} = V$, and denote $V_0 := W \cap W'$, $V_1 := W^\omega$ and $V_2 := W'^{\omega}$. As in the proof of Claim (1) it follows that V is the direct sum of the V_i 's. We define $P_i : V \rightarrow V_i$ to be the linear projection along the subspace $\bigoplus_{j \neq i} V_j$, and we denote $\Psi_{ij} := P_i \Psi|_{V_j}$, for $i, j = 0, 1, 2$. We fix $t \in \mathbb{R}$ and define, using the splitting $V = \bigoplus_{i=0,1,2} V_i$,

$$(22) \quad \Psi^t := \begin{pmatrix} \Psi_{00} & 0 & t\Psi_{02} \\ t\Psi_{10} & \Psi_{11} & t^2\Psi_{12} \\ 0 & 0 & \Psi_{22} \end{pmatrix} : V \rightarrow V.$$

Claim 3. *We have $\Psi^1 = \Psi$.*

Proof of Claim 3. Since $\Psi W = W$, we have $\Psi_{20} = 0$, and since $\Psi W^\omega = W^\omega$, we have $\Psi_{01} = \Psi_{21} = 0$. Hence Ψ has the form (22) with $t = 1$. This proves Claim 3. \square

Claim 4. *The map Ψ^t is an ω -symplectic.*

Proof of Claim 4. Since Ψ is symplectic, we have for $v_0 \in V_0$, $w_2 \in V_2$,

$$(23) \quad 0 = \omega(v_0, w_2) = \omega(\Psi v_0, \Psi w_2) = \omega(\Psi_{00}v_0, \Psi_{02}w_2) + \omega(\Psi_{10}v_0, \Psi_{22}w_2).$$

Furthermore, for $v_2, w_2 \in V_2$,

$$(24) \quad 0 = \omega(v_2, w_2) = \omega(\Psi_{02}v_2, \Psi_{02}w_2) + \omega(\Psi_{12}v_2, \Psi_{22}w_2) + \omega(\Psi_{22}v_2, \Psi_{12}w_2).$$

Hence, for every $v = v_0 + v_1 + v_2$, $w = w_0 + w_1 + w_2 \in V = V_0 \oplus V_1 \oplus V_2$,

$$\begin{aligned} \omega(\Psi^t v, \Psi^t w) &= \omega(\Psi_{00}v_0, \Psi_{00}w_0) + \omega(\Psi_{11}v_1, \Psi_{22}w_2) + \omega(\Psi_{22}v_2, \Psi_{11}w_1) + \\ &\quad t \left(\omega(\Psi_{00}v_0, \Psi_{02}w_2) + \omega(\Psi_{02}v_2, \Psi_{00}w_0) + \right. \\ &\quad \left. \omega(\Psi_{10}v_0, \Psi_{22}w_2) + \omega(\Psi_{22}v_2, \Psi_{10}w_0) \right) + \\ &\quad + t^2 \left(\omega(\Psi_{02}v_2, \Psi_{02}w_2) + \omega(\Psi_{12}v_2, \Psi_{22}w_2) + \omega(\Psi_{22}v_2, \Psi_{12}w_2) \right) \\ &= \omega(\Psi^1 v, \Psi^1 w) + (t-1)(0-0) + (t^2-1)0 \\ &= \omega(v, w). \end{aligned}$$

Here in the second equality we used equalities (23) and (24), and in the last equality we used Claim 3 and the fact that Ψ is symplectic. This proves Claim 4. \square

Claim 5. *We have*

$$\rho_\omega(\Psi^t) = \rho_\omega(\Psi^0).$$

Proof of Claim 5. We denote by $\sigma(\Phi)$ the set of eigenvalues of an endomorphism Φ of any vector space. We define

$$S := \{ \pm \prod_{\lambda \in \sigma(\Psi^0) \cap S^1} \lambda^{m_\lambda} \mid m_\lambda \in \{0, \dots, \dim V\}, \text{ for } \lambda \in \sigma(\Psi^0) \} \subseteq S^1.$$

The block form (22) implies that $\det(\lambda \mathbf{1} - \Psi^t) = \det(\lambda \mathbf{1} - \Psi^0)$. Hence $\sigma(\Psi^t) = \sigma(\Psi^0)$. Therefore, by the formula (29) of Lemma 43 we have

$$f(t) := \rho_\omega(\Psi^t) \in S.$$

By Proposition 42 the map $\rho_\omega : \text{Iso}(\omega) \rightarrow S^1$ is continuous, so the same holds for the map $f : \mathbb{R} \rightarrow S$. Since the set S is finite, it follows that f is constant. This proves Claim 5. \square

Since $W' = V_0 \oplus V_2$ and Ψ^0 leaves the subspaces V_i invariant, we have $\Psi^0 W' = W'$. Therefore, by what we already proved, $\rho_\omega(\Psi^0) = \pm \rho_{\omega_W}(\Psi_W^0)$. Combining this with Claims 3 and 5 and the fact $\Psi_W^0 = \Psi_W$, we get $\rho_\omega(\Psi) = \pm \rho_{\omega_W}(\Psi_W)$. Similarly, if $\det \Psi|_W > 0$ then it follows that $\rho_\omega(\Psi) = \rho_{\omega_W}(\Psi_W)$. This proves Proposition 28. \square

Proof of Proposition 27. Consider the map $f : [0, 1] \times [0, 1] \rightarrow S^1 \subseteq \mathbb{C}$, $f(s, t) := \Phi([x(s, \cdot)]|_{[0, t]})$. Let $s \in [0, 1]$. Proposition 28 implies that $\rho_\omega(\Psi(s, 0)) = \pm f(s, 0) = \pm 1$. Since $\Psi(s, 1)W_{x(s, 1)} = W_{x(s, 1)}$ and $\Psi(s, 1)_{W_{x(s, 1)}} = \Phi([x(s, \cdot)])$, Proposition 28 implies that $\rho_\omega(\Psi(s, 1)) = \pm f(s, 1)$. We define $\tilde{x} : [0, 1] \rightarrow X$ to be the concatenation of the paths $[0, 1] \ni t \mapsto x(s(1-t), 0)$, $x(0, \cdot)$ and $[0, 1] \ni t \mapsto x(st, 0)$. Then \tilde{x} is homotopic with fixed endpoints to $x(s, \cdot)$, and therefore $\Phi([x(s, \cdot)]) = \Phi([\tilde{x}]) = \Phi([x'])\Phi([x(0, \cdot)])\Phi([x'])^{-1}$. By naturality of ρ , it follows that $f(s, 1) = f(0, 1)$, and hence $\rho_\omega(\Psi(s, 1)) = \pm f(0, 1)$. Combining this with the equality $\rho_\omega(\Psi(s, 0)) = \pm 1$, it follows that the map $[0, 1] \ni s \mapsto m(\Psi(s, \cdot))$ is constant. This proves Proposition 27. \square

Proof of Lemma 26. **Statement (i)** follows from Proposition 27, and **statement (ii)** follows from an elementary argument. This proves Lemma 26. \square

For the proof of Theorem 24(v) we need the following remark. We denote by ω_0 the standard symplectic form on \mathbb{R}^{2n} , and by $\mathrm{Sp}(2n) = \mathrm{Aut}(\omega_0)$ the linear symplectic group. We identify $S^1 \cong \mathbb{R}/\mathbb{Z}$.

Remark 35. We define $m_0 : C(S^1, \mathrm{Sp}(2n)) \rightarrow \mathbb{Z}$ by $m_0(\Psi) := m_{\omega_0}([0, 1] \ni t \mapsto \Psi(t + \mathbb{Z}) \in \mathrm{Sp}(2n))/2 \in \mathbb{Z}$. This map equals the usual Maslov index of Ψ , as defined for example axiomatically in the book [MS]. To see this, note that on $U(n) = \mathrm{Sp}(2n) \cap O(2n)$, m_0 agrees with the map \tilde{m}_0 constructed in the proof of Theorem 2.29 in that book. Furthermore, $\mathrm{Sp}(2n)$ deformation retracts onto $U(n)$ (see Proposition 2.22 in [MS]). Since m_0 and \tilde{m}_0 are invariant under homotopy, the statement follows.

Lemma 36. Let X be a topological space, $Y \subseteq X$, (E, ω) a symplectic vector bundle over X , $\Phi : Y \times Y \rightarrow \mathrm{GL}(\omega)$ a morphism of topological groupoids whose composition with the canonical projection $\mathrm{GL}(\omega) \rightarrow X \times X$ is the identity, and (V, Ω) a symplectic vector space of dimension $\mathrm{rank} E$. If there is a homeomorphism $f : [0, 1] \times Y \rightarrow X$ such that $f(0, x) \in Y$, for every $x \in Y$, then there exists $\Psi \in \mathrm{Iso}(X \times V, \Omega; E, \omega)$ such that $\Phi_x^{x'} \Psi^x = \Psi^{x'}$, for every $x, x' \in Y$.

Proof of Lemma 36. Assume without loss of generality that $Y \neq \emptyset$. We choose a homeomorphism $f : [0, 1] \times Y \rightarrow X$ as above, a point $x_0 \in Y$, and $\Psi_0 \in \mathrm{Iso}(V, \Omega; (E, \omega)|_{x_0})$. We denote by $\mathrm{pr} : [0, 1] \times Y \rightarrow Y$ the canonical projection. By Lemma 60 there exists $\tilde{\Psi} \in \mathrm{Iso}(\mathrm{pr}^* f^*(E, \omega), f^*(E, \omega))$ such that $\tilde{\Psi}|_{\{0\} \times Y} = \mathrm{id}$. We define $\Psi : X \times V \rightarrow E$ by $\Psi^x := \tilde{\Psi}^{f^{-1}(x)} \Phi_{x_0}^{\mathrm{pr} \circ f^{-1}(x)} \Psi_0$:

$V \rightarrow E_x$, for $x \in X$. This map has the required properties. This proves Lemma 36. \square

The next remark will be used in the proof of Theorem 24(v). Let X be a closed oriented curve. We denote by \overline{X} the curve X with the opposite orientation.

Remark 37. *Let (E, ω) a symplectic vector bundle over X and $\Phi, \Psi \in \mathcal{T}(E, \omega)$, with Φ regular. Then $m_{\overline{X}, \omega}(\Psi, \Phi) = -m_{X, \omega}(\Psi, \Phi)$. This follows from directly from the definition.*

For the proof of Theorem 24(viii) we need the following.

