FUNCTORIALITY FOR GROMOV-WITTEN INVARIANTS UNDER SYMPLECTIC QUOTIENTS

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Contents

Preliminary version.

1. Introduction

This manifesto describes conjectural generalizations of the Kirwan map, non-abelian localization, and abelianization to the setting of genus zero Gromov-Witten theory, and their relationship with conjectures of Bertram, Ciocan-Fontanine, and Kim [2]. Let G be a compact, connected Lie group, X a Hamiltonian G-manifold with proper moment map and $X/\!\!/G$ its symplectic quotient; we suppose throughout that $X/\!\!/G$ is a free quotient. The *Kirwan map* refers to the surjection in cohomology

$$\kappa_G : H_G(X, \mathbb{Q}) \to H(X/\!\!/G, \mathbb{Q}) \quad (1)$$

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induced by restriction to the zero level set of the moment map and quotient by the action [22]. By *non-abelian localization* we mean the description of failure of commutativity of the diagram

$$\begin{array}{ccc} H_G(X,\mathbb{Q}) & \xrightarrow{\kappa_G} & H(X/\!\!/G,\mathbb{Q}) \\ & {\scriptstyle\int_{X\times\mathfrak{g}}}\searrow \quad \swarrow{\scriptstyle\int_{X/G}} & \\ & \mathbb{Q} & \end{array} \tag{2}$$

by a sum of fixed point contributions from one-parameter subgroups, as first suggested by Witten [38] and described in Paradan [33] and Woodward [40]. For let T be a maximal torus of G with Weyl group $W = N(T)/T$. By *abelianization*, we mean the existence of a commutative square

$$\begin{array}{ccc} & H_G(X,\mathbb{Q}) & \\ \swarrow & & \searrow \\ H(X/\!\!/G,\mathbb{Q}) & & H(X/\!\!/T,\mathbb{Q}) \\ {\scriptstyle\int_{X/G}}\searrow & & \swarrow{\scriptstyle\int^{twist}_{X/T}} \\ & \mathbb{Q} & \end{array} \tag{3}$$

described by Martin [27], where the right integration map is twisted by a certain Euler class, see Section 2.4 below.

Our basic proposal is that the diagrams (1), (2), and (3) have generalizations to the quantum (that is, Gromov-Witten) setting that are *diagrams of cohomological field theories*. To explain the basic idea, recall that Gromov-Witten theory count holomorphic maps of curves to smooth projective varieties, or in its symplectic version, to compact symplectic manifolds equipped with compatible almost complex structures. These invariants satisfy a splitting axiom which describes what happens when the domain degenerates. Restricting to genus zero, the resulting invariants can be organized into a *cohomological field theory* (CohFT), a collection of *correlators*

$$\tau^n : QH(X)^{\otimes n} \otimes H(\overline{M}_{0,n}) \to \Lambda_X \tag{4}$$

where Λ_X is a Novikov ring, satisfying certain properties described below in Definition 3.1.1. Alternatively, by duality, one can define a CohFT as a collection of *composition maps*

$$\mu^n : QH(X)^{\otimes n} \otimes H(\overline{M}_{0,n+1}) \to QH(X).$$

In this formulation, CohFT's can be viewed as complex analogs of A_∞ algebras, see Ma'u-Woodward [28], with the splitting axiom taking the place of the associativity relations. The proposal of [28] is that the notion of morphism of cohomological field theory should be based on the analog with A_∞ morphisms. Namely, one should take the *multiplihedron* $\overline{M}_{n,1}(\mathbb{R})^+$ which appears in Stasheff's definition of A_∞ morphism, and embed it as the positive real part of some normal projective variety $\overline{M}_{n,1}(\mathbb{C})$ of *scaled marked lines*, that is, marked lines equipped with translation-invariant area forms. Then, a *morphism of CohFT's* $V = (V, \mu^\bullet_V)$ to $W = (W, \mu^\bullet_W)$ is a collection of maps

$$\phi^n : V^n \otimes H(\overline{M}_{n,1}(\mathbb{C})) \to W$$

satisfying a certain splitting axiom which "complexifies" that for A_∞ morphisms. We call a collection of maps τ^n as in (4) a *cohomological trace*. In the string theory picture, a morphism of CohFT's resp. cohomological trace is obtained by a sum of surfaces with one resp. zero outgoing boundary component. A *diagram of CohFT's* consists of an oriented graph where each node is either a CohFT or a Novikov ring and each edge is either a morphism of CohFT's or a trace. The notion of *commutative diagram of CohFT's* is somewhat subtle; composition of morphisms of CohFT's, or a morphism of CohFT's with a cohomological trace, is not canonically defined but depends on additional cohomological information in the same way that the composition maps in a CohFT depend on a choice of cohomology class on the moduli space of curves. Nevertheless, the failure of commutative of a given triangle or square to commute can be measured on cohomological classes in some slightly more complicated moduli space, such as the moduli space of affine lines equipped with *two* area forms (in the case of compositions of morphisms of CohFT's) or the moduli space of projective lines equipped with a *varying* area form, in the case of the composition of a morphism with a trace. We say that the a diagram of CohFT's *commutes* if the failure to commute vanishes on all such classes.

In this manifesto, we describe how these ideas should give appear in the case of functoriality of Gromov-Witten theory under the symplectic quotient construction. The second author constructed in his thesis a moduli space of *symplectic vortices* over the affine line, obtained from the symplectic quotient construction and a certain twisting of the usual Cauchy-Riemann equation. Integrating over these moduli spaces formally defines a collection of maps

$$Q\kappa^n : QH_G(X)^{\otimes n} \otimes H(\overline{M}_{n,1}(\mathbb{C})) \to QH(X/\!\!/G)$$

which we call the *quantum Kirwan morphism* and denote

$$Q\kappa : GW_G(X) \to GW(X/\!\!/G).$$

By studying symplectic vortices on the projective line and allowing the area form that appears in the vortex equation to vary one should obtain a *quantum generalization of non-abelian localization*, namely that the diagram

$$\begin{array}{ccc} GW_G(X) & \longrightarrow & GW(X/\!\!/G) \\ & \searrow \quad \swarrow & \\ & \Lambda_X^G & \end{array}$$

.

fails to commute by an explicit sum of wall-crossing terms involving extended vortices of groups with smaller rank. Finally, an examination of wall-crossing terms and induction on the dimension of the group lead to the *quantum abelianization conjecture* that the diagram

$$\begin{array}{ccccc} & & GW_G(X) & & \\ & \swarrow & & \searrow & \\ GW(X/\!\!/G) & & & & GW(X/\!\!/T) \\ & \searrow & & \swarrow & \\ & & \Lambda_X^G & & \end{array}$$

with correlators resp. twisted correlators as the southwest and southeast arrows, commutes. Composition of the two sides of the diagram is defined similarly to composition of A_∞ morphisms and involves the higher compositions $Q\kappa^\bullet$. This conjecture is a modified version of that of Bertram et al [2].

This is *not* a research announcement, rather a collection of conjectures. Some rigorous results (in addition to those of [2]) can be found in [12] and, as relates to the quantum Kirwan morphism, [41].

2. Equivariant symplectic geometry

2.1. **Equivariant cohomology and localization.** Let G be a compact, connected group and $EG \to BG$ a universal G-bundle. Let X be a G-space, say with the homotopy type of $G - CW$ complex. We denote by $X_G = (X \times EG)/G$ the *homotopy quotient* of X, and by

$$H_G(X) := H(X_G)$$

the *equivariant cohomology* of X. If X has a free G-action, then the map $X \to X/G$ induces a pull-back isomorphism $H(X/G) \to H_G(X)$. More generally, if $K \subset G$ is a normal subgroup acting freely then pull-back under $X \to X/K$ induces an isomorphism

$$H_G(X) \to H_{G/K}(X/K). \tag{5}$$

Suppose that $T \subset G$ is a maximal torus with Weyl group $W = N(T)/T$. The group W naturally acts on ET, hence X_T, by homeomorphisms, and induces an action of W on the equivariant cohomology $H_T(X)$. We denote by $H_T(X)^W$ the invariant part. The inclusion $T \to G$ induces a map $H_G(X) \to H_T(X)$; with image contained in $H_X(T)^W$; over the rationals the map

$$H_G(X, \mathbb{Q}) \to H_T(X, \mathbb{Q})^W$$

is an isomorphism, see [19].

The G-space X is *equivariantly formal* if the natural map

$$H(X) \to H_G(X) \tag{6}$$

is an injection or equivalently if $H_G(X) \cong H(X) \otimes H(BG)$ as $H(BG)$-modules.

Suppose that X is a compact, oriented manifold. X defines an *equivariant fundamental class*

$$[X]_G \in H^G_{\dim(X)}(X).$$

Product with $[X]_G$ defines a map

$$\int_X : H_G^{\bullet}(X,\mathbb{Z}) \to H_G^{\bullet - \dim(X)}(\mathrm{pt},\mathbb{Z}). \tag{7}$$

If X is equivariantly formal, this induces an isomorphism over the rationals (fiberwise Poincaré duality)

$$H_G^{\bullet}(X,\mathbb{Q}) \to \mathrm{Hom}_{H_G(\mathrm{pt},\mathbb{Q})}(H_G^{\bullet}(X,\mathbb{Q}), H_G^{\bullet-\dim(X)}(\mathrm{pt},\mathbb{Q})) \tag{8}$$

see for example [14], [3].