Lemma 38. *Let X be a closed oriented curve, (E, ω) a symplectic vector bundle over X , $W \subseteq E$ an ω -coisotropic subbundle, and $\Phi, \Psi \in \mathcal{T}(E, \omega)$. Assume that Φ and Ψ leave W invariant, and that Φ is regular. Then $m_{X, \omega}(\Psi, \Phi) = m_{X, \omega_W}(\Psi_W, \Phi_W)$.*

Proof of Lemma 38. Without loss of generality we may assume that X is connected. We choose $z \in C([0, 1], X)$ such that $z(0) = z(1)$ and the map $S^1 \cong [0, 1]/\{0, 1\} \ni [t] \mapsto z(t) \in X$ has degree one. For $s \in [0, 1]$ we define $z_s \in C([0, 1], X)$ by $z_s(t) := z(st)$. We define $F : [0, 1] \rightarrow \text{Aut}_{\omega_{z(0)}}$ by $F(s) := \Phi([z_s])^{-1} \Psi([z_s])$. Since $F(0) = \text{id}$ and F is continuous. It follows that $\det F(s) > 0$, for every $s \in [0, 1]$. Hence Proposition 28 implies that $\rho_{\omega}(F(s)) = \rho_{\omega_W}(F(s)_{W_{z(0)}})$, for every $s \in [0, 1]$. The statement of Lemma 38 follows. \square

For the proof of Theorem 24(ix) we need the following.

Lemma 39. *Let X be a closed curve, (E, ω) a vector bundle over X , and $\Phi, \Psi \in \mathcal{T}(E, \omega)$, with Φ regular. Assume that there exists a coisotropic subbundle $W \subseteq E$ that is invariant under Ψ , such that Ψ_W is regular. Then $m_{X, \omega}(\Psi, \Phi) \in \mathbb{Z}$. Furthermore, if there is an orientable such W then $m_{X, \omega}(\Psi, \Phi) \in 2\mathbb{Z}$.*

Proof of Lemma 39. Let X, E, ω, Φ, Ψ and W be as in the hypothesis. We choose a path $z \in C([0, 1], X)$ such that $z(0) = z(1)$ and the map $S^1 \cong [0, 1]/\{0, 1\} \ni [t] \mapsto z(t) \in X$ has degree one. By our regularity assumptions, we have $\Phi([z]) = \text{id}$ and $\Psi_W([z]) = \text{id}$. Hence by the first assertion of Proposition 28, we have $\rho_{\omega}(\Phi([z])^{-1} \Psi([z])) = \pm \rho_{\omega_W}(\Psi_W([z])) = \pm 1 \in S^1$. It follows that $m_{X, \omega}(\Psi, \Phi) \in \mathbb{Z}$.

To prove the second assertion, for $s \in [0, 1]$ we define $z_s \in C([0, 1], X)$ by $z_s(t) := z(st)$. We define S to be the set of all $s \in [0, 1]$ such that $\Psi([z_s])$ maps the orientation of $W_{z(0)}$ to the orientation of $W_{z(s)}$. This set is non-empty, since $0 \in S$, open and closed. It follows that $S = [0, 1]$, and therefore $\det \Psi([z]) > 0$. Therefore, by the second assertion of Proposition

28 we have $\rho_\omega(\Phi([z])^{-1}\Psi([z])) = \rho_{\omega_{W_{z(0)}}}(\Psi([z])_{W_{z(0)}}) = 1 \in S^1$. It follows that $m_{X,\omega}(\Psi, \Phi) \in 2\mathbb{Z}$.

This proves Lemma 39. □

For the proof of Theorem 24(x) we need the following lemma. Let X be a closed curve, (E, ω) a symplectic vector bundle over X , (V', ω') a symplectic vector space, $W \subseteq E$ an ω -coisotropic subbundle, $\Phi_0 \in \mathcal{T}(E, \omega)$ a regular transport, and $F : W \rightarrow X \times V'$ a surjective homomorphism such that $F^*\omega' = \omega$. We denote by $F_W : W_\omega \rightarrow E'$ the map induced by F . We define $\Phi \in \mathcal{T}(W_\omega, \omega_W)$ by $\Phi([z]) := (F_W)_{z(1)}^{-1}(F_W)_{z(0)}$, $\tilde{E} := E \oplus (S^1 \times V')$, $\tilde{\omega} := \omega \oplus (-\omega')$, and $\tilde{W} := \{(z, v, Fv) \mid z \in C, v \in W\}$. Then \tilde{W} is an $\tilde{\omega}$ -Lagrangian subbundle of \tilde{E} . Furthermore, we define $\Phi'_0 \in \mathcal{T}(X \times V', \omega')$ to be the trivial transport $\Phi'_0 \equiv \text{id}$, and $\tilde{\Phi}_0 := \Phi_0 \oplus \Phi'_0 \in \mathcal{T}(\tilde{E}, \tilde{\omega})$. Let $\Psi \in \mathcal{T}(E, \omega)$ be a lift of (W, Φ) . We define $\tilde{\Psi} := \Psi \oplus \Phi'_0 \in \mathcal{T}(\tilde{E}, \tilde{\omega})$. Then $\tilde{\Psi}$ is a lift of $(\tilde{W}, 0)$.

Lemma 40. *We have $m_{X,\tilde{\omega}}(\tilde{\Psi}, \tilde{\Phi}_0) = m_{X,\omega}(\Psi, \Phi_0)$.*

Proof of Lemma 40. This follows from a straight-forward argument. □

4.1. Proof of Theorem 24 (Properties of the coisotropic Maslov map for bundles).

Proof of Theorem 24. **Assertion (i)** follows directly from the definitions and **assertion (ii)** from Remark 25. **Assertions (iii,iv)** follow from Lemma 26 and Theorem 9(iii,iv).

To prove **statement (v)**, let $\Sigma, X, E, \omega, W, \Phi, W'$ and Φ' be as in the hypothesis. Without loss of generality, we may assume that Σ is connected and $X \neq \emptyset$. We choose a symplectic vector space (V, Ω) of dimension $\text{rank} E$. Assume first that $\partial\Sigma \neq \emptyset$. We choose $\Psi \in \text{Iso}(\Sigma \times V, \Omega; E, \omega)$. We define $\tilde{\Psi} := (\text{pr}_X^\Sigma)^*\Psi \in \text{Iso}(\Sigma_X \times V, \Omega; (\text{pr}_X^\Sigma)^*(E, \omega))$.

Claim 1. *We have $m_{X^\Sigma, \Omega}(\tilde{\Psi}^* \text{pr}_X^{\Sigma*}(W', \Phi')) = 0$.*

Proof of Claim 1. Without loss of generality we may assume that X is connected. We denote by X_1 and X_2 the two connected components of $\text{pr}_X^{\Sigma^{-1}}(X) \subseteq \Sigma_X$. We denote by $f : X_1 \rightarrow X_2$ the unique map such that $\text{pr}_X^{\partial\Sigma}|_{X_1} = \text{pr}_X^{\partial\Sigma} \circ f$. Furthermore, for $i = 1, 2$ we define $(W_i, \Phi_i) := \tilde{\Psi}^* \text{pr}_X^{\Sigma*}(W', \Phi')|_{X_i}$. Then $f^*(W_2, \Phi_2) = (W_1, \Phi_1)$. Furthermore, the canonical orientation of X_1 (induced by the orientation of Σ_X) is opposite to the pullback under f of the canonical orientation of X_2 . Therefore, Claim 1 follows from Remark 37 and statement (i). □

Using Claim 1, it follows that

$$\begin{aligned} & m((\text{pr}_X^\Sigma)^*(E, \omega, (W, \Phi) \amalg (W', \Phi'))) \\ &= m_{\partial\Sigma_X, \Omega}(\tilde{\Psi}^* \text{pr}_X^{\Sigma^*}((W, \Phi) \amalg (W', \Phi'))) \\ &= m_{\partial\Sigma, \Omega}(\Psi^*(W, \Phi)). \end{aligned}$$

Since $m_\Omega(\Psi^*(W, \Phi)) = m_\Sigma(E, \omega, W, \Phi)$, equality (11) follows.

Assume now that $\partial\Sigma = \emptyset$. We choose a connected closed curve $X' \subseteq \Sigma \setminus X$, such that $\Sigma_{X'}$ is disconnected. We denote by X_1 and X_2 the connected components of $\Sigma_{X'}$, and by $f : X_1 \rightarrow X_2$ the canonical map. We also choose a trivialization $\tilde{\Psi} \in \text{Iso}(\Sigma_{X'} \times \mathbb{R}^{2n}, \omega_0; E, \omega)$. We define $\Psi : X_1 \rightarrow \text{Aut}(\omega_0)$ by $\Psi(z) := \tilde{\Psi}_z^{-1} \tilde{\Psi}_{f(z)}$. Furthermore, we define m_0 as in Remark 35. As explained in the proof of Theorem 2.69 in [MS], we have $c_1(E, \omega) = m_0(\Psi)$. (That theorem is stated for smooth surfaces, however, the proof carries over to topological surfaces.) We define $\Phi'' \in \mathcal{T}((E, \omega)|_{X'})$ to be the unique transport such that $\text{pr}_{X'}^{\Sigma^*} \Phi''([z]) = \tilde{\Psi}_{z(1)} \tilde{\Psi}_{z(0)}^{-1}$, for $z \in C([0, 1], X_1)$. It follows that

(25)

$$\tilde{\Psi}^* \text{pr}_{X'}^{\Sigma^*} \Phi''([z]) = \begin{cases} \text{id}_{\mathbb{R}^{2n}}, & \text{for } z \in C([0, 1], X_1), \\ \Psi(f^{-1}(z(1)))^{-1} \Psi(f^{-1}(z(0))), & \text{for } z \in C([0, 1], X_2). \end{cases}$$

We choose $z \in C([0, 1], X_2)$ such that the map $z(0) = z(1)$ and the map $S^1 \cong \mathbb{R}/\mathbb{Z} \ni t + \mathbb{Z} \mapsto z(t) \in X_2$ is an orientation reversing homeomorphism (with respect to the orientation on X_2 induced by the orientation of $\Sigma_{X'}$). Equality (25) implies that $m_{\text{pr}_{X'}^{\Sigma^*} \Phi''|_{X_1}} = 0$ and

$$\begin{aligned} m_{\text{pr}_{X'}^{\Sigma^*} \Phi''|_{X_2}} &= -2m_0((\Psi \circ f^{-1} \circ z)^{-1} \Psi(f^{-1}(z(0)))) \\ &= -2m_0((\Psi \circ f^{-1} \circ z)^{-1}) \\ &= 2m_0(\Psi). \end{aligned}$$

It follows that

$$(26) \quad m(\text{pr}_{X'}^{\Sigma^*}(E, \omega, E|_{X'}, \Phi'')) = m_{\text{pr}_{X'}^{\Sigma^*} \Phi''|_{X_2}}(\tilde{\Psi}^* \text{pr}_{X'}^{\Sigma^*} \Phi'') = 2m_0(\Psi).$$

On the other hand, using the canonical identifications $(\Sigma_{X'})_X \cong \Sigma_X \amalg X'$ $(\Sigma_X)_{X'}$, by what we already proved, we have

$$\begin{aligned} m(\text{pr}_{X'}^{\Sigma^*}(E, \omega, E|_{X'}, \Phi'')) &= m(\text{pr}_X^\Sigma \amalg \text{pr}_{X'}^{\Sigma^*}(E, \omega, (W', \Phi') \amalg (E|_{X'}, \Phi''))) \\ &= m(\text{pr}_X^\Sigma(E, \omega, W', \Phi')). \end{aligned}$$

Combining this with (26) and the fact $m_0(\Psi) = c_1(E, \omega)$, equality (11) follows. This proves statement (v).