Suppose that G is a torus. The equivariant integration map (7) can be computed using localization techniques as follows. Let $\beta \in \mathfrak{g}$, and X^β the fixed point set of β. Let ν^β be the normal bundle of the embedding $X^\beta \to X$. The intersection pairings on X are related to those on X^β by

$$\int_X \alpha = \int_{X^\beta} \iota_\beta^* \alpha \wedge \mathrm{Eul}(\nu^\beta)^{-1}. \tag{9}$$

In particular, localization implies that the restriction map $\iota_\beta : H_G(X,\mathbb{Q}) \to H_G(X^\beta,\mathbb{Q})$ is injective.

Let X be a smooth G-manifold. Equivariant cohomology with real coefficients has a *Cartan model* given by equivariant differential forms, as follows. The action of G on X gives rise to a homomorphism

$$\mathfrak{g} \to \mathrm{Vect}(X), \qquad \xi \mapsto \xi_X.$$

We denote by

$$\Omega_G(X) = \mathrm{Hom}(\mathfrak{g}, \Omega(X))^G$$

the space of equivariant forms on X with smooth coefficients, and $H_G^{dR}(X,\mathbb{R})$ the cohomology of the equivariant de Rham operator

$$\mathrm{d}_G \in \mathrm{End}(\Omega_G(X))[1], \quad (\mathrm{d}_G\alpha)(\zeta) := (\mathrm{d}\alpha)(\zeta) + (\iota(\zeta_X)\alpha)(\zeta).$$

Then $H_G^{dR}(X,\mathbb{R})$ is naturally isomorphic to $H_G(X,\mathbb{R})$, see [17].

2.2. **Hamiltonian group actions.** Let G be a compact, connected Lie group with complexification $G_\mathbb{C}$ and Lie algebra $\mathfrak{g}$. Let X denote a compact Hamiltonian G-manifold with symplectic form ω and moment map $\Phi : X \to \mathfrak{g}^*$, that is, an equivariant map satisfying

$$\iota(\xi_X)\omega = -\mathrm{d}(\Phi,\xi), \qquad \forall \xi \in \mathfrak{g}.$$

The *equivariant symplectic form*

$$\omega_G(\zeta) := \omega + (\Phi,\xi) \in \Omega_G^2(X)$$

is equivariantly closed. We denote by $[\omega_G]$ its class in $H_G^2(X)$.

The *symplectic quotient* of X by G is

$$X/\!\!/G := \Phi^{-1}(0)/G.$$

If G acts freely on $\Phi^{-1}(0)$, then $X/\!\!/G$ naturally has the structure of a symplectic manifold with symplectic form $\omega/\!\!/G \in \Omega^2(X/\!\!/G)$ defined by

$$p^*(\omega/\!\!/G) = \iota^*\omega$$

where

$$\begin{array}{ccc} \Phi^{-1}(0) & \xrightarrow{\iota} & X \\ \downarrow{\scriptstyle p} & & \\ X/\!\!/G & & \end{array}$$

are the inclusion and projection respectively [30],[25].

A *polarization* of X is a G-equivariant line bundle $\pi : L \to X$ equipped with a connection $\alpha \in \Omega^1(L_1)^G$ with $\mathrm{d}_G\alpha = -\pi^*\omega_G$. In particular, the curvature of α is ω. If $L \to X$ is a polarization, then the line bundle

$$L/\!\!/G := (L|\Phi^{-1}(0))/G$$

with connection defined by

$$(\alpha/\!\!/G) \in \Omega^1((L/\!\!/G)_1), \quad \hat{p}^*(\alpha/\!\!/G) = \hat{\iota}^*\alpha$$

is a polarization of $X/\!\!/G$; here $\hat{p}, \hat{\iota}$ are the lift of p, ι to $L|\Phi^{-1}(0)$.

Suppose that $K \subset G$ is a normal subgroup of G. In this case, we can quotient first by K, and then by G/K. The result identifying the two-fold quotient with the quotient by G is known as *reduction in stages* [26]: There exists a homeomorphism (symplectomorphism on the smooth locus)

$$X/\!\!/G \to (X/\!\!/K)/\!\!/(G/K).$$

In particular, if all symplectic quotients are free then there is a commutative diagram in equivariant cohomology

$$\begin{array}{ccc} H_G(X) & \xrightarrow{\kappa_G} & H(X/\!\!/G) \\ & {\scriptstyle \kappa_K}\searrow \quad \nearrow{\scriptstyle \kappa_{G/K}} & \\ & H_{G/K}(X/\!\!/K) & \end{array}.$$

Let X be a smooth, projectively embedded $G_{\mathbb{C}}$-variety, that is, embedded in the projectivization of some $G_{\mathbb{C}}$-representation. Recall the geometric invariant theory quotient [31] of Mumford: A point x is *semistable* if there exists an invariant section of some power of the hyperplane bundle L that is non-vanishing at x:

$$X^{\mathrm{ss}} = \{x \in X | \exists n \in \mathbb{N}, s \in H^0(L^n)^G, \, s(x) \neq 0\}.$$

Let X^{ss} denote the subset of semistable points and define the geometric invariant theory quotient

$$X/\!\!/_{\mathrm{git}}G = X^{\mathrm{ss}}/\sim$$

where $x \sim x'$ if the orbit closures intersect in X^{ss}:

$$x \sim x' \iff \overline{Gx} \cap \overline{Gx'} \cap X^{\mathrm{ss}} \neq \emptyset.$$

A semistable point is called *polystable* if its orbit is closed in X^{ss}, and *stable* if in addition its automorphism group is finite. Any semistable orbit contains a unique polystable orbit in its closure. If every semistable point is stable, or equivalently every polystable point is stable, then the quotient is a smooth projective variety. The Fubini-Study form on projective space induces on X a symplectic form and moment map $\Phi : X \to \mathfrak{g}^*$, giving X the structure of a Hamiltonian G-manifold, with polarization given by the restriction of the hyperplane bundle. Inclusion of the zero level set in X induces a homeomorphism

$$X/\!\!/_{\mathrm{symp}} G \to X/\!\!/_{\mathrm{git}} G$$

by results of Guillemin-Sternberg [16], Kirwan [22] and Heinzner-Loose [18].

2.3. **Kirwan homomorphism and non-abelian localization.** The relation between the cohomology of X and the cohomology of the quotient $X/\!\!/G$ is studied by Kirwan [22], via the use of the stratification induced by the norm-square of of the moment map as follows. Fix an invariant inner product on the Lie algebra $\mathfrak{g}$. This induces an equivariant identification $\mathfrak{g} \to \mathfrak{g}^*$. Consider the smooth invariant function

$$\|\Phi\|^2 : X \to \mathbb{R}.$$

The critical set of $\|\Phi\|^2$ consists of points x stabilized by $\Phi(x)$; we denote by C_λ the component of the critical set intersecting $\Phi^{-1}(\lambda)$. The stable manifold X_λ consists of points flowing to C_λ, and λ is called the *type* of the stratum X_λ, and we may write X as a disjoint union

$$X = \bigcup_\lambda X_\lambda .$$

Kirwan's main result is that the stratification is equivariantly perfect: the spectral sequence degenerates after the first page and hence

$$H_G^\bullet(X) \cong \bigoplus_\lambda H_G^{\bullet - \mathrm{codim}(X_\lambda)}(X_\lambda).$$

This implies that any compact Hamiltonian G-manifold is equivariantly formal. Restriction to the zero level set, followed by the quotient defines a surjection called the *Kirwan map*

$$\kappa_G : \ H_G(X) \to H(X/\!\!/G).$$

The intersection pairings over X and $X/\!\!/G$ are related by a *non-abelian localization formula* [40], also [33],

$$\int_X \alpha = \sum_\lambda \int_{X^{\lambda,\circ}} \iota_\lambda^* \alpha \wedge \mathrm{Eul}(\nu^\lambda)^{-1}$$

where $X^{\lambda,\circ}$ is an open neighborhood of $\Phi^{-1}(\lambda) \cap X^\lambda$ in the fixed point set X^λ of λ, the integrals on the right-hand side are interpreted as a distributional limit. In other words, there is a commutative diagram

$$\begin{array}{ccc} H_G(X) & \longrightarrow & \bigoplus_\lambda H_G(X^{\lambda,\circ}) \\ & \searrow \quad \swarrow & \\ & H_G(\mathrm{pt}) & \end{array} .$$

In particular, the diagram obtained by restricting to the symplectic quotient and taking invariant parts (that is, integrating over $\mathfrak{g}$; of course this is only defined by a suitable limit, see [40])

$$\begin{array}{ccc} H_G(X) & \xrightarrow{\kappa_G} & H(X/\!\!/G) \\ & \searrow^{\int_{X\times\mathfrak{g}}} \quad {}^{\int_{X/G}}\swarrow & \\ & \mathbb{Q} & \end{array}$$

fails to commute by an explicit sum of fixed point contributions from the one-parameter subgroups of G.

2.4. **Abelianization.** Let T be a maximal torus of G with Weyl group $W = N(T)/T$, and $X/\!\!/T = \Phi_T^{-1}(0)/T$ the symplectic quotient by T. Over $X/\!\!/T$ define a vector bundle $(\mathfrak{g}/\mathfrak{t})^{\mathbb{C}}/\!\!/T := (\Phi_T^{-1}(0) \times (\mathfrak{g}/\mathfrak{t})^{\mathbb{C}})/T$. The *abelianization formula* of S. Martin [27] reads

$$\int_{X/G} \kappa_G(\alpha) = \frac{1}{|W|} \int_{X/T} \kappa_T(\alpha) \wedge \operatorname{Eul}((\mathfrak{g}/\mathfrak{t})^{\mathbb{C}}/\!\!/T).$$

In other words, we have a commutative square

$$\begin{array}{ccccc} & & H_G(X) & & \\ & \swarrow & & \searrow & \\ . \; H(X/\!\!/G) & & & & H(X/\!\!/T) \\ & \searrow^{\int_{X/G}} & & {}^{|W|^{-1}\int^{\text{twist}}_{X/T}}\swarrow & \\ & & \mathbb{Q}. & & \end{array}$$

The proof follows from considering the diagram

$$\begin{array}{ccc} \Phi_G^{-1}(0)/T & \xrightarrow{\iota_G/T} & X/\!\!/T \\ \Big\downarrow{\scriptstyle p/G} & & \\ X/\!\!/G. & & \end{array}$$

Note that

(a) the vertical arrow is a fiber bundle with fiber G/T, which has Euler characteristic the order $|W|$ of the fixed point set of T acting on G/T;
(b) $\Phi_G^{-1}(0)/T$ is the zero set of the transverse section of $(\mathfrak{g}^*/\mathfrak{t}^*)/\!\!/T$ defined by Φ;
(c) the pull-back of $\kappa_T(\alpha)$ to $\Phi_G^{-1}(0)/T$ is the pull-back of $\kappa_G(\alpha)$.

Hence

$$\begin{aligned}\int_{X/\!\!/G} \kappa_G(\alpha) &= \frac{1}{|W|}\int_{\Phi_G^{-1}(0)/T} (\iota_G/T)^*\kappa_T(\alpha) \wedge \mathrm{Eul}((\mathfrak{g}/\mathfrak{t})/\!\!/T) \\ &= \frac{1}{|W|}\int_{X/\!\!/T} \kappa_T(\alpha) \wedge \mathrm{Eul}((\mathfrak{g}/\mathfrak{t})/\!\!/T) \wedge \mathrm{Eul}((\mathfrak{g}^*/\mathfrak{t}^*)/\!\!/T) \\ &= \frac{1}{|W|}\int_{X/\!\!/T} \kappa_T(\alpha) \wedge \mathrm{Eul}((\mathfrak{g}/\mathfrak{t})_{\mathbb{C}}/\!\!/T)\end{aligned}$$

where $(\mathfrak{g}/\mathfrak{t})/\!\!/T$ denotes the vector bundle $(\Phi^{-1}(0) \times \mathfrak{g}/\mathfrak{t})/T$ etc. An alternative proof uses a formula of Harish-Chandra relating distributions on $\mathfrak{t}$ and $\mathfrak{g}$, see [40].

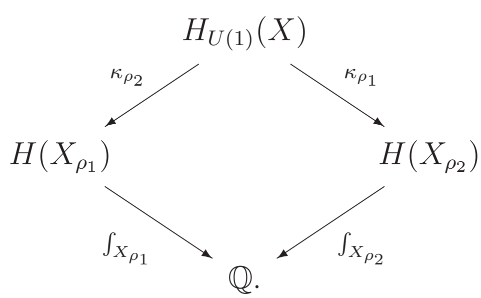
2.5. **Wall-crossing formulae.** Suppose that $G = U(1)$. We denote by κ_ρ the Kirwan map

$$\kappa_\rho : H_{U(1)}(X) \to H(X_\rho).$$

The integrals of the forms $\kappa_\rho(\alpha)$ over the quotients X_ρ satisfy the wall-crossing formula (see Kalkman [21] or Lerman [23])

$$\int_{X_{\rho_2}} \kappa_{\rho_2}(\beta) - \int_{X_{\rho_1}} \kappa_{\rho_1}(\beta) = \mathrm{Res}_\xi \left(\sum_{F \subset \Phi^{-1}(\rho_1,\rho_2)^{U(1)}} \int_F \iota_F^* \beta \wedge \mathrm{Eul}(\nu^F)^{-1} \right) \tag{10}$$

where ξ is the equivariant parameter, and Res_ξ denotes the residue at 0. In other words, failure of the following square to commute is measured by an explicit sum of wall-crossing terms:



A proof suggested by Lerman proceeds as follows. For any interval $[\rho_1, \rho_2]$ let $\mathbb{P}^1_{[\rho_1,\rho_2]}$ be the two-sphere equipped with equivariant symplectic form with moment image $[\rho_1, \rho_2]$. Define the *symplectic cut* of X at $[\rho_1, \rho_2]$ by

$$X_{[\rho_1,\rho_2]} := (X \times \mathbb{P}^1_{-[\rho_1,\rho_2]})/\!\!/U(1) \cong \phi^{-1}(\rho_1)/U(1) \cup \phi^{-1}((\rho_1, \rho_2)) \cup \phi^{-1}(\rho_2)/U(1). \tag{11}$$

Assuming these quotients are locally free, $X_{[\rho_1,\rho_2]}$ has the structure of a Hamiltonian $U(1)$-manifold, with fixed point set

$$X^{U(1)}_{[\rho_1,\rho_2]} = X_{\rho_1} \cup \phi^{-1}((\rho_1, \rho_2))^{U(1)} \cup X_{\rho_2}.$$

We denote by

$$\kappa_{[\rho_1,\rho_2]} : H_{U(1)}(X) \to H_{U(1)}(X_{[\rho_1,\rho_2]})$$

the canonical map given by the Cartan construction. Taking the residue of the integral of $\kappa_{[\rho_1,\rho_2]}(\alpha)$ over $X_{[\rho_1,\rho_2]}$ gives

$$\mathrm{Res}_\xi \int_{X_{[\rho_1,\rho_2]}} \kappa_{[\rho_1,\rho_2]}\alpha = 0$$

since the integral is polynomial. On the other hand, localization (9) expresses the residue as

$$\int_{X_{\rho_2}} \kappa_{\rho_2}(\alpha)\,\mathrm{Eul}(\nu^{\xi_2})^{-1} + \int_{X_{\rho_1}} \kappa_{\rho_1}(\alpha)\,\mathrm{Eul}(\nu^{\xi_1})^{-1} + \mathrm{Res}_\xi \left(\sum_{F \subset \Phi^{-1}(\rho_1,\rho_2)^{U(1)}} \int_F \iota_F^* \alpha \wedge \mathrm{Eul}(\nu^F)^{-1} \right) \tag{12}$$

where ν^{ξ_j} is the normal bundle of X_{ξ_j} in $X_{[\xi_1,\xi_2]}$ for $j = 1, 2$. The residues of the first two terms come exclusively from the first term in the expansion of the inverted Euler classes, since α is top degree, and the wall-crossing formula follows.

Kalkman's wall-crossing formula in particular applies to the family of sympletic quotients obtained from a Hamiltonian action of a non-abelian group with varying symplectic form. Suppose that $\omega_0 \in \Omega^2(X)$ is the symplectic form on a compact Hamiltonian G-manifold X with moment map Φ_0, $\omega_1 \in \Omega^2(X)$ is a closed two-form with moment map Φ_1. Consider the one-parameter family of Hamiltonian G-manifolds

$$X_\rho = (X, \omega_\rho, \Phi_\rho) = (X, \omega_0 + \rho\omega_1, \Phi_0 + \rho\Phi_1).$$

The symplectic quotients of X_ρ by G are

$$X_{0,\rho} = X_\rho /\!\!/ G = \Phi_\rho^{-1}(0)/G.$$

One can realize the family $X_{0,\rho}$ as the symplectic quotients of a Hamiltonian $U(1)$-manifold by the following trick. Suppose that (ω_1, Φ_1) is the equivariant curvature of an equivariant Hermitian line bundle-with-connection $\pi : L \to M$ with connection $\alpha_1 \in \Omega^1(L_1)$; here L_1 is the unit circle bundle L. That is,

$$\pi^*\omega_1 = \mathrm{d}\alpha_1, \quad (\Phi_1, \xi) = \alpha_1(\xi_L), \quad \xi \in \mathfrak{g}.$$