We prove **(vi)**. For each natural number n we denote by ω_0 and J_0 the standard symplectic form and complex structure on \mathbb{R}^{2n} , and by $R(n)$

the set of all totally real subspaces of \mathbb{R}^{2n} , and for $W_0 \in R(n)$ we define $\rho_{W_0} : R(n) \rightarrow S^1$ by $\rho_{W_0}(W) := \det(\Psi)^2 / |\det(\Psi)|^2$, where $\Psi : \mathbb{C}^n = \mathbb{R}^{2n} \rightarrow \mathbb{C}^n$ is a complex linear map such that $\Psi W_0 = W$. For a closed oriented topological curve X and a map $W \in C(X, R(n))$ we define $m^{\text{real}}(W) := \deg(\partial\Sigma \ni z \mapsto \rho_{W_0}(W(z)) \in S^1)$, where $W_0 \in R(n)$ is arbitrary. Let $\Sigma \in \mathcal{S}$. We define $\mathcal{E}_\Sigma^{\text{real}}$ and $m_\Sigma^{\text{real}} : \mathcal{E}_\Sigma^{\text{real}} \rightarrow \mathbb{Z}$ as in the paragraph before (32) in the appendix. Let $(E, \omega, W) \in \mathcal{E}_\Sigma^L$. We denote $2n := \text{rank}E$. We choose $\Psi \in \text{Iso}(\Sigma \times \mathbb{R}^{2n}, \omega_0; E, \omega)$, and define $W' : \partial\Sigma \rightarrow R(n)$ by $W'(z) := \Psi|_{\partial\Sigma}^{-1} W_z$ and view this also as a subbundle of $\partial\Sigma \times \mathbb{R}^{2n}$. By [MS], the (Isomorphism) condition in Theorem C.3.5. and the (Trivial bundle) property in Theorem C.3.6., we have

$$m_\Sigma^L(E, \omega, W) = m_\Sigma^{\text{real}}(E, \Psi_* J_0, W) = m_\Sigma(\Sigma \times \mathbb{R}^{2n}, J_0, W') = m(W').$$

On the other hand, $m_\Sigma(E, \omega, W, 0) = m_\Sigma(\Sigma \times \mathbb{R}^{2n}, \omega_0, W', 0) = m_{\partial\Sigma, \omega_0}(W', 0)$. Therefore, (vi) is a consequence of the following claim.

Claim 2. *We have $m(W') = m_{\partial\Sigma, \omega_0}(W', 0)$.*

Proof of Claim 2. We fix a connected component X of $\partial\Sigma$. We choose a path $z \in C([0, 1], X)$ such that the map $S^1 \cong \mathbb{R}/\mathbb{Z} \ni t + \mathbb{Z} \mapsto z(t) \in X$ has degree one. We denote by $U(n) \subseteq \mathbb{C}^{n \times n}$ and $O(n) \subseteq \mathbb{R}^{n \times n}$ the unitary and orthogonal groups. Note that $G(\omega_0, n)$ is the Grassmannian of Lagrangian subspaces of \mathbb{R}^{2n} . For every $W_0 \in G(\omega_0, n)$ the map $U(n)/O(n) \ni \Psi O(n) \mapsto \Psi W_0 \in G(\omega_0, n)$ is a well-defined diffeomorphism. Since the map $U(n) \rightarrow U(n)/O(n)$ is a smooth fiber bundle, setting $W_0 := W'_{z(0)}$, it follows that there exists a path $\tilde{\Psi} \in C([0, 1], U(n))$ such that $\tilde{\Psi}(0) = \text{id}$ and $\tilde{\Psi}(t)W'_{z(0)} = W'_{z(t)}$, for every $t \in [0, 1]$. By definition, we have $m_{\partial\Sigma, \omega_0}(W', 0) = m(\tilde{\Psi}) = 2\alpha([0, 1] \ni t \mapsto \rho_{\omega_0}(\tilde{\Psi}(t)))$. Let $t \in [0, 1]$. Since $\Psi(t) \in U(n)$, by the Determinant property of ρ_{ω_0} we have $\rho_{\omega_0}(\Psi(t))^2 = \det(\Psi(t))^2 = \rho_{W'_{z(0)}}(W'_{z(t)})$. Claim 2 follows. \square

We prove (vii). Let $\Sigma, E, \omega, W, \Phi$ and X be as in the hypothesis. Without loss of generality we may assume that Σ is connected. Assume first also that Σ is homeomorphic to $[0, 1] \times X$. We choose a symplectic vector space (V, Ω) of dimension $\text{rank}E$. Since Φ is regular, we may apply Lemma 36 with $(X, Y) := (\Sigma, X)$ and Φ replaced by the map $X \times X \ni (z_0, z_1) \mapsto \Phi([z])$, where $z \in C([0, 1], X)$ is an arbitrary path satisfying $z(i) = z_i$, for $i = 0, 1$. It follows that there exists $\Psi \in \text{Iso}(\Sigma \times V, \Omega; E, \omega)$ such that $\Psi^{z(1)} = \Phi([z])\Psi^{z(0)}$, for every $z \in C([0, 1], X)$. We define $\Psi' : \Sigma/X \times V \rightarrow E/\Phi$ by

$\Psi'_{[z]}v := [z, \Psi_z v]$. It follows that

$$\begin{aligned} m(E, \omega, W, \Phi) &= m_\Omega(\Psi^*(W, \Phi)) \\ &= m_\Omega(\Psi'^*(W, \Phi)|_{\partial\Sigma \setminus X}) \\ &= m((E, \omega, W, \Phi)/X). \end{aligned}$$

Here in the second step we used the fact that $\Psi^*\Phi|_C([z]) = \text{id} : V \rightarrow V$, for every $z \in C([0, 1], X)$. This proves the statement if Σ is homeomorphic to $[0, 1] \times X$.

In the general case we choose a curve $X' \subseteq \Sigma$ such that $\Sigma_{X'}$ is the disjoint union of two surfaces Σ_0 and Σ_1 , such that Σ_1 is homeomorphic to $[0, 1] \times X$. By statement (v) we have

$$\begin{aligned} m(E, \omega, W, \Phi) &= m((E, \omega, W, \Phi)^{\Phi'}) \\ (27) \quad &= m((E, \omega, W, \Phi)^{\Phi'}|_{\Sigma_0}) + m((E, \omega, W, \Phi)^{\Phi'}|_{\Sigma_1}). \end{aligned}$$

By what we already proved, we have

$$(28) \quad m((E, \omega, W, \Phi)^{\Phi'}|_{\Sigma_1}) = m((E, \omega, W, \Phi)^{\Phi'}|_{\Sigma_1}/X).$$

Using again statement (v), we have $m((E, \omega, W, \Phi)^{\Phi'}|_{\Sigma_0}) + m((E, \omega, W, \Phi)^{\Phi'}|_{\Sigma_1}/X) = m((E, \omega, W, \Phi)/X)$. Combining this with (27,28), statement (vii) follows.

We prove **assertion (viii)**. We choose a symplectic vector space (V, Ω) of dimension $\text{rank}E$ and a coisotropic subspace $W^0 \subseteq V$ of dimension $\text{rank}W$.

Claim 3. *There exists $\Psi \in \text{Iso}(\mathbb{D} \times V, \Omega; E, \omega)$ such that $\Psi(\mathbb{D} \times W^0) = W$.*

Proof of Claim 3. We choose an arbitrary $\tilde{\Psi} \in \text{Iso}(\mathbb{D} \times V, \Omega; E, \omega)$ and define $\tilde{W} := \tilde{\Psi}^{-1}W \subseteq \mathbb{D} \times V$. We choose a map $f_0 \in \text{Aut}(\Omega)$ such that $f_0 W^0 = \tilde{W}_0$. It follows from Lemma 11 and the homotopy lifting property for smooth fiber bundles that there exists $f \in C(\mathbb{D}, \text{Aut}(\Omega))$ such that $f(z)W^0 = \tilde{W}_z$, for every $z \in \mathbb{D}$. We define $\Psi := \tilde{\Psi}f$. Claim 3 follows. \square

We choose Ψ as in Claim 3. The assertion (viii) follows from Lemma 38 with ω, W, Φ replaced by $\Omega, W^0, \Psi^*\Phi$. **Assertion (ix)** follows from Lemma 39.

We prove **assertion (x)**. We choose a symplectic vector space (V, Ω) of dimension $\text{rank}E$ and $f \in \text{Iso}(\mathbb{D} \times V, \Omega; E, \omega)$. Then the hypotheses of Lemma 40 are satisfied with $X := S^1$ and ω, W, Φ replaced by $\Omega, f|_{S^1}^*(W, \Phi)$. We denote $\widehat{W} := \{(z, fv, \Psi fv) \mid (z, v) \in W\}$. By the conclusion of Lemma 40 we have $m_{S^1, \Omega}(f|_{S^1}^*(W, \Phi)) = m_{S^1, \Omega \oplus \omega'}(\widehat{W}, 0)$. Combining this with Lemma 17, it follows that $m(E, \omega, W, \Phi) = m_{S^1, \omega}(W, \Phi) = m_{S^1, \omega \oplus \omega'}(\widehat{W}, 0)$. This proves assertion (x) and completes the proof of Theorem 24. \square

Appendix A. Auxiliary results

The following result was used in Section 1.

Lemma 41. *The winding map $\alpha : C([0, 1], \mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}$ is continuous.*

Proof of Lemma 41. We denote by d the standard metric on \mathbb{R}/\mathbb{Z} . By Lemma 63(iv) $C([0, 1], \mathbb{R}/\mathbb{Z})$ is metrized by the metric d' defined as in (50). Let $z_0 \in C([0, 1], \mathbb{R}/\mathbb{Z})$. We denote by $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ the canonical projection. We choose a path $\tilde{z}_0 \in C([0, 1], \mathbb{R})$ such that $\pi \circ \tilde{z}_0 = z_0$. We define the map $\varphi : B_{1/2}^{d'}(z_0) \rightarrow C([0, 1], \mathbb{R})$ by defining $\varphi(z)(t)$ to be the unique point in $(\tilde{z}_0(t) - 1/2, \tilde{z}_0(t) + 1/2)$ such that $\pi(\varphi(z)(t)) = z(t)$. This map is continuous. Furthermore, by Lemma 63(ii) the map $C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ given by $\tilde{z} \mapsto \tilde{z}(1) - \tilde{z}(0)$ is continuous. Since $\alpha|_{B_{1/2}^{d'}(z_0)}$ is the composition of φ with this map, it is continuous. It follows that α is continuous. This proves Lemma 41. \square

The next result was used in Section 1 for the definition of the map $m_{C,\omega} : \mathcal{T}(C \times V, \omega) \rightarrow \mathbb{R}$.