Let $\phi : L \to \mathbb{R}$ denote the norm function. Let $L(T^*S^1)$ denote the associated fiber bundle with fiber T^*S^1,

$$L(T^*S^1) = L_1 \times_{U(1)} T^*S^1.$$

Equip $L(T^*S^1)$ with the closed two-form $\pi^*\omega_0 + \mathrm{d}(\phi, \alpha_1)$, where ϕ is function induced by the moment map $T^*S^1 \to \mathbb{R}$ giving the moment map for the action of $U(1)$ on T^*S^1. The action of G on $L(T^*S^1)$ is Hamiltonian with moment map given by $\Phi_L := \pi^*\Phi_0 + \phi\pi^*\Phi_1$. Suppose that the symplectic quotient $L(T^*S^1)/\!\!/G$ is smooth. Then the action of $U(1)$ on $L(T^*S^1)$ descends to an action of $U(1)$ on the quotient $L(T^*S^1)/\!\!/G$ with proper moment map. One checks easily that there is a symplectomorphism of symplectic quotients

$$(L(T^*S^1)/\!\!/G)/\!\!/_\rho U(1) \to X_\rho, \quad l \mapsto \pi(l)$$

if these are smooth orbifolds. Failure of the quotient $L(T^*S^1)/\!\!/G$ to be smooth is caused by fixed points of the action. An element $\xi \in \mathfrak{g}$ acts trivially on an element $y \in \Phi_L^{-1}(0)$ if and only if $\pi(y)$ is ξ-fixed and ξ acts trivially on the fiber over $\pi(y)$, that is,

$(\Phi_1(\pi(y)), \xi) = 0$. Hence $L(T^*S^1)/\!\!/G$ is a smooth orbifold if for each element $\xi \in G$, the fixed point locus X_ξ satisfies $(\Phi_1(X_\xi), \xi) \neq 0$. For example, this condition holds whenever M is a flag variety and ω_1 is a sufficiently generic invariant closed 2-form. Kalkman's wall-crossing formula compares the pairings on X_{0,ρ_1} and X_{0,ρ_2}:

$$\int_{X_{0,\rho_2}} \kappa_{0,\rho_2}(\beta) - \int_{X_{0,\rho_1}} \kappa_{0,\rho_1}(\beta) = \sum_{\rho \in (\rho_1,\rho_2)} \sum_{\zeta} \frac{|W_\zeta|}{|W|} \operatorname{Res}_\zeta \sum_{F \subset X^\zeta} \int_F \iota_F^* \beta \wedge \operatorname{Eul}(\nu_F)^{-1} \tag{13}$$

where the sum is over fixed point components $F \subset X$ with $F \cap \Phi_\rho^{-1}(0) \neq \emptyset$. In other words, the failure of the following square to commute is measured by an explicit sum of wall-crossing terms:

$$\begin{array}{ccccc} & & H_G(X) & & \\ & \swarrow^{\kappa_{0,\rho_1}} & & \searrow^{\kappa_{0,\rho_2}} & \\ H(X_{0,\rho_1}) & & & & H(X_{0,\rho_2}) \\ & \searrow_{\int_{X_{0,\rho_1}}} & & \swarrow_{\int_{X_{0,\rho_2}}} & \\ & & \mathbb{Q}. & & \end{array}$$

3. Equivariant Gromov-Witten theory

3.1. **Cohomological field theory.** Let $\overline{M}_{g,n}$ denote the moduli space of isomorphism classes of n-marked, genus g stable curves. $\overline{M}_{g,n}$ is a normal projective variety, and carries the structure of a smooth Deligne-Mumford stack [7]. The boundary of $\overline{M}_{g,n}$ consists of the following divisors:

(a) if $2g + n > 3$, a divisor

$$\iota_{g-1,n+2} : D_{g-1,n} \to \overline{M}_{g,n}$$

equipped with an isomorphism

$$\varphi_{g-1,n+2} : D_{g-1,n} \to \overline{M}_{g-1,n+2}.$$

The inclusion is obtained by identifying the last two marked points.

(b) for each splitting $g = g_1 + g_2, \{1, \ldots, n\} = I_1 \cup I_2$ with $2g_j + |I_j| \geq 3, j = 1, 2$, a divisor

$$\iota_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2} \to \overline{M}_{g,n}$$

corresponding to the formation of a separating node, splitting the surface into pieces of genus g_1, g_2 with markings I_1, I_2, equipped with an isomorphism

$$\varphi_{g_1+g_2,I_1\cup I_2} : D_{g_1+g_2,I_1\cup I_2} \to \overline{M}_{g_1,|I_1|+1} \times \overline{M}_{g_2,|I_2|+1}$$

(except in the cases $I_1 = I_2 = \emptyset$ and $g_1 = g_2$ in which case there is an additional automorphism.)

The pull-back $\iota_{g_1+g_2,I_1\cup I_2}^* \beta$ of any class $\beta \in H(\overline{M}_{g,n})$ has a Kunneth decomposition

$$\iota_{g_1+g_2,I_1\cup I_2}^* \beta = \sum_{j \in J} \beta_{1,j} \otimes \beta_{2,j} \tag{14}$$

for some index set J and classes $\beta_{k,j} \in H^\bullet(\overline{M}_{g_1,|I_1|+1})$.

Definition 3.1.1. (Kontsevich-Manin) An (even) cohomological field theory (CohFT) with values in a ring R is a vector space V equipped with a symmetric non-degenerate bilinear form and collection of *correlators*

$$V^n \times H^\bullet(\overline{M}_{g,n}) \to R, \quad (\alpha, \beta) \mapsto \langle \alpha; \beta \rangle_{g,n}$$

satisfying the following two splitting axioms:

$$\begin{aligned} \langle \alpha; \beta \wedge \gamma_{g-1,n+2} \rangle_{g,n} &= \sum_k \langle \alpha, \delta_k, \delta^k; \iota^*_{g-1,n}\beta \rangle_{g-1,n+2} \\ \langle \alpha, ; \beta \wedge \gamma_{g_1+g_2,I_1 \cup I_2} \rangle_{g,n} &= \sum_k \langle \alpha_{I_1}, \delta_k; \cdot \rangle_{g_1,|I_1|+1} \langle \alpha_{I_2}, \delta^k; \cdot \rangle_{g_2,|I_2|+1} (\iota^*_{g_1+g_2,I_1\cup I_2}\beta) \end{aligned}$$

where the dots indicate insertion of the Kunneth components of $\iota^*_{g_1+g_2,I_1\cup I_2}\beta$, $\delta_k \in V$ is a basis and $\delta^k \in V$ a dual basis. (There is an additional factor of 2 in the exceptional case $g_1 = g_2$, $I_1 = I_2 = \emptyset$ arising from the additional automorphism.)

That is, if β is as in (14) then

$$\langle \alpha; \beta \wedge \gamma_{g_1+g_2,I_1\cup I_2} \rangle_{g,n} = \sum_{j \in J} \langle \alpha_{I_1}, \delta_k; \beta_{1,j} \rangle_{g_1,|I_1|+1} \langle \alpha_{I_2}, \delta^k; \beta_{2,j} \rangle_{g_2,|I_2|+1}.$$

By duality, any set of correlators determine maps

$$\mu^{g,n} : V^n \times H^\bullet(\overline{M}_{g,n+1}) \to V, \quad (\mu^{g,n}(\alpha_1, \ldots, \alpha_n; \beta), \alpha_0) = \langle \alpha_0, \ldots, \alpha_n; \beta \rangle_{g,n+1}$$

which are the complex analogs of the A_∞ -structure maps. The various relations on the divisors in $\overline{M}_{g,n}$ give rise to relations on the maps $\mu^{g,n}$. In particular the relation

$$[D_{0,\{0,3\}\cup\{1,2\}}] = [D_{0,\{0,1\}\cup\{2,3\}}]$$

in $H^2(\overline{M}_{0,4})$ implies that

$$\mu^{0,2} : V \otimes V \to V$$

is associative. We call the data $(V, \mu^{\bullet,\bullet})$ a CohFT, and the data $(V, \mu^{0,\bullet})$ a genus zero CohFT. We have considered here only the even case; in the full definition V is $\mathbb{Z}_2$-graded.

3.2. **Equivariant Gromov-Witten theory.** For any smooth projective variety X and class $d \in H_2(X, \mathbb{Z})$ let $\overline{M}_{g,n}(X, d)$ denote the moduli space of n-pointed, degree d, genus g stable maps to X, equipped with evaluation maps

$$\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_n) : \overline{M}_{g,n}(X, d) \to X^n.$$

For any cohomology classes $\alpha \in H(X, \mathbb{Q})^n$ integration over X using the virtual fundamental class $[\overline{M}_{g,n}(X, d)] \in H(\overline{M}_{g,n}(X, d))$ defines a *Gromov-Witten invariant*

$$\langle \alpha; \beta \rangle_{X,d} = \int_{[\overline{M}_{g,n}(X,d)]} \mathrm{ev}^* \alpha \wedge f^*\beta.$$

We define the Novikov ring Λ_X for X as the set of all maps $a : H_2(X) := H_2(X, \mathbb{Z})/\text{torsion} \to \mathbb{Z}$ such that for every constant c, the set of classes

$$\{d \in H_2(X), \ \langle [\omega], d \rangle \leq c\}$$

is finite. Addition is defined in the usual way and multiplication is convolution. Define as vector spaces

$$QH(X,\mathbb{Q}) := H(X,\mathbb{Q}) \otimes \Lambda_X$$

and

$$\mu^n : QH(X,\mathbb{Q})^n \otimes H(\overline{M}_{0,n+1},\mathbb{Q}) \to QH(X,\mathbb{Q})$$

by

$$(\mu^n(\alpha_1,\ldots,\alpha_n;\beta),\alpha_0) = \sum_{d\in H_2(X,\mathbb{Z})} e^d\langle\alpha_0,\ldots,\alpha_n;\beta\rangle_{X,d} \in \Lambda_X.$$

For the following, see e.g. [24].

Theorem 3.2.1. *The data $GW(X) := (QH(X,\mathbb{Q}),\mu^\bullet)$ define a CohFT.*

Gromov-Witten theory can be extended to the equivariant setting as follows. Suppose that $G_\mathbb{C}$ is a complex reductive group with maximal compact subgroup G, and X is a smooth projective $G_\mathbb{C}$-variety. Equivariant versions of Gromov-Witten invariants introduced by Givental [11]:

$$QH_G(X) := H_G(X,\mathbb{Q}) \otimes \Lambda_X$$

and structure maps are defined by equivariantintegration over the same moduli spaces. Using equivariant Poincaré duality, these give higher composition maps

$$\mu^{g,n}_{X,G} : QH_G(X,\mathbb{Q})^n \times H(\overline{M}_{g,n+1}) \to QH_G(X,\mathbb{Q}).$$

by $GW_G(X)$; see for example [15] for references on foundations from the algebraic viewpoint. For general symplectic manifolds the resulting theory is still under construction, see e.g. [4].

4. Gauged Gromov-Witten theory

4.1. **Cohomological traces.** Let $\overline{M}_n(\Sigma)$ denotes the moduli space of n-marked stable curves with principal component parametrized by Σ, that is, with a distinguished isomorphism with Σ. Each curve comes with a distinguished component Σ_0, called its *root component*, and a distinguished isomorphism $\Sigma_0 \to \Sigma$. In the case $\Sigma = \mathbb{P}^1$ we may identify $\overline{M}_n(\Sigma)$ with $\overline{M}_{0,n+3}$, where the three additional points give the parametrization. For curves Σ without automorphisms, $\overline{M}_n(\Sigma)$ is isomorphic to the fiber of the forgetful morphism $\overline{M}_{g,n} \to \overline{M}_{g,0}$ over $[\Sigma] \in \overline{M}_{g,0}$, where $g = \text{genus}(\Sigma)$. Alternative, $\overline{M}_n(\Sigma)$ may be viewed as the moduli space of stable maps of genus $g = \text{genus}(\Sigma)$ to Σ of degree 1.

The boundary structure of $\overline{M}_n(\Sigma)$ is similar to that for the usual Deligne-Mumford spaces. For each subset $I \subset \{1,\ldots,n\}$ of order at least two there is a boundary divisor

$$\iota_I : D_I \to \overline{M}_n(\Sigma)$$

where the markings for $i \in I$ have bubbled off onto an (unparametrized) sphere bubble, and a homeomorphism

$$\varphi_I : D_I \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1}(\Sigma).$$

In particular, for any $\beta \in \overline{M}_n(\Sigma)$, the pull-back $\iota_I^*\beta$ has a Kunneth decomposition

$$\iota_I^*\beta = \sum_{j\in J} \beta_{1,j} \otimes \beta_{2,j} \tag{15}$$

for some index set J and classes $\beta_{1,j} \in H(\overline{M}_{0,|I|+1})$ and $\beta_{2,j} \in \overline{M}_{n-|I|+1}(\Sigma)$. Let Λ be a ring.

Definition 4.1.1. A Σ-*trace* on a CohFT V is a collection of *correlators*

$$\langle \ ; \rangle_n : V^n \times H^\bullet(\overline{M}_n(\Sigma)) \to \Lambda$$

satisfying a splitting axiom

$$\langle \alpha; \beta \wedge \gamma_I \rangle_n = \langle \alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \cdot); \cdot \rangle_{n-|I|+1}(\iota_I^*\beta)$$

where γ_I is the dual class to D_I and the $\cdot$'s denote insertion of the Kunneth components of β.

That is, with β as in (15),

$$\langle \alpha; \beta \wedge \gamma_I \rangle_n = \sum_{j\in J} \langle \alpha_i, i \notin I, \mu^{|I|}(\alpha_i, i \in I; \beta_{1,j}); \beta_{2,j} \rangle_{n-|I|+1}.$$

Let $\mathrm{Tr}_\Sigma(V)$ denote the space of cohomological Σ-traces on V.

4.2. **Gauged pseudoholomorphic maps.** Let Σ be a compact Riemann surface, and $\pi : P \to \Sigma$ a smooth principal G-bundle. Given any left G-manifold F we have a left action of G on $P \times F$ given by $g(p, f) = (pg^{-1}, gf)$ and we denote by $P(F) = (P \times F)/G$ the quotient, that is, the associated fiber bundle with fiber F. We denote by $\mathcal{A}(P)$ the space of smooth connections on P, and by $P(\mathfrak{g}) := (P \times \mathfrak{g})/G$ the adjoint bundle. For any $A \in \mathcal{A}(P)$, we denote $F_A \in \Omega^2(\Sigma, P(\mathfrak{g}))$ the curvature of A.

Let X denote a compact Hamiltonian G-manifold with symplectic form ω and moment map $\Phi : X \to \mathfrak{g}^*$. Via the Cartan construction the equivariant symplectic form ω_G descends to a closed, fiber-wise symplectic two-form $\omega_{P(X),A} \in \Omega^2(P(X))$. Its cohomology class $[\omega_{P(X)}] \in H^2(P(X))$ is independent of the choice of connection. Let $\psi : \Sigma \to BG$ be a classifying map for $P \to \Sigma$. Sections $u : \Sigma \to P(X)$ are in one-to-one correspondence with lifts of ψ to X_G. The degree $\deg(u)$ of the section u is defined to be the degree $\deg(u) \in H_2^G(X, \mathbb{Z})$ of the corresponding lift. The projection of $\deg(u)$ onto $H_2^G(\mathrm{pt}, \mathbb{Z}) = H_2(BG, \mathbb{Z})$ is the first Chern class of P.

A *gauged map* from Σ to X is a datum (P, A, u) where $P \to \Sigma$ is a principal G-bundle, $A \in \mathcal{A}(P)$ and $u : \Sigma \to P(X)$ is a section. For simplicity we drop P from the notation. The *energy* of a gauged map (A, u) is given by

$$E(A, u) = \frac{1}{2} \int_\Sigma |\mathrm{d}_A u|^2 + |F_A|^2 + |u^* P(\Phi)|^2.$$

The *equivariant symplectic area* of a pair (A, u) is pairing of the degree of u with the class $[\omega_G]$,

$$D(A, u) = (\deg(u), [\omega_G]) = ([\Sigma], u^*[\omega_{P(X)}]).$$

More concretely,

$$D(A,u) = \int_\Sigma u^* \omega_{P(X),A}$$

independent of the choice of connection A.

Gauged pseudoholomorphic maps are defined as follows. Denote by $\mathcal{J}(X)$ the space of almost complex structures on X compatible with ω. The action of G induces an action on $\mathcal{J}(X)$, and we denote by $\mathcal{J}(X)^G$ the invariant subspace. Any connection $A \in \mathcal{A}(P)$ induces a map of spaces of almost complex structures

$$\mathcal{J}(X)^G \to \mathcal{J}(P(X)), \quad J \mapsto J_A$$

by combining the almost complex structure on X and Σ using the splitting defined by the connection. Let $\Gamma(\Sigma, P(X))$ denote the space of sections of $P(X)$. We denote by

$$\overline{\partial}_A : \Gamma(\Sigma, P(X)) \to \Omega^{0,1}(\Sigma, (\cdot)^* T^{\mathrm{vert}} P(X))$$

the Cauchy-Riemann operator defined by J_A. If ω_Σ is the area form determined by a choice of metric on Σ then the energy and equivariant symplectic area are related by

$$E(A,u) = D(A,u) + \int_\Sigma |\overline{\partial}_A u|^2 + \frac{1}{2} \int |F_A + u^* P(\Phi) \omega_\Sigma|^2. \tag{16}$$

A *gauged pseudoholomorphic map* is a pair (A,u) satisfying $\overline{\partial}_A u = 0$. Let $\mathcal{H}(P,X)$ be the space of gauged pseudoholomorphic maps,

$$\mathcal{H}(P,X) = \{(A,u) \in \mathcal{A}(P) \times \Gamma(\Sigma, P(X)), \quad \overline{\partial}_A u = 0\}.$$

Formally the space $\mathcal{H}(P,X)$ has a closed two-form induced from the sum of the symplectic form on the affine space of connections and the space of maps to X. Namely, let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be an invariant inner product, and

$$\Omega^1(P(\mathfrak{g}))^2 \to \mathbb{R}, \quad (a_1, a_2) \mapsto \int_\Sigma \langle a_1 \wedge a_2 \rangle$$

the symplectic form on the affine space of connections $\mathcal{A}(P)$. On the other hand, let $P(\omega)$ denote the fiber-wise two-formon $P(X)$ defined by ω. Choose a two-form $\omega_\Sigma \in \Omega^2(\Sigma)$ and consider the formal two-form on $\mathcal{H}(P,X)$

$$\Omega^0(\Sigma, u^* T^{\mathrm{vert}} P(X))^2 \to \mathbb{R}, \quad (v_1, v_2) \mapsto \int_\Sigma u^* P(\omega)(v_1, v_2) \omega_\Sigma.$$