Proposition 42. *[D. A. Salamon and E. Zehnder, Theorem 3.1. in [SZ]] There is a unique collection of continuous mappings $\rho_\omega : \text{Iso}(\omega) \rightarrow S^1$ (one for every symplectic vector space (V, ω)) satisfying the following conditions:*

- (i) *(Naturality:)* If (V, ω) and (V', ω') are symplectic vector spaces, $\Phi \in \text{Iso}(\omega, \omega')$ and $\Psi \in \text{Iso}(\omega)$ then $\rho_{\omega'}(\Phi\Psi\Phi^{-1}) = \rho_\omega(\Psi)$.
- (ii) *(Direct sum:)* If (V, ω) and (V', ω') are symplectic vector spaces and $\Phi \in \text{Iso}(\omega)$ and $\Phi' \in \text{Iso}(\omega')$ then $\rho_{\omega \oplus \omega'}(\Phi \oplus \Phi') = \rho_\omega(\Phi)\rho_{\omega'}(\Phi')$.
- (iii) *(Determinant:)* If $\Phi \in \text{Sp}(2n) \cap \text{O}(2n)$ then $\rho_{\omega_0}(\Phi) = \det(X + iY)$, where $X, Y \in \mathbb{R}^{n \times n}$ are such that

$$\Phi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

- (iv) *(Normalization:)* If $\Phi \in \text{Iso}(\omega)$ has no eigenvalue on the unit circle then $\rho_\omega(\Phi) = \pm 1$.

The maps ρ_ω in the collection of this proposition are called Salamon-Zehnder maps.

The next lemma was used in the proof of Proposition 28. We fix $\lambda \in \mathbb{C}$ and denote

$$E_\lambda^\Psi := \ker \left((\lambda \text{id} - \Psi)^{\dim V} : V \otimes \mathbb{C} \rightarrow V \otimes \mathbb{C} \right).$$

(If λ is an eigenvalue of Ψ then this is the generalized eigenspace of λ , otherwise it is $\{0\}$.) We fix $\lambda \in S^1 \setminus \{\pm 1\}$ and define

$$m_+(\omega, \Psi, \lambda) := \max \{ \dim_{\mathbb{C}} W \mid W \subseteq E_\lambda^\Psi \text{ complex subspace: } i\omega(\bar{v}, v) < 0, \forall 0 \neq v \in W \},$$

$$\text{and we denote } m_-(\Psi) := \frac{1}{2} \sum_{\lambda \in (-\infty, 0)} \dim_{\mathbb{C}} E_\lambda^\Psi.$$

Lemma 43. *The number $m_-(\Psi)$ is an integer, and*

$$(29) \quad \rho_\omega(\Psi) = (-1)^{m_-(\Psi)} \prod_{\lambda \in S^1 \setminus \{\pm 1\}} \lambda^{m_+(\omega, \Psi, \lambda)}.$$

Proof of Lemma 43. This follows from the proof of Theorem 3.1. in [SZ]. \square

Remark 44. *By Remark 30 we have*

$$(30) \quad \rho_{-\omega}(\Psi) = \overline{\rho_\omega(\Psi)}.$$

The following lemma was used in Section 1.

Lemma 45. *Let (M, ω, G, μ) be a Hamiltonian G -manifold, and let $\Sigma \in \mathcal{S}$. Assume that the action of G on $N := \mu^{-1}(0)$ is free. Then $m_{\Sigma, \omega, \mu} = m_{\Sigma, \omega, N}$.*

Proof of Lemma 45. Let $a \in [\mathbb{D}, M; N, \omega]$. We choose a representative $u \in C(\mathbb{D}, M)$ of a , and define $g \in C(S^1, G)$ to be the unique map satisfying $u(z) = g(z)u(1)$, for every $z \in S^1$. We choose a continuous symplectic trivialization $\Psi : \mathbb{D} \times \mathbb{R}^{2n} \rightarrow u^*TM$. We define $z \in C([0, 1], S^1)$ by $z(t) := e^{2\pi it}$ and $\tilde{\Psi} : [0, 1] \rightarrow \text{Sp}(2n)$ by $\tilde{\Psi}(t) := \Psi_{z(t)}^{-1}g(z(t))\Psi_1$. It follows that $m_{\text{GS}}(u) = m(\tilde{\Psi})$. We define the coisotropic subbundle $W \subseteq \mathbb{R}^{2n}$ by $W_z := \Psi_z^{-1}T_{u(z)}N$, and $\Phi := \Psi|_{S^1}^* \text{hol}^{N, \omega}$. We have $\tilde{\Psi}_{z(t)}W_1 = W_{z(t)}$, for every $t \in [0, 1]$. Lemma 45 follows now from the following claim.

Claim 1. *We have $\tilde{\Psi}(t)_{W_1} = \Phi([z|_{[0, t]})$, for every $t \in [0, 1]$.*

Proof of Claim 1: We choose a smooth map $f : TN_\omega \rightarrow N$ such that $f(0) = u(1)$ and $\text{pr}_N df(0) = \text{id}$. We define $\tilde{f} : [0, 1] \times TN_\omega \rightarrow N$ by $\tilde{f}(t, v) := g(z(t))f(v)$. It follows that $\text{hol}^{N, \omega}([u \circ z|_{[0, t]}) = \text{pr}_N d\tilde{f}(t, \cdot)(0) = (g(t) \cdot)_{T_{u(1)}}$, for every $t \in [0, 1]$. This implies that $\Phi([z|_{[0, t]}) = (\Psi_{z(t)}^{-1})_{T_{u \circ z(t)}N} (g(t) \cdot)_{T_{u(1)}} (\Psi_1)_{W_1} = \tilde{\Psi}(t)_{W_1}$. This proves Claim 1. \square

The following result was used in Section 1. Let (M, ω) be a closed connected symplectic manifold. Assume that there exists $a \in \mathbb{R}$ such that $[\omega] = 2ac_1(TM, \omega)$ on $[S^2, M]$, and that \mathcal{T}^k acts on M with moment map μ . The mixed action-Maslov index is a homomorphism $I : \pi_1(\text{Ham}(M, \omega)) \rightarrow \mathbb{R}$, where homotopy is taken with respect to the C^∞ -topology on $\text{Ham}(M, \omega)$. It is defined as follows (see [EP]). Let $A \in \pi_1(\text{Ham}(M, \omega))$. We choose a representative $\varphi \in C^\infty(S^1, \text{Ham}(M, \omega))$ of A . By Floer-theory there exists $u \in C^\infty(\mathbb{D}, M)$ such that $\varphi_z \circ u(1) = u(z)$, for every $z \in S^1$. We define $m(u, \varphi) \in \mathbb{Z}$ as follows. We choose a symplectic vector space (V, Ω) of dimension $\dim M$, and a trivialization $\Psi \in \text{Iso}(\mathbb{D} \times V, \Omega; u^*(TM, \omega))$. We define $m(u, \varphi) := m(S^1 \ni z \mapsto \Psi_z^{-1}d\varphi_z(u(1))\Psi_1 \in \text{Aut}(\Omega))$. By a standard homotopy argument this number does not depend on the choices of V, Ω

and Ψ . We define $I(A) := \int_{\mathbb{D}} u^* \omega - \int_{S^1} F \circ u d\theta - am(u, \varphi)$, where $\theta \in S^1$ denotes the angular coordinate, and $F \in C^\infty([0, 1] \times M, \mathbb{R})$ is the unique map whose flow is φ and that satisfies $\int_M F(t, \cdot) \omega^{\wedge n} = 0$, for every $t \in [0, 1]$. This number does not depend on the choice of φ and u (see [EP]).

The exponential map $\exp : \mathfrak{t} \rightarrow \mathcal{T}$ induces an isomorphism $\text{Hom}(\pi_1(\mathcal{T}), \mathbb{R}) \cong \mathfrak{t}^*$. The action of \mathcal{T} on M induces a homomorphism $\text{Hom}(\pi_1(\text{Ham}(M, \omega)), \mathbb{R}) \rightarrow \text{Hom}(\pi_1(\mathcal{T}), \mathbb{R})$. We define $\xi_{\text{spec}} \in \mathfrak{t}^*$ to be the image of $-I$ under the composition of these two maps, and the special fiber $N := \mu^{-1}(\xi_{\text{spec}} \subseteq M$.

Lemma 46. *If \mathbb{T} acts freely on N then for every $u \in C^\infty(\mathbb{D}, M; N, \omega)$ we have $\int_{\mathbb{D}} u^* \omega = am_{\mathbb{D}, N, \omega}([u])$.*

Proof of Lemma 46. Let $a \in \langle \mathbb{D}, M; N, \omega \rangle$. We choose a representative $\tilde{u} \in C(\mathbb{D}, M)$ of a . We define $\tilde{g} \in C(S^1, G)$ by $\tilde{u}(z) = \tilde{g}(z)\tilde{u}(1)$. Let $\xi \in \Gamma \subseteq \mathfrak{t}$. By definition we have

$$(31) \quad I(\langle \mu, \xi \rangle) = -\mathcal{A}_{\langle \mu, \xi \rangle}(u) + \frac{a}{2}m(u, \varphi_{\langle \mu, \xi \rangle}).$$

Furthermore, $\int_{S^1} \langle \mu \circ u(e^{2\pi i t}), \xi \rangle dt = \langle \mu \circ u(1), \xi \rangle = -I \circ \varphi_{\#}([S^1 \cong \mathbb{R}/\mathbb{Z} \ni t + \mathbb{Z} \mapsto \exp(t\xi)]) = -I(\varphi_{\langle \mu, \xi \rangle})$, since $u(1) \in \mu^{-1}(p_{\text{spec}})$. Combining this with (31), we obtain $\int_{\mathbb{D}} u^* \omega = am_{\mathbb{D}, \omega, \mu}([u]) = am_{\mathbb{D}, N, \omega}([u])$. Here in the last step we used Lemma 45. This proves Lemma 46. \square

In order to define the collection of maps m_{Σ}^L occurring in Theorem 24, for $\Sigma \in \mathcal{S}$ we define $\mathcal{E}_{\Sigma}^{\text{real}}$ to be the class of all triples (E, J, W) , where (E, J) is a complex vector bundle over Σ and $W \subseteq E|_{\partial\Sigma}$ is a totally real subbundle. By Theorem C.3.5. in the book [MS] by D. McDuff and D. A. Salamon there exists a unique collection of maps $m_{\Sigma}^{\text{real}} : \mathcal{E}_{\Sigma}^{\text{real}} \rightarrow \mathbb{Z}$, where $\Sigma \in \mathcal{S}$, satisfying suitable (Isomorphism), (Direct sum), (Composition) and (Normalization) conditions. Let $\Sigma \in \mathcal{S}$. We define the map $m_{\Sigma}^L : \mathcal{E}_{\Sigma}^L \rightarrow \mathbb{Z}$ as follows. Let $(E, \omega, W) \in \mathcal{E}^L$. We choose a fiberwise complex structure J on E that is ω -compatible, and we define

$$(32) \quad m_{\Sigma}^L(E, \omega, W) := m_{\Sigma}^{\text{real}}(E, J, W)$$

This number is well-defined, i.e. it does not depend on the choice of J .

In the following we define the linear holonomy along a leaf in a foliation. This was used in Section 1, in order to define the Maslov map. Let M be a manifold and \mathcal{F} a foliation on M , i.e. a maximal atlas of foliation charts. We denote by $T\mathcal{F} \subseteq TM$ and $N\mathcal{F} := TM/T\mathcal{F}$ the tangent and normal bundles of \mathcal{F} , by $\text{pr}^{\mathcal{F}} : TM \rightarrow N\mathcal{F}$ the canonical projection, by $\mathcal{F}_x \subseteq M$ the leaf through a point $x \in M$, and by $R^{\mathcal{F}} := \{(x, y) \in M \times M \mid y \in \mathcal{F}_x\}$ the leaf relation. For $x \in M$ we write $T_x\mathcal{F} := (T\mathcal{F})_x$ and $N_x\mathcal{F} := (N\mathcal{F})_x$. Let F be a leaf of \mathcal{F} , $a \leq b$, and $x \in C([a, b], F)$. The *linear holonomy of \mathcal{F} along x* is the linear map $\text{hol}_x^{\mathcal{F}} : N_{x(a)}\mathcal{F} \rightarrow N_{x(b)}\mathcal{F}$, whose definition is based on the following result.

Proposition 47. *Let M, \mathcal{F}, F, a, b and x be as above, N a manifold, and $y_0 \in N$. Then the following statements hold.*

(i) *For every linear map $T : T_{y_0}N \rightarrow T_{x(a)}M$ there exists a map $u \in C([a, b] \times N, M)$ such that*

$$(33) \quad u(\cdot, y_0) = x,$$

$$(34) \quad u(t, y) \in \mathcal{F}_{u(a, y)}, \forall t \in [a, b], y \in N,$$

$$(35) \quad u(t, \cdot) \text{ is differentiable at } y_0, \forall t \in [a, b],$$

$$(36) \quad d(u(a, \cdot))(y_0) = T.$$

(ii) *Let $u, u' \in C([a, b] \times N, M)$ be maps satisfying (33,34,35), such that*

$$(37) \quad \text{pr}^{\mathcal{F}} d(u(a, \cdot))(y_0) = \text{pr}^{\mathcal{F}} d(u'(a, \cdot))(y_0).$$

Then $\text{pr}^{\mathcal{F}} d(u(b, \cdot))(y_0) = \text{pr}^{\mathcal{F}} d(u'(b, \cdot))(y_0)$.

For the proof of Proposition 47 we need the following lemmas.

Lemma 48. *Let X be a connected topological space and $R \subseteq X \times X$ an equivalence relation on X . Assume that every equivalence class is open. Then $R = X \times X$.*

Proof of Lemma 48. This follows from an elementary argument. \square

By a foliation chart we mean a pair (U, φ) , where $U \subseteq M$ is an open subset and $\varphi : U \rightarrow \mathbb{R}^n$ is a smooth chart satisfying $d\varphi(x_0)T_{x_0}\mathcal{F} = \{0\} \times \mathbb{R}^k$, for every $x_0 \in U$. We denote by $\text{pr}_1 : \mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ and $\text{pr}_2 : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the canonical projections.

Lemma 49. *Let $F \subseteq M$ be a leaf of \mathcal{F} and (U, φ) a foliation chart. Then the subset $\text{pr}_1 \circ \varphi(U \cap F) \subseteq \mathbb{R}^{n-k}$ is at most countable.*

Proof of Lemma 49. Let (M, \mathcal{F}) be a foliated manifold. By definition, the leaf topology on F is the topology $\tau_F^{\mathcal{F}}$ generated by the sets $\varphi^{-1}(\{0\} \times \mathbb{R}^k)$, where $(U, \varphi) \in \mathcal{F}$ is such that $\varphi^{-1}(\{0\} \times \mathbb{R}^k) \subseteq F$. It is second countable, see for example Lemma 1.3. on p. 11 in the book [Mol]. It follows that there exists a countable collection of surjective foliation charts $\varphi_i : U_i \rightarrow \mathbb{R}^n$ ($i \in \mathbb{N}$), such that $(\varphi_i^{-1}(\{0\} \times \mathbb{R}^k))_{i \in \mathbb{N}}$ is a basis for $\tau_F^{\mathcal{F}}$. Let $(U, \varphi) \in \mathcal{F}$. Then $U \cap F \in \tau_F^{\mathcal{F}}$, and therefore there exists a subset $S \subseteq \mathbb{N}$ such that $U \cap F = \bigcup_{i \in S} U_i$. For each $i \in S$ compatibility of φ and φ_i implies that φ^ξ is constant on U_i . It follows that $\varphi^\xi(U \cap F) \subseteq \mathbb{R}^{n-k}$ is at most countable. The statement of Lemma 49 follows from this. \square

Lemma 50. *Let $M, \mathcal{F}, F, a, b, N$ and y_0 be as above, and $u \in C([0, 1] \times [a, b] \times N, F)$ be such that (35,37) hold and $u(s, i, y_0) = u(0, i, y_0)$, for every $s \in [0, 1]$ and $i = 0, 1$. If there exists a surjective foliation chart (U, φ) such that $u(s, t, y_0) \in U$ for every $s \in [0, 1]$ and $t \in [a, b]$, then $\text{pr}^{\mathcal{F}} d(u(0, b, \cdot))(y_0) = \text{pr}^{\mathcal{F}} d(u(1, b, \cdot))(y_0)$.*

For the proof of Lemma 50 we need the following.

Lemma 51. *Let M, \mathcal{F}, a, b, N and y_0 be as above, (U, φ) a foliation chart, and $u \in C([a, b] \times N, U)$ be such that (33,34,35) hold. Then $\text{pr}_1 d\varphi d(u(t, \cdot))(y_0) = \text{pr}_1 d\varphi d(u(a, \cdot))(y_0)$, for every $t \in [a, b]$.*

Proof of Lemma 51. We choose an open neighborhood $V \subseteq N$ of y_0 , such that $u([a, b] \times V) \subseteq U$ and $u'([a, b] \times V) \subseteq U$. Let $y \in V$. Since $u([a, b] \times \{y\}) = \{\mathcal{F}_{u(a,y)}\}$ and $\text{pr}_1 \circ \varphi \circ u([a, b] \times \{y\}) \subseteq \mathbb{R}^{n-k}$ is connected, it follows from Lemma 49 that $\text{pr}_1 \circ \varphi \circ u(t, y) = \text{pr}_1 \circ \varphi \circ u(a, y)$, for every $t \in [a, b]$. The statement of Lemma 51 follows from this. \square

Remark 52. *Let (U, φ) be a foliation chart and $x_0 \in U$. Then there exists a unique linear isomorphism $\Psi_{x_0} : \mathbb{R}^{n-k} \rightarrow N_{x_0}\mathcal{F} = T_{x_0}M/T_{x_0}\mathcal{F}$ satisfying $\text{pr}_{x_0}^{\mathcal{F}} = \Psi_{x_0}\text{pr}_1 d\varphi(x_0)$. To see this, observe that the map $\text{pr}_1 d\varphi(x_0) : T_{x_0}M \rightarrow \mathbb{R}^{n-k}$ is surjective and has kernel $T_{x_0}\mathcal{F}$.*

Proof of Lemma 50. For $x_0 \in U$ we define Ψ_{x_0} as in Remark 52. It follows from Lemma 51 that

$$\text{pr}^{\mathcal{F}} d(u(b, \cdot))(y_0) = \Psi_{x(b)} \text{pr}_1 d\varphi d(u(a, \cdot))(y_0) = \Psi_{x(b)} \Psi_{x(a)}^{-1} \text{pr}^{\mathcal{F}} d(u(a, \cdot))(y_0),$$

and $\text{pr}^{\mathcal{F}} d(u'(b, \cdot))(y_0) = \Psi_{x(b)} \Psi_{x(a)}^{-1} \text{pr}^{\mathcal{F}} d(u'(a, \cdot))(y_0)$. Using equality (37), the conclusion of (ii) follows. This proves Lemma 50. \square

We will use the following notations and conventions. Let $a, b \in \mathbb{R}$. If $a \leq b$ then we equip the interval $[a, b]$ with the positive orientation. If $a > b$ then we define $[a, b] := [b, a]$ and equip this interval with the negative orientation. We call a the initial point and b the end-point of I . Let I and I' be closed oriented intervals. We define the equivalence relation \sim on $I \amalg I'$ by $t \sim t'$ iff $t = t'$ or t is the end-point of I and t' is the initial point of I' . Furthermore, we define the connected sum $I \# I'$ to be the oriented topological one-manifold $(I \amalg I') / \sim$. Let now X and Y be sets, and $u : I \times Y \rightarrow X$ and $u' : I' \times Y \rightarrow X$ maps. We define the concatenation $u \# u' : (I \# I') \times Y \rightarrow X$ by $u \# u'([t], y) := u(t, y)$, if $t \in I$, and $u \# u'([t], y) := u'(t, y)$, otherwise. If $r > 0$ and $t \in [a, b]$ then we denote $\bar{B}_r(t) := [t - r, t + r] \cap [a, b]$.

Proof of Proposition 47. We prove **statement (i)**. We define \tilde{R} to be the set of all $(t_1, t_2) \in [a, b] \times [a, b]$ such that for every linear map $T : T_{y_0}N \rightarrow T_{x(t_1)}M$ there exists a map $u \in C([t_1, t_2] \times N, M)$ such that $u(\cdot, y_0) = x|_{[t_1, t_2]}$, $u(t, y) \in \mathcal{F}_{u(t_1, y)}$, for every $t \in [t_1, t_2]$ and $y \in N$, $u(t, \cdot)$ is differentiable at y_0 , for every $t \in [t_1, t_2]$, and $d(u(t_1, \cdot))(y_0) = T$. This is a relation on $[a, b]$.

Claim 1. *The relation \tilde{R} is reflexiv and transitive.*

Proof of Claim 1. To prove reflexivity, let $t_1 \in [a, b]$. We show that $(t_1, t_1) \in \tilde{R}$. Let $T : T_{y_0}N \rightarrow T_{x(t_1)}M$ be a linear map. We choose local parametrizations $\varphi : \mathbb{R}^n \rightarrow N$ and $\psi : \mathbb{R}^m \rightarrow M$ such that $\varphi(0) = y_0$ and $\psi(0) = x(t_1)$, and a smooth function $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ with compact support, such that $\rho = 1$ in a neighborhood of 0. We define $u : \{t_1\} \times N \rightarrow M$ by $u(t_0, y) := \psi(\rho \circ \varphi^{-1}(y) d\psi(0)^{-1} T d\varphi(0) \varphi^{-1}(y))$, if $y \in \varphi(\mathbb{R}^n)$, and $u(t_1, y) := x(t_1)$, otherwise. This map satisfies the condition in the definition of \tilde{R} with $t_2 = t_1$. This proves reflexivity.