Choose a constant $\rho > 0$ and consider the formal two-form

$$((a_1, v_1), (a_2, v_2)) \to \int_\Sigma \langle a_1 \wedge a_2 \rangle + (u^* P(\omega))(v_1, v_2) \rho^{-1} \omega_\Sigma. \tag{17}$$

The form (17) is the formally restriction of a closed two-form on the space of all sections $\Sigma \to P(X)$, and so formally restricts to a closed two-form on the smooth locus of $\mathcal{H}(P,X)$. If X is Kähler, then the moduli space of pseudoholomorphic sections is an almost complex manifold, and this can be used to show non-degeneracy of (17). In general, we know of no argument which implies that (17) is non-degenerate. Let $\mathcal{G}(P)$ denote the group of gauge transformations

$$\mathcal{G}(P) = \{a : P \to P, a(pg) = a(p)g, \quad \pi \circ a = \pi\}.$$

The Lie algebra of $\mathcal{G}(P)$ is the space of sections $\Omega^0(\Sigma, P(\mathfrak{g}))$ of the adjoint bundle. The action of $\mathcal{G}(P)$ on $\mathcal{H}(P, X)$ has generating vector fields given by the covariant derivative and infinitesimal action

$$\xi_{\mathcal{H}(P,X)}(A, u) = (-\mathrm{d}_A\xi, \xi_X(u)) \in \Omega^1(\Sigma, P(\mathfrak{g})) \times \Omega^0(\Sigma, u^*T^{\mathrm{vert}}P(X)), \quad \xi \in \Omega^0(\Sigma, P(\mathfrak{g})).$$

The action preserves the two-form (17) and has moment map given by the curvature plus pull-back of the moment map for X,

$$\mathcal{H}(P, X) \to \Omega^2(\Sigma, P(\mathfrak{g})), \quad (A, u) \mapsto F_A + \rho^{-1}\omega_\Sigma u^*P(\Phi).$$

Motivated by these formal considerations we have:

Definition 4.2.1. A gauged map $(A, u) \in \mathcal{H}(P, X)$ is an *ρ-vortex* if it satisfies the moment map condition

$$F_A + \rho^{-1}\omega_\Sigma u^*P(\Phi) = 0.$$

We say that $(A, u) \in \mathcal{H}(P, X)$ is *stable* if it has a finite automorphism group, and *regular* if the linearized operator $\tilde{D}_{A,u}$ is surjective, see [12].

In particular, for any vortex (A, u) the energy-area relation (16) simplifies to

$$E(A, u) = D(A, u).$$

Symplectic vortices were introduced by Mundet and Salamon, and further studied with R. Gaio, K. Cieliebak, and others, see [32], [5], [10]. The corresponding notion of *gauged sigma model* in the physics literature goes back at least to Witten [39].

Let $M(P, X)_\rho$ be the quotient of the zero level set of the moment map by gauge transformations, the *moduli space of ρ-vortices*,

$$M(P, X)_\rho := \mathcal{A}(P, X, d) /\!\!/ \mathcal{G}(P) = \{F_A + \rho^{-1}\omega_\Sigma u^*P(\Phi) = 0\}/\mathcal{G}(P).$$

Let $M(\Sigma, X)_\rho$ denote the union over types, and $M(\Sigma, X, d)_\rho$ the component of degree $d \in H_2^G(X, \mathbb{Z})$. The locus $M^{\mathrm{reg}}(\Sigma, X)_\rho$ of regular, stable vortices has the structure of a smooth orbifold [5].

Definition 4.2.2. An *n-marked* symplectic vortex is a vortex (A, u) together with n-tuple $\underline{z} = (z_1, \ldots, z_n)$ of distinct points on Σ. A *framed ρ-vortex* is a collection $(A, u, \underline{z}, \underline{\phi})$, where $(A, u, \underline{z})$ is a marked ρ-vortex and $\underline{\phi} = (\phi_1, \ldots, \phi_n)$ are trivializations of the fibers of P at $z_1, \ldots, z_n$, that is, each $\phi_j : P_{z_j} \to G$ is a G-equivariant isomorphism.

Let $M_n(P, X)_\rho$ denote the moduli space of isomorphism classes of n-marked ρ-vortices and $M_n(\Sigma, X)_\rho$ the union over types of bundles $P \to \Sigma$. The moduli space $M_n(\Sigma, X)_\rho$ is homeomorphic to the product $M(\Sigma, X)_\rho \times M_n(\Sigma)$ where $M_n(\Sigma)$ denotes the configuration space of n-tuple of distinct points on Σ. Let $M_n^{\mathrm{fr}}(\Sigma, X)_\rho$ denote the moduli space of framed n-marked ρ-vortices. We have an evaluation map

$$\mathrm{ev}^{\mathrm{fr}} : M_n^{\mathrm{fr}}(\Sigma, X)_\rho \to X^n, \quad (A, \underline{u}, \underline{z}) \mapsto (\phi_1(u(z_1)), \ldots, \phi_n(u(z_n)))$$

defined by combining the framings with evaluation at the marked points. Forgetting the framings defines a map

$$\pi : M_n^{\mathrm{fr}}(\Sigma, X)_\rho \to M_n(\Sigma, X)_\rho.$$

Over the locus $M_n^{\mathrm{fr},s}(\Sigma, X)_\rho$ of stable framed vortices the action of G^n is locally free. It has has the structure of a topological orbifold principal G-bundle, and a smooth orbifold principal G-bundle over the stable locus. Suppose first that all finite stabilizers for the gauge group action are trivial, so that π is an honest G^n-bundle. Let

$$\psi : M_n^{\mathrm{fr}}(\Sigma, X)_\rho \to EG^n$$

be a classifying map for the bundle $\pi : M_n^{\mathrm{fr}}(\Sigma, X)_\rho \to M_n(\Sigma, X)_\rho$. Combining ψ with the evaluation map ev gives rise to a G^n-equivariant map

$$\mathrm{ev}^{\mathrm{fr}} \times \psi : M_n^{\mathrm{fr}}(\Sigma, X)_\rho \to X^n \times EG^n.$$

This induces a map

$$\mathrm{ev} : \ M_n(\Sigma, X)_\rho \to (X^n \times EG^n)/G^n = X^n_G.$$

In particular, pull-back by ev induces a map in equivariant cohomology

$$\mathrm{ev}^* : H_G(X, \mathbb{Z})^n \to H(M_n(\Sigma, X)_\rho, \mathbb{Z}).$$

More generally, non-triviality of finite stabilizers means that the classifying map exists only after passing to the classifying space of $M_n^{\mathrm{fr}}(\Sigma, X)_\rho$, and the pull-back map is defined over rational coefficients

$$\mathrm{ev}^* : H_G(X, \mathbb{Q})^n \to H(M_n(\Sigma, X)_\rho, \mathbb{Q}).$$

4.3. **Compactification.** The moduli space $M_n(\Sigma, X)_\rho$ admits a partial compactification given by allowing the sections to develop bubbles in the fibers of $P(X)$. If $\underline{u} : \underline{\Sigma} \to P(X)$ is a nodal map consisting of a section and a collection of sphere bubbles in the fibers, then the degree $\deg(\underline{u})$ is the sum of the degree of the section u_0 and the degree of the bubbles $u_1, \ldots, u_k$.

Definition 4.3.1. A *nodal gauged (pseudoholomorphic) map* consists of a datum $(\underline{\Sigma}, \underline{z}, A, \underline{u})$ where

(a) $\underline{\Sigma}$ is a connected nodal curve consisting of $\Sigma_0 = \Sigma$ as *principal component* and *bubble components* $\Sigma_1, \ldots, \Sigma_k$. We denote by $w_1^\pm, \ldots, w_k^\pm$ the nodes. For each $i = 1, \ldots, k$, we denote by $w_i^0 \in \Sigma$ the attaching point to the principal component.
(b) a gauged pseudoholomorphic map $(A, u_0) \in \mathcal{H}(P, X)$ on Σ.
(c) for each sphere bubble Σ_i, a pseudoholomorphic map $u_i : \Sigma_i \to P(X)_{w_i^0}$.
(d) $\underline{z} = (z_1, \ldots, z_n) \in \underline{\Sigma}$ are distinct, smooth points of $\underline{\Sigma}$.

A nodal gauged map is *polystable* if each sphere bubble Σ_i on which u_i is constant has at least three marked or singular points, and *stable* if $(A, \underline{u})$ has finite automorphism group. There is no condition for points on the principal component. In particular, nodal gauged maps with no markings can be stable. A *(poly)stable* ρ-vortex is a (poly)stable gauged map such that the principal component is an ρ-vortex.

An *isomorphism* of nodal ρ-vortices $(\underline{\Sigma}, A, \underline{u}, \underline{z}), (\underline{\Sigma}', A', \underline{u}', \underline{z}')$ consists of an isomorphism $\underline{\Sigma} \to \underline{\Sigma}'$ acting trivially on the principal component, and a lift on the principal component to P, intertwining with the connections A, A' and sections $\underline{u}, \underline{u}'$. In particular, the markings on the principal component Σ must be equal.

The *combinatorial type* $\Gamma(\underline{\Sigma}, \underline{z}, A, \underline{u})$ of is the rooted graph whose vertices represent the components of $\underline{\Sigma}$, whose finite edges represent the nodes, semi-infinite edges represent the markings, and whose root vertex represents the principal component.

We denote by $M_{n,\Gamma}(P,X)_\rho$ of the moduli space of isomorphism classes of polystable nodal ρ-vortices of combinatorial type Γ. Denote by $M_{n,\Gamma}(P,X,d)_\rho$ the subset of degree $\deg(u) = d \in H_2^G(X,\mathbb{Z})$. The union over combinatorial types is

$$\overline{M}_n(P,X)_\rho = \bigcup_{n,\Gamma} \overline{M}_\Gamma(P,X)_\rho.$$

We denote by $\overline{M}_n(\Sigma,X)$ the union over topological types of bundles $P \to \Sigma$.

We equip $\overline{M}_n(\Sigma,X)$ with a topology induced by the C^0-topology on the space of connections, and the Gromov topology on the space of nodal maps to $P(X)$. Namely, we say that a sequence $(A_\alpha, \underline{u}_\alpha)$ converges to a limiting stable vortex $(A_\infty, \underline{u}_\infty)$ if A_α converges to A_∞ uniformly, and $\underline{u}_\alpha$ Gromov converges to $\underline{u}_\infty$ as in [29, p.114]. The following is essentially due to Mundet [32]:

Theorem 4.3.2. *For any $E > 0$, $J \in \mathcal{J}(X)^G$, the union of components $\overline{M}_n(\Sigma,X,d)_\rho$ over $d \in H_2^G(X,\mathbb{Z})$ with $(d,[\omega]) < E$ is a compact, Hausdorff space.*

A more general result, for the cases of surfaces with cylindrical ends, is proved in [8].

Each marking z_i determines a point z_i^0 where the bubble tree containing it is attached to the principal component. A *framed polystable ρ-vortex* consists of a stable vortex together with framings $\underline{\phi} = (\phi_1,\dots,\phi_n)$ at the attaching points of the bubbles

$$\phi_i : P_{z_i^0} \to G, \quad i = 1,\dots,n.$$

Let $\overline{M}_n^{\mathrm{fr}}(\Sigma,X)_\rho$ denote the moduli space of gauge equivalence classes of stable framed nodal ρ-vortices. Since G and $\overline{M}_n(\Sigma,X,d)$ are compact, so is $\overline{M}_n^{\mathrm{fr}}(\Sigma,X,d)$, for any degree $d \in H_2^G(X,\mathbb{Z})$. The group G^n acts on $\overline{M}_n^{\mathrm{fr}}(\Sigma,X)_\rho$ by changing the framings at the attaching points. Let

$$\pi : \overline{M}_n^{\mathrm{fr}}(\Sigma,X)_\rho \to \overline{M}_n(\Sigma,X)_\rho$$

denote the map forgetting the framings. Since the action of the group $\mathcal{G}(P)$ admits slices, so does the action of G on $\overline{M}_n(\Sigma,X)_\rho$, and the restriction of π to the stable, regular locus forms an orbifold principal G-bundle. Let

$$\psi : B\overline{M}_n^{\mathrm{fr}}(\Sigma,X)_\rho \to EG^n$$

be a classifying map for the bundle. The space $\overline{M}_n^{\mathrm{fr}}(\Sigma,X)_\rho$ admits *evaluation maps*

$$\mathrm{ev}^{\mathrm{fr}} : \overline{M}_n^{\mathrm{fr}}(\Sigma,X)_\rho \to X^n, \quad (\underline{\Sigma}, A, \underline{u}, \underline{\phi}, \underline{z}) \mapsto \underline{\phi} \circ \underline{u}(\underline{z}).$$

Combining the evaluation maps with the classifying map gives rise to an equivariant evaluation map

$$\mathrm{ev} : B\overline{M}_n(\Sigma,X)_\rho \to X_G^n.$$

Using the equivalence in rational cohomology

$$H^*(B\overline{M}_n(\Sigma,X)_\rho,\mathbb{Q}) \to H(\overline{M}_n(\Sigma,X)_\rho,\mathbb{Q})$$

we obtain a pull-back map

$$\mathrm{ev}^* : H_G(X,\mathbb{Q})^n \to H(\overline{M}_n(\Sigma,X)_\rho,\mathbb{Q}).$$

We denote by

$$f : \overline{M}_n(\Sigma, X)_\rho \to \overline{M}_n(\Sigma)$$

the forgetful morphism obtained by forgetting the data (A, u) and collapsing any unstable component. In particular, we obtain a pull-back map in cohomology

$$f^* : H(\overline{M}_n(\Sigma)) \to H(\overline{M}_n(\Sigma, X)_\rho).$$

4.4. **Vortex invariants.** From now on, we assume the existence of virtual fundamental classes with standard properties. Suppose that $\overline{M}_n(\Sigma, X)_\rho$ consists entirely of stable vortices. We define the *ρ-vortex invariants*

$$H_G(X, \mathbb{Q})^n \times H(\overline{M}_n(\Sigma), \mathbb{Q}) \to \mathbb{Q}, \quad (\alpha, \beta) \mapsto \langle \alpha, \beta \rangle_{d,\rho}$$

$$\langle \alpha; \beta \rangle_{d,\rho} := \int_{\overline{M}(\Sigma, X, d)_\rho} \mathrm{ev}^* \alpha \wedge f^* \beta.$$

We denote by Λ_X^G the *equivariant Novikov ring* for X, the set of all maps $a : H_2^G(X) := H_2^G(X, \mathbb{Z})/\operatorname{torsion} \to \mathbb{Z}$ such that for every constant c, the set of classes

$$\left\{ d \in H_2^G(X),\ \langle [\omega_G], d \rangle \le c \right\}$$

is finite. Addition is defined in the usual way and multiplication is convolution. Summing over equivariant degrees gives a map

$$\tau_{X,\rho}^n : QH_G(X)^n \otimes H(\overline{M}_n(\Sigma)) \to \Lambda_X^G, \qquad \sum_{d \in H_2^G(X, \mathbb{Z})} e^d \langle \alpha; \beta \rangle_{d,\rho}.$$

Conjecture 4.4.1. *For any vortex parameter ρ with the property that every polystable ρ-vortex is stable, the maps $\tau_{X,\rho}^n, n \ge 0$ form a cohomological trace on $GW_G(X)$.*

Some cases are proved in [12].

4.5. **The small area limit.** In this section, we present some conjectures about the small area limit of the genus zero gauged Gromov-Witten invariants. These conjectures are partly proved, for convex target, in [13]. We consider $H_2(X)$ a subset of $H_2^G(X)$ by equivariant formality (6). Let Σ have genus zero.

Definition 4.5.1. An *zero-area vortex* is a solution $u : \Sigma \to X$ to the equations

$$\int_\Sigma u^* \Phi \, \omega_\Sigma = 0, \quad \overline{\partial} u = 0. \tag{18}$$

The group G acts on the space of zero-area vortices; two zero-area vortices are *isomorphic* if they are related by the action of G. We denote by $M(\Sigma, X, d)_\infty$ the moduli space of zero-area-vortices of degree $d \in H_2(X, \mathbb{Z})$. The moduli space $M(\Sigma, X, d)_\infty$ admits a compactification by allowing bubbling in the fibers, as in the case of finite vortex parameter. We say that a nodal pseudoholomorphic map $\underline{u}$ from a rooted nodal curve $\underline{\Sigma}$ is a *nodal zero-area-vortex* if the restriction u to the principal component Σ is a nodal zero-area-vortex, see Definition 4.3.1. Thus, the restriction of the map to a bubble is *not* required to have zero average moment map.

A nodal zero-area-vortex is *polystable* if each bubble has at least three special (marked or nodal) points, and *stable* if it has finite automorphism group. An *isomorphism* of nodal zero-area-vortices u, u' consists of an automorphism of the domain, acting trivially on the principal component, intertwining u with u'. Let $M_\Gamma(\Sigma, X, d)_\infty$ denote the moduli space of stable zero-area vortices of combinatorial type Γ. Let $\overline{M}(\Sigma, X, d)_\infty$ denote the union over combinatorial types Γ, and $\overline{M}(\Sigma, X)_\infty$ the union over degrees. If $d \in H_2^G(X, \mathbb{Z})$ is not in the image of $H_2(X)$, we define $\overline{M}(\Sigma, X, d)_\infty$ to be empty. For the following, see [13]:

Theorem 4.5.2. *For any $E > 0$, the union of components $\overline{M}_n(\Sigma, X, d)_\infty$ with $(d, [\omega]) < E$ is a compact, Hausdorff space. The regular, stable locus $\overline{M}_n^{\text{reg}}(\Sigma, X, d)_\infty$ has the structure of a topological orbifold.*