To prove transitivity, let $t_1, t_2, t_3 \in [a, b]$ be such that $(t_1, t_2) \in \tilde{R}$ and $(t_2, t_3) \in \tilde{R}$, and $T : T_{y_0}N \rightarrow T_{x(t_1)}M$ be a linear map. We choose u as in the definition of \tilde{R} , and v as in this definition, with t_1, t_2 and T replaced by t_2, t_3 and $d(u(t_2, \cdot))(y_0)$. Then the map $u \# v$ satisfies the conditions in the definition of \tilde{R} , with t_1, t_2 replaced by t_1, t_3 . It follows that $(t_1, t_3) \in \tilde{R}$. This proves transitivity and completes the proof of Claim 1. \square

Claim 2. For every $t_1 \in [a, b]$ the set $S_{t_1} := \{t_2 \in [a, b] \mid (t_1, t_2) \in \tilde{R}\}$ is open.

Proof of Claim 2. Let $t_2 \in S_{t_1}$. We choose a map $u \in C([t_1, t_2] \times N, M)$ as in the definition of \tilde{R} and a pair (U, φ) , where $U \subseteq M$ is an open neighborhood of $x(t_2)$, and $\varphi : U \rightarrow \mathbb{R}^m$ is a surjective foliation chart. We also choose a number $\varepsilon > 0$ so small that $x(B_\varepsilon(t_2)) \subseteq U$. Let $t_3 \in B_\varepsilon(t_2)$. We define $v : [t_2, t_3] \times N \rightarrow M$ by $v(t, y) := \varphi^{-1}(\varphi \circ u(t_2, y) + \varphi \circ x(t) - \varphi \circ x(t_2))$. Since $x([t_2, t_3]) \subseteq F$ and $\varphi \circ x([t_2, t_3]) \subseteq \mathbb{R}^{n-k}$ is connected, it follows from Lemma 49 that $\text{pr}_1 \circ \varphi \circ x([t_2, t_3]) = \text{pr}_1 \circ \varphi \circ x(t_2)$. Hence $v(t, y) \in \mathcal{F}_{u(t_2, y)} = \mathcal{F}_{u(t_1, y)}$, for every $t \in [t_2, t_3]$ and $y \in N$. It follows that the concatenation $u \# v$ satisfies the conditions in the definition of \tilde{R} , with t_1, t_2 replaced by t_2, t_3 . Therefore, $t_3 \in S_{t_1}$. This proves that S_{t_1} is open. This proves Claim 2. \square

We define $R := \tilde{R} \cap \{(t_1, t_2) \mid (t_2, t_1) \in \tilde{R}\}$. By Claim 1 this is an equivalence relation on $[a, b]$. Let $t_1 \in [a, b]$. We define $S^{t_1} := \{t_2 \in [a, b] \mid (t_2, t_1) \in \tilde{R}\}$. Interchanging the roles of t_1 and t_2 , Claim 2 implies that S^{t_1} is open. The R -equivalence class of t_1 equals $S_{t_1} \cap S^{t_1}$ and hence is open. Therefore, by Lemma 48 $R = [a, b] \times [a, b]$. Statement (i) follows.

We prove **assertion (ii)**. Let u and u' be as in the hypothesis. We define \tilde{R} to be the set of all pairs $(t_1, t_2) \in [a, b] \times [a, b]$ such that

$$\text{pr}^{\mathcal{F}} d(u(t_1, \cdot))(y_0) = \text{pr}^{\mathcal{F}} d(u'(t_1, \cdot))(y_0) \Rightarrow \text{pr}^{\mathcal{F}} d(u(t_2, \cdot))(y_0) = \text{pr}^{\mathcal{F}} d(u'(t_2, \cdot))(y_0).$$

This is a reflexive and transitive relation on $[a, b]$.

Claim 3. For every $t_1 \in [a, b]$ the set $S_{t_1} := \{t_2 \in [a, b] \mid (t_1, t_2) \in \tilde{R}\}$ is open.

Proof of Claim 3. We choose $\varepsilon > 0$ so small that there exists a surjective foliation chart (U, φ) such that $x(\bar{B}_\varepsilon(t_1)) \subseteq U$. Let $t_2 \in \bar{B}_\varepsilon(t_1)$. Lemma 50 with a, b, u replaced by $t_1, t_2, u|_{[t_1, t_2]}$ implies that $t_2 \in S_{t_1}$. This proves Claim 3. \square

We define $R := \tilde{R} \cap \{(t_1, t_2) \mid (t_2, t_1) \in \tilde{R}\}$. This is an equivalence relation on $[a, b]$. Let $t_1 \in [a, b]$. We define $S^{t_1} := \{t_2 \in [a, b] \mid (t_2, t_1) \in \tilde{R}\}$. Interchanging the roles of t_1 and t_2 , Claim 3 implies that S^{t_1} is open. Since the R -equivalence class of t_1 equals $S_{t_1} \cap S^{t_1}$, the hypotheses of Lemma 48 are satisfied. It follows that $R = [a, b] \times [a, b]$. Assertion (ii) follows.

This completes the proof of Proposition 47. \square

We define $N := N_{x(a)}\mathcal{F}$ and $y_0 := 0$, and we canonically identify $T_0(N_{x(a)}\mathcal{F}) = N_{x(a)}\mathcal{F}$. We choose a linear map $T : N_{x(a)}\mathcal{F} \rightarrow T_{x(a)}M$, such that $\text{pr}^{\mathcal{F}}T = \text{id}_{N_{x(a)}\mathcal{F}}$, and a map $u \in C^\infty([a, b] \times N_{x(a)}\mathcal{F}, M)$ such that (33,34,35,36) hold. We define

$$\text{hol}_x^{\mathcal{F}} := \text{pr}^{\mathcal{F}}d(u(b, \cdot))(0) : N_{x(a)}\mathcal{F}(= T_0(N_{x(a)}\mathcal{F})) \rightarrow N_{x(b)}\mathcal{F}.$$

It follows from Proposition 47 that this map is well-defined. Consider now the set

$$\mathcal{X}^{\mathcal{F}} := \{x \in C([0, 1], M) \mid \exists F : \text{leaf of } \mathcal{F} : x([0, 1]) \subseteq F\}.$$

We define the map $\text{hol}^{\mathcal{F}} : \mathcal{X}^{\mathcal{F}} \rightarrow \text{GL}(N\mathcal{F})$ by $\text{hol}^{\mathcal{F}}(x) := \text{hol}_x^{\mathcal{F}}$.

Remark 53. *This map is a morphism of groupoids. To see this, observe that if $x : [a, b] \rightarrow M$ is constant then $\text{hol}_x^{\mathcal{F}} = \text{id}_{N_{x(a)}\mathcal{F}}$. Furthermore, If $a \leq b$ and $a' \leq b'$ are numbers, $x \in C([a, b], F)$ and $x' \in C([a', b'], M)$ are such that $x(b) = x'(a')$, then $\text{hol}_{x \# x'}^{\mathcal{F}} = \text{hol}_{x'}^{\mathcal{F}} \text{hol}_x^{\mathcal{F}}$. These assertions follow immediately from the definition of the holonomy along a path.*

Denoting by \bar{x} the map x together with the reversed orientation of $[a, b]$, it follows from Remark 53 that $\text{hol}_{\bar{x}}^{\mathcal{F}} = (\text{hol}_x^{\mathcal{F}})^{-1}$.

Proposition 54. *The map $\text{hol}^{\mathcal{F}}$ is continuous with respect to the compact open topology on $\mathcal{X}^{\mathcal{F}}$.*

Remark 55. *If $x_0 \in \mathcal{X}^{\mathcal{F}}$ is such that there exists a surjective foliation chart (U, φ) such that $x_0([a, b]) \subseteq U$ then $\text{hol}^{\mathcal{F}}$ is continuous at x_0 . To see this, we define Ψ_{x_0} as in Remark 52. Let $x \in \mathcal{X}^{\mathcal{F}}$ be such that $x([a, b]) \subseteq U$. It follows from Lemma 51 that $\text{hol}_x^{\mathcal{F}} = \Psi_{x(b)}\Psi_{x(a)}^{-1}$. This depends continuously on x .*

Proof of Proposition 54. Let $x \in \mathcal{X}^{\mathcal{F}}$. We define R to be the set of all pairs $(t_1, t_2) \in [a, b] \times [a, b]$ such that $\text{hol}^{\mathcal{F}}$ is continuous at the restriction $x|_{[t_1, t_2]} \in \mathcal{X}^{\mathcal{F}}$. It follows from Remark 55 that R is a reflexive relation. Remark 53 implies that it is symmetric and transitive. Furthermore, Remarks 53 and

55 imply that the R -equivalence classes are open. Therefore, by Lemma 48 we have $R = [a, b] \times [a, b]$. It follows that $\text{hol}^{\mathcal{F}}$ is continuous at x . This proves Proposition 54. \square

Proposition 56. *If F is a leaf of \mathcal{F} and $u \in C([0, 1] \times [a, b], F)$ is such that $u(s, i) = u(0, i)$, for every $s \in [0, 1]$ and $i = a, b$, then $\text{hol}_{u(0, \cdot)}^{\mathcal{F}} = \text{hol}_{u(1, \cdot)}^{\mathcal{F}}$.*

For the proof of Proposition 56 we need the following. Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$. We define x_{s_1, s_2, t_1, t_2} to be the concatenation of the paths $[s_1, s_2] \ni s \mapsto (s, t_1) \in \mathbb{R}^2$, $[t_1, t_2] \ni t \mapsto (s_2, t) \in \mathbb{R}^2$, $[s_2, s_1] \ni s \mapsto (s, t_2) \in \mathbb{R}^2$ and $[t_2, t_1] \ni t \mapsto (s_1, t) \in \mathbb{R}^2$.

Remark 57. *Let (U, φ) be a surjective foliation chart and $u \in C([s_1, s_2] \times [t_1, t_2] \rightarrow U \cap F)$. Then $\text{hol}_{u \circ x_{s_1, s_2, t_1, t_2}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, t_1)}M}$. This follows from Lemma 50 and the fact that x_{s_1, s_2, t_1, t_2} is homotopic in $[s_1, s_2] \times [t_1, t_2]$ to a constant path.*

Let F be a leaf of \mathcal{F} , $a, b, c, d \in \mathbb{R}$, $s_1 \in [a, b]$, and $u \in C([a, b] \times [c, d], F)$. We define R_{s_1} to be the set of all pairs $(t_1, t_2) \in [c, d] \times [c, d]$ such that there exists $\varepsilon > 0$ such that $\text{hol}_{u \circ x_{s_1, s_2, t_1, t_2}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, t_1)}M}$, for every $s_2 \in [s_1 - \varepsilon, s_1 + \varepsilon] \cap [a, b]$. It follows from Remark 53 that R_{s_1} is an equivalence relation on $[c, d]$.

Lemma 58. *We have $R_{s_1} = [c, d] \times [c, d]$.*

Proof of Lemma 58. By Lemma 48 it suffices to prove that for every $t_1 \in [c, d]$ the R_{s_1} -equivalence class of t_1 is open. To see this, let $t_2 \in [c, d]$ be such that $(t_1, t_2) \in R_{s_1}$. We choose $\varepsilon_1 := \varepsilon$ as in the definition of R_{s_1} , a surjective foliation chart (U, φ) such that $u(s_1, t_2) \in U$, and $\varepsilon_2 > 0$ so small that $u([s_1 - \varepsilon_2, s_1 + \varepsilon_2] \times [t_2 - \varepsilon_2, t_2 + \varepsilon_2]) \subseteq U$. We define $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Let $s_2 \in [s_1 - \varepsilon, s_1 + \varepsilon]$ and $t_3 \in [t_2 - \varepsilon, t_2 + \varepsilon]$. It follows that $\text{hol}_{u \circ x_{s_1, s_2, t_1, t_2}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, t_1)}M}$. Furthermore, by Remark 57 we have $\text{hol}_{u \circ x_{s_1, s_2, t_2, t_3}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, t_2)}M}$. Using Remark 53, it follows that $\text{hol}_{u \circ x_{s_1, s_2, t_1, t_2}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, t_1)}M}$. Therefore, $(t_1, t_3) \in R_{s_1}$. This proves Lemma 58. \square

Proof of Proposition 56. We define $R := \{(s_1, s_2) \in [0, 1] \times [0, 1] \mid \text{hol}_{u \circ x_{s_1, s_2, a, b}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, a)}M}\}$. It follows from Remark 53 that this is an equivalence relation on $[0, 1]$.

Claim 1. *For every $s_1 \in [0, 1]$ the R -equivalence class of s_1 is open.*

Proof of Claim 1. Let $s_2 \in [0, 1]$ be such that $(s_1, s_2) \in R$. Thus we have $\text{hol}_{u \circ x_{s_1, s_2, a, b}}^{\mathcal{F}} = \text{id}_{N_{u(s_1, a)}M}$. By Lemma 58 we have $(a, b) \in R_{s_2}$. Hence there exists $\varepsilon > 0$ such that for $s_3 \in [s_2 - \varepsilon, s_2 + \varepsilon]$ we have $\text{hol}_{u \circ x_{s_2, s_3, a, b}}^{\mathcal{F}} =$

$\text{id}_{N_{u(s_2,a)}M}$. Let $s_3 \in [s_2 - \varepsilon, s_2 + \varepsilon]$. Using Remark 53, it follows that $\text{hol}_{u \circ x_{s_1, s_3, a, b}}^{\mathcal{F}} = \text{id}_{N_{u(s_1,a)}M}$, i.e. $(s_1, s_3) \in R$. This proves Claim 1. \square

Claim 1 and Lemma 48 imply that $R = [0, 1] \times [0, 1]$. Using Remark 53, the statement of Proposition 56 follows. \square

It follows from Proposition 56 that the map $\text{hol}^{\mathcal{F}} : \mathcal{X}^{\mathcal{F}} \rightarrow \text{GL}(N\mathcal{F})$ descends to a morphism of topological groupoids

$$(38) \quad \text{hol}^{\mathcal{F}} : \mathcal{X}^{\mathcal{F}} / \sim_{\mathcal{X}^{\mathcal{F}}} \rightarrow \text{GL}(N\mathcal{F}).$$

(We use the same notation for this map.) We call this map the *linear holonomy of \mathcal{F}* .

The following result was used in the proof of Theorem 19 in Section 4. Let X be a topological manifold, $Y \subseteq X$, $E \rightarrow X$ a vector bundle, $\Phi : Y \times Y \rightarrow \text{GL}(E)$ a morphism of topological groupoids whose composition with the canonical projection $\text{GL}(E) \rightarrow X \times X$ is the identity, $k \in \mathbb{N}$ and $T : E^{\oplus k} \rightarrow \mathbb{R}$ be a (continuous) tensor. Assume that $T(\Phi_x^{x'} v_1, \dots, \Phi_x^{x'} v_k) = T(v_1, \dots, v_k)$, for every $x, x' \in Y$ and $v_1, \dots, v_k \in E_x$. We define \sim_{Φ} , E_{Φ} , π_{Φ} and T_{Φ} as in Section 4.

Lemma 59. *Assume that there exists a continuous injective map $f : [0, 1] \times Y \rightarrow X$ such that $f(\{0\} \times Y) = Y$. Then the pair (E_{Φ}, π_{Φ}) is a vector bundle and T_{Φ} is continuous.*

The next result is used in the proof of Lemmas 59 and 36.

Lemma 60. *Let X be a paracompact topological space and $E \rightarrow [0, 1] \times X$ a vector bundle. We denote by $\text{pr} : [0, 1] \times X \rightarrow X$ the canonical projection. Then there exists $\Psi \in \text{Iso}(\text{pr}^*E, E)$ such that $\Psi_{(0,x)} = \text{id}_{E_{(0,x)}}$, for every $x \in X$. Furthermore, if ω is a fiberwise symplectic structure on E then there exists Ψ as above that preserves ω .*

Proof of Lemma 60. To prove the first assertion, note that there exists $\tilde{\Psi} \in \text{Iso}(\text{pr}^*E, E)$, see for example [Hu], Part I Chap. 3, 4.4 Corollary (p. 28). We define $\Psi^{(t,x)} := \tilde{\Psi}^{(t,x)}(\tilde{\Psi}^{(0,x)})^{-1}$. The second assertion follows from a version of that corollary for symplectic vector bundles. This version is proved by the argument in [Hu], by choosing the local trivializations to be symplectic. \square

Proof of Lemma 59. To show that (E_{Φ}, π_{Φ}) is a vector bundle, let $x_0 \in X$. Assume that $x \notin Y$. We denote by n the rank of E and choose a pair (U, Ψ) , where $U \subseteq X$ is an open neighborhood of x_0 and $\Psi : U \times \mathbb{R}^n \rightarrow E$ is a local trivialization. Then viewing $U \setminus Y$ as a subset of X/Y the map $(U \setminus Y) \times \mathbb{R}^n \rightarrow E_{\Phi}$, $(x, v) \mapsto [x, v]$, is a local trivialization for E_{Φ} around the point x_0 . Assume now that $x_0 \in Y$. We denote $n := \text{rank}E$ and $U := f([0, 1] \times Y) \subseteq X$.

Claim 1. *There exists $\Psi \in \text{Iso}(U \times \mathbb{R}^n, E|_U)$ such that $\Psi^{(0,x')} = \Phi_x^{x'} \Psi^{(0,x)}$, for $x, x' \in Y$.*

Proof of Claim 1. We denote by $\text{pr} : [0, 1] \times Y \rightarrow Y$ the canonical projection, and choose f as in the hypothesis. We denote $E' := f^*E \rightarrow [0, 1] \times Y$. By an elementary argument there exists $\tilde{\Psi} \in \text{Iso}(\text{pr}^*E', E')$ such that $\tilde{\Psi}_{(0,x)} = \text{id}_{E'_x}$, for every $x \in Y$. We denote $n := \text{rank}E$ and choose a point $x_0 \in Y$ and $\Psi_0 \in \text{Iso}(\mathbb{R}^n, E_{x_0})$. We define $\Psi : U \times \mathbb{R}^n \rightarrow E$ by $\Psi^x := \tilde{\Psi}_{f^{-1}(x)} \Phi_{x_0}^{\text{pr} \circ f^{-1}(x)} \Psi_0$, for $x \in U$. This map has the required properties. This proves Claim 1. \square

We choose a map Ψ as in Claim 1, and define $\Psi' : U/Y \times \mathbb{R}^n \rightarrow E_\Phi$, $\Psi'([x], v) := [x, v]$. This is a local trivialization for E_Φ around x_0 . Furthermore, the map $U/Y \times (\mathbb{R}^n)^k \rightarrow \mathbb{R}$, $([x], v_1, \dots, v_k) \mapsto T_\Phi(\Psi'_{[x]}v_1, \dots, \Psi'_{[x]}v_k) = T_x(\Psi_x v_1, \dots, \Psi_x v_k)$ is continuous. It follows that (E_Φ, π_Φ) is a vector bundle and T_Φ is continuous. This proves Lemma 59. \square

The following result was used in the proof of Theorem 3. Let X be a topological space, $Y \subseteq X$ a subset, and \sim an equivalence relation on Y . We denote by $\iota : Y \rightarrow X$ the inclusion and by $\pi : Y \rightarrow Y/\sim$ the canonical projection. We fix a positive integer k , and we denote by $\bar{B}^k \subseteq \mathbb{R}^k$ and $S^{k-1} \subseteq \mathbb{R}^k$ the closed unit ball and the unit sphere. We define the map

$$(39) \quad \{u \in C(\bar{B}^k, X) \mid u \text{ is } (S^{k-1}, Y/\sim)\text{-compatible}\} \rightarrow C(\bar{B}^k, X \times Y/\sim),$$

$$(40) \quad \hat{\varphi}(u) := (u, \pi \circ u(z_0)),$$

where $z_0 \in S^{k-1}$ is an arbitrary point and we view $\pi \circ u(z_0)$ as a constant map from \bar{B}^k to Y/\sim . Note that $(S^{k-1}, Y/\sim)$ -compatibility of u implies that the right hand side of (40) does not depend on the choice of z_0 . Furthermore, the map $\hat{\varphi}(u)$ is $(S^{k-1}, \{\text{im}(\iota, \pi)\})$ -compatible, and the $(S^{k-1}, \{\text{im}(\iota, \pi)\})$ -compatible homotopy class of this map is invariant under $(S^{k-1}, Y/\sim)$ -compatible homotopies. Hence $\hat{\varphi}$ descends to a map

$$(41) \quad \varphi : [\bar{B}^k, S^{k-1}; X, Y/\sim] \rightarrow [\bar{B}^k, S^{k-1}; X \times Y/\sim, \{\text{im}(\iota, \pi)\}],$$

defining $\varphi([u]) := [\hat{\varphi}(u)]$. Recall that a Serre fibration is a continuous map with the homotopy lifting property for all CW-complexes.

Proposition 61. *Let X, Y, \sim, ι, π and k be as above. Assume that the map $\pi : Y \rightarrow Y/\sim$ is a Serre fibration. Then the map φ above is a bijection.*

Remark 62. *Let $\pi : X \rightarrow X'$ be a Serre fibration, Y be a CW-complex, and $u_0, u_1, v : [0, 1] \times Y \rightarrow X$ be continuous maps. Assume that*

$$(42) \quad v(i, \cdot) = u_i(0, \cdot), \quad i = 0, 1,$$

and there exists a continuous map $u' : [0, 1] \times [0, 1] \times Y \rightarrow X'$ such that

$$(43) \quad \pi \circ u_i = u'(i, \cdot, \cdot), \quad i = 0, 1, \quad \pi \circ v = u'(\cdot, 0, \cdot).$$

Then there exists a continuous map $u : [0, 1] \times [0, 1] \times Y \rightarrow X$ such that $\pi \circ u = u'$ and

$$u(i, \cdot, \cdot) = u_i, \quad i = 0, 1, \quad u(\cdot, 0, \cdot) = v.$$

This follows from the homotopy lifting property for $(\pi, [0, 1] \times Y)$, applied to the map $u' \circ (\varphi \times \text{id}_Y) : [0, 1] \times [0, 1] \times Y \rightarrow X'$, where $\varphi : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ is a homeomorphism that maps $[0, 1] \times \{0\}$ to $\{0, 1\} \times [0, 1] \cup [0, 1] \times \{0\}$.

Proof of Proposition 61. Let X, Y, \sim, ι, π and k be as in the hypothesis. We define the map

$$\psi : [\bar{B}^k, S^{k-1}; X \times Y/\sim, \{\text{im}(\iota, \pi)\}] \rightarrow [\bar{B}^k, S^{k-1}; X, Y/\sim]$$

as follows. Namely, let $\tilde{a} \in [\bar{B}^k, S^{k-1}; X \times Y/\sim, \{\text{im}(\iota, \pi)\}]$. We fix a representative $(u, v') : \bar{B}^k \rightarrow X \times Y/\sim$ of \tilde{a} .

Claim 1. *There exists a continuous map $f : \bar{B}^k \rightarrow X$ such that*

$$(44) \quad f(z) = u(2z), \quad \text{if } |z| \leq \frac{1}{2},$$

$$(45) \quad f(z) \in Y, \quad \pi \circ f(z) = v'((2/|z| - 2)z), \quad \text{if } \frac{1}{2} < |z| \leq 1.$$

Proof of Claim 1. We define $w' : [0, 1] \times S^{k-1} \rightarrow Y/\sim$ by $w'(r, z) := v'((1 - r)z)$. By $(S^{k-1}, \{\text{im}(\iota, \pi)\})$ -compatibility of (u, v') we have $w'(0, z) = \pi \circ u(z)$, for every $z \in S^{k-1}$. Therefore, by the homotopy lifting property of π there exists a continuous map $w : [0, 1] \times S^{k-1} \rightarrow Y$ such that $\pi \circ w = w'$ and $w(0, z) = u(z)$, for every $z \in S^{k-1}$. We define

$$f : \bar{B}^k \rightarrow X, \quad f(z) := \begin{cases} u(2z), & \text{if } |z| \leq \frac{1}{2}, \\ w(2|z| - 1, z/|z|), & \text{if } \frac{1}{2} < |z| \leq 1. \end{cases}$$

This proves Claim 1. \square

Note that if f is as in Claim 1 then $\pi \circ f(z) = v'(0)$, for $z \in S^{k-1}$, and therefore the map f is $(S^{k-1}, Y/\sim)$ -compatible.

Claim 2. *If (u_0, v'_0) and (u_1, v'_1) are two representatives of \tilde{a} and $f_0, f_1 : \bar{B}^k \rightarrow X$ are continuous maps satisfying (44,45) with u, v', f replaced by u_i, v'_i, f_i , for $i = 0, 1$, then the maps f_0 and f_1 are homotopic compatibly with $(S^{k-1}, Y/\sim)$.*

Proof of Claim 2. Let $(u, v') : [0, 1] \times \bar{B}^k \rightarrow X \times Y/\sim$ be continuous maps such that

$$(46) \quad (u, v')(i, \cdot) = (u_i, v'_i), \quad (u, v')(\{s\} \times S^{k-1}) \subseteq \text{im}(\iota, \pi), \quad \forall s \in [0, 1].$$

We define

$$(47) \quad w_i : [0, 1] \times S^{k-1} \rightarrow Y, \quad w_i(t, z) := f_i\left(\frac{t+1}{2}z\right),$$

for $i = 0, 1$, and $v := u|_{[0,1] \times S^{k-1}}$. Then the conditions of Remark 62 with $Y := S^{k-1}$ and X, X', u_i replaced by Y, Y', w_i are satisfied. To see this, note that (42) follows from (44). We define

$$(48) \quad w' : [0, 1] \times [0, 1] \times S^{k-1} \rightarrow Y/\sim, \quad w'(s, t, \cdot) := v'(s, (1-t)z).$$

Then for every $t \in [0, 1]$, $z \in S^{k-1}$, we have

$$\begin{aligned} \pi \circ w_i(t, z) &= \pi \circ f_i\left(\frac{t+1}{2}z\right) \\ &= v'_i((1-t)z) \\ &= v'(i, (1-t)z) \\ &= w'(i, t, z). \end{aligned}$$

Here in the first step we used (44) with f, u replaced by f_i, u_i and $z \in S_{1/2}^{k-1}$, and in the second step we used (45) with f, v' replaced by f_i, v'_i . So the second hypothesis of Remark 62 is also satisfied, with $Y := S^{k-1}, u' := w'$ and X, X', u_i replaced by Y, Y', w_i . It follows that there exists a continuous map $w : [0, 1] \times [0, 1] \times S^{k-1} \rightarrow Y$ such that

$$(49) \quad \pi \circ w = w', \quad w(i, \cdot, \cdot) = w_i, \quad i = 0, 1, \quad w(\cdot, 0, \cdot) = v = u|_{[0,1] \times S^{k-1}}.$$

We define $f : [0, 1] \times \bar{B}^k \rightarrow X$ by

$$f(s, z) := \begin{cases} u(s, 2z), & \text{if } |z| \leq \frac{1}{2}, \\ w(s, 2|z| - 1, z/|z|), & \text{if } \frac{1}{2} < |z| \leq 1. \end{cases}$$

By the third equality in (49) the map f is continuous. Furthermore, (44) with f, u replaced by f_i, u_i , (47) and the second equality in (49) imply that $f(i, z) = f_i(z)$, for $i = 0, 1$ and every $z \in \bar{B}^k$. Finally, let $s \in [0, 1]$. Then by (48) and the first equality in (49) we have, $\pi \circ f(s, z) = v'(s, 0)$, for $z \in S^{k-1}$. Hence $f(s, \cdot)$ is $(S^{k-1}, Y/\sim)$ -compatible, and therefore f is a $(S^{k-1}, Y/\sim)$ -compatible homotopy from f_0 to f_1 . This proves Claim 2. \square

We choose a map $f : \bar{B}^k \rightarrow X$ as in Claim 1 and define $\psi(\tilde{a})$ to be the $(S^{k-1}, Y/\sim)$ -compatible homotopy class of f . By Claim 2 this definition does not depend on the choice of f .

Claim 3. *The maps φ and ψ are inverses of each other.*

Proof of Claim 3: To see that $\psi \circ \varphi = \text{id}$ let $a \in [\bar{B}^k, S^{k-1}; X, Y/\sim]$.

We choose a representative u of a , and define $f : \bar{B}^k \rightarrow X$ by

$$f(z) := \begin{cases} u(2z), & \text{if } |z| \leq \frac{1}{2}, \\ u(z/|z|), & \text{otherwise.} \end{cases}$$

Note that f is continuous and $(S^{k-1}, Y/\sim)$ -compatible. Note that $(S^{k-1}, Y/\sim)$ -compatible homotopy classes of u and f agree. The identity $\psi \circ \varphi = \text{id}$ follows now from the next claim.

Claim 4. *The $(S^{k-1}, Y/\sim)$ -compatible homotopy class of f equals $\psi \circ \varphi(a)$.*

Proof of Claim 4. By definition $\varphi(a) = [u, \pi \circ u(z_0)]$, where $z_0 \in S^{k-1}$ is an arbitrary point. Furthermore, equalities (44,45) are satisfied, with $v' := \pi \circ u(z_0)$. Hence f represents $\psi \circ \varphi(a)$. This proves Claim 4. \square

To see that $\varphi \circ \psi = \text{id}$ let $\tilde{a} \in [\bar{B}^k, S^{k-1}; X \times Y/\sim, \{\text{im}(\iota, \pi)\}]$. We choose a representative (u, v') of \tilde{a} and a continuous map $f : \bar{B}^k \rightarrow X$ such that the conditions (44,45) hold. We fix $z_0 \in S^{k-1}$. Then by definition the maps $(f, \pi \circ f(z_0))$ and $\varphi \circ \psi(\tilde{a})$ are homotopic compatibly with $(S^{k-1}, \{\text{im}(\iota, \pi)\})$. The identity $\varphi \circ \psi = \text{id}$ follows now from the next claim.

Claim 5. *The $(S^{k-1}, \{\text{im}(\iota, \pi)\})$ -compatible homotopy class of $(f, \pi \circ f(z_0))$ equals \tilde{a} .*

Proof of Claim 5: We define the map $h : [0, 1] \times \bar{B}^k \rightarrow X \times Y/\sim$ by

$$h(s, z) := \left(f\left(\left(1 - \frac{s}{2}\right)z\right), v'(sz) \right).$$

Then

$$\begin{aligned} h(0, \cdot) &= (f, \pi \circ f(z_0)), & h(1, \cdot) &= (u, v'), \\ h(s, z) &\in \text{im}\{\iota, \pi\}, \quad \forall s \in [0, 1], z \in S^{k-1}. \end{aligned}$$

Here in the first equality we used (45), in the second equality we used (44), and in the condition we used (45) again. Hence h is a $(S^{k-1}, \{\text{im}(\iota, \pi)\})$ -compatible homotopy from $(f, \pi \circ f(z_0))$ to (u, v') . This proves Claim 5 and hence Claim 3, and concludes the proof of Proposition 61. \square

Open compact topology. The following lemma was used in the proofs of Lemmas 10, 41. For two topological spaces X and Y we equip the set of continuous maps $C(X, Y)$ with the compact open topology.

Lemma 63. *Let X, Y and Z be topological spaces. Then the following statements hold.*

- (i) *If Y is locally compact and Hausdorff then the composition map $C(X, Y) \times C(Y, Z) \ni (f, g) \mapsto g \circ f \in C(X, Z)$ is continuous.*
- (ii) *If X is locally compact and Hausdorff then the evaluation map $C(X, Y) \times X \ni (f, x) \mapsto f(x) \in Y$ is continuous.*
- (iii) *If X is Hausdorff and Y is locally compact and Hausdorff then the map $\varphi : C(X, C(Y, Z)) \rightarrow C(X \times Y, Z)$ defined by $\varphi(f)(x, y) := f(x)(y)$, for $(x, y) \in X \times Y$, is well-defined and a homeomorphism.*
- (iv) *If X is compact Hausdorff and Y is metrized by a metric d , then $C(X, Y)$ is metrized by the metric d' defined by*

$$(50) \quad d'(f, g) := \sup \{d(f(x), g(x)) \mid x \in X\}.$$

Proof of Lemma 63. These are standard results, see for example the book [Hat]. \square

Lemma 64. *For every symplectic vector space (V, ω) the map $m_\omega : C([0, 1], \text{Aut}\omega) \rightarrow \mathbb{R}$ is continuous with respect to the compact open topology.*

Proof of Lemma 64. Since ρ_ω is continuous, by Lemma 63(i) the map $C([0, 1], \text{Aut}\omega) \ni \Phi \mapsto \rho_\omega \circ \Phi \in C([0, 1], S^1)$ is continuous. By Lemma 41 the winding map $\alpha : C([0, 1], S^1) \rightarrow \mathbb{R}$ is continuous. Since the map m_ω is the composition of these two maps, the statement of Lemma 64 follows. \square

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