This moduli space may be viewed as a quotient of the moduli space of stable maps, as follows. Given an element $(\Sigma, u : \Sigma \to X)$ of $\overline{M}_{0,3+n}(X, d)$, the first three marked points determine a distinguished component of the domain Σ_0, equipped with a canonical isomorphism $\psi : \mathbb{P}^1 \to \Sigma_0$ mapping the three marked points to $0, 1, \infty$ respectively. Define a map

$$\phi : \overline{M}_{0,3+n}(X, d) \to \mathfrak{g}^*, \quad (\Sigma, u) \mapsto \int_{\mathbb{P}^1} \psi^*(u|\Sigma_0)^* \Phi \, \omega_{\mathbb{P}^1}.$$

ϕ can be considered a moment map for the action of G on $\overline{M}_{0,3+n}(X, d)$ for a closed two-form given by integrating the pull-back of $\omega \in \Omega^2(X)$ over the principal component as in (17); however, one should first prove that this form extends over the boundary, and we do not do that here. The following is immediate from the definitions:

Proposition 4.5.3. $\overline{M}_n(\Sigma, X)_\infty = \phi^{-1}(0)/G = \overline{M}_{0,3+n}(X, d)/\!/G.$

Let $\overline{M}_n^{\text{fr}}(\Sigma, X)_\infty$ denote the moduli space of framed zero-area vortices. It admits an *evaluation map* $\text{ev}^{\text{fr}} : \overline{M}_n^{\text{fr}}(\Sigma, X)_\infty \to X^n$. If the action of G^n is locally free, ev^{fr} induces a pull-back map in cohomology

$$\text{ev}^* : H_G^*(X, \mathbb{Q})^n \to H^*(\overline{M}_n(\Sigma, X)_\infty, \mathbb{Q}).$$

Let $f : \overline{M}_n(\Sigma, X)_\infty \to \overline{M}_n(\Sigma)$ denote the forgetful morphism. Suppose that $\overline{M}_n(\Sigma, X)_\infty$ has a virtual fundamental class with the usual properties. Define

$$\langle \cdot ; \cdot \rangle_X^G : H_G(X, \mathbb{Q})^n \otimes H(\overline{M}_n(\Sigma), \mathbb{Q}) \to \mathbb{Q}$$

by

$$\langle \alpha; \beta \rangle_X^G = \int_{\overline{M}_n(\Sigma, X)_\infty} \text{ev}^* \alpha \wedge f^* \beta.$$

On the basis of the quotient description of the previous section, we call these invariants the *invariant part of the equivariant Gromov-Witten invariants.* As before, summing over degrees yields maps

$$\tau_X^{n,G} : QH_G(X)^n \otimes H(\overline{M}_n(\Sigma)) \to \Lambda_X^G, \qquad \sum_{d \in H_2^G(X, \mathbb{Z})} e^d \langle \alpha; \beta \rangle_{d,X}^G.$$

Conjecture 4.5.4. *The maps $\tau_X^{n,G}$ form a cohomological trace on $GW_G(X)$ with values in Λ_X^G.*

Note that these correlators take values in Λ_X^G, whereas Givental's equivariant correlators take values in $\Lambda_X^G \otimes H(BG)$. We compare this trace with that arising from the vortex invariants for ρ small. The following is proved in [13]:

Theorem 4.5.5. *Let $P \to \Sigma$ be a principal G-bundle over the sphere $\Sigma = \mathbb{P}^1$. Suppose that (A_ν, u_ν) is a sequence of ρ_ν vortices of constant degree $d \in H_2^G(X)$, with $\rho_\nu \to \infty$. Then (i) P is isomorphic to the trivial bundle $\Sigma \times G$ and (ii) after passing to a subsequence there exists a sequence of gauge transformations g_ν such that (a) $g_\nu A_\nu$ converges to the trivial connection and (b) u_ν converges to a polystable zero-area vortex u. Furthermore, any polystable regular zero-area vortex is the limit of such a sequence.*

In particular, in the case that d is not in the image of $H_2(X)$ in $H_2^G(X)$, the moduli space $\overline{M}(\Sigma, X, d)_\rho$ is empty for ρ sufficiently large. Naturally one expects that for ρ sufficiently large the virtual fundamental classes for $\overline{M}(\Sigma, X)_\rho$ and $\overline{M}(\Sigma, X)_\infty$ are equivalent.

Conjecture 4.5.6 (Small Area Conjecture)**.** *There exists an $\rho_0 > 0$ such that if $\rho > \rho_0$, we have $\langle \cdot ; \cdot \rangle_{X,\rho} = \langle \cdot ; \cdot \rangle_X^G$.*

5. Polarized vortices and wall-crossing formulae

In this section we discuss the dependence of the gauged Gromov-Witten invariants on the vortex parameter, that is, the area of the surface. The main result says that the traces are equal up to an explicit (but rather difficult to compute) sum of wall-crossing terms, describing the failure of commutativity of the triangle

$$\begin{array}{ccc} GW_G(X) & \longrightarrow & GW_G(X) \\ & \searrow^{\rho_1} \quad \swarrow^{\rho_2} & \\ & \Lambda_X^G & \end{array}$$

where the top arrow is the identity and the remaining two arrows are the ρ_1 and ρ_2-vortex traces.

5.1. **Polarized vortices.** The wall-crossing formula depends on the construction of a Hamiltonian $U(1)$-space whose symplectic quotients are the moduli spaces of vortices for varying vortex parameter. Let $\tilde{\mathcal{A}}(P) \to \mathcal{A}(P)$ denote the Chern-Simons line bundle; a $\mathcal{G}(P)$-equivariant line bundle-with-connection whose equivariant curvature is the $\mathcal{G}(P)$-equivariant symplectic form on $\mathcal{A}(P)$ [34]. Let $\tilde{\mathcal{H}}(P, X)$ denote the pull-back of $\tilde{\mathcal{A}}(P)$ to $\mathcal{H}(P, X)$, that is,

$$\tilde{\mathcal{H}}(P, X) = \{(\tilde{A}, u, z), \overline{\partial}_A u = 0\} \subset \tilde{\mathcal{A}}(P) \times \mathrm{Map}(\Sigma, P(X)).$$

The action of $\mathcal{G}(P)$ on $\tilde{\mathcal{H}}(P, X)$ has formal moment map given by

$$\tilde{\mathcal{H}}(P, X) \to \Omega^2(\Sigma, P(\mathfrak{g})), \quad \tilde{A} \mapsto \phi(\tilde{A}) F_A + \omega_\Sigma \, u^* P(\Phi).$$

Let $L(P, X)$ denote the formal symplectic quotient

$$L(P, X) = \{(\tilde{A}, u) \in \tilde{\mathcal{H}}(P, X), \quad \phi(\tilde{A}) F_A + \omega_\Sigma \, u^* \Phi = 0, \quad \overline{\partial}_A u = 0\} / \mathcal{G}(P).$$

We call a pair $(\tilde{A}, u)$ in the zero level set a *polarized vortex*, and $L(P, X)$ the moduli space of isomorphism classes of polarized vortices. We denote by $L(\Sigma, X)$ the sum over types of principal bundles P, and $L(\Sigma, X, d)$ the component of equivariant degree $d \in H_2^G(X, \mathbb{Z})$.

Theorem 5.1.1. *The stable, regular locus $L^{\mathrm{reg}}(\Sigma, X, d)$ is a smooth, finite dimensional manifold with dimension*

$$\dim L^{\mathrm{reg}}(\Sigma, X, d) = 2((1-g)\dim(X) + (c_1(TX), d) - \dim(G) + 1).$$

Every polarized vortex is stable if the symplectic form on X is "sufficiently irrational", more precisely, if for every rational element $\xi \in \mathfrak{g}$, the pairing $(\Phi(X^\xi), \xi)$ does not contain 0. The fixed points of the $U(1)$-action on $L(P, X)$ correspond to reducible pairs in $M(P, X)_\rho$ as ρ ranges from 0 to ∞: A point $[\tilde{A}, u] \in L(P, X)$ is $U(1)$-fixed if and only if the pair (A, u) is fixed by some one-parameter subgroup of gauge transformations. We denote by $L(\Sigma, X)$ the union over types of bundles P, and $L_n(\Sigma, X)$ the moduli space of polarized, n-marked vortices.

We can partially compactify $L_n(\Sigma, X)$ by passing to *polystable polarized vortices*; the definition is the same as that for polystable vortices but on the principal component we have a lift $\tilde{A}$ of A to $\tilde{\mathcal{A}}(P)$. Let $\overline{L}_n(\Sigma, X)$ denote the moduli space of stable polarized vortices. The map ϕ extends to $\overline{L}_n(\Sigma, X)$ and the relevant version of the compactness theorem is the first part of the following:

Theorem 5.1.2. *$\overline{L}_n(\Sigma, X)$ is a Hausdorff space. For any $C > 0$, the moment map $\phi : \bigcup_{(d,[\omega_G])<C} \overline{L}_n(\Sigma, X, d) \to (0, \infty)$ is proper. The symplectic quotient of $\overline{L}_n(\Sigma, X, d)$ at ρ is canonically homeomorphic to $\overline{M}(\Sigma, X, d)_\rho$. The subset $\overline{L}_n^{\mathrm{reg}}(\Sigma, X)$ of stable regular polarized vortices has (non-canonically) the structure of a C^1-orbifold.*

The restriction of ϕ should be a moment map for a closed two form on $\overline{L}_n^{\mathrm{reg}}(\Sigma, X)$; however, we have not shown that the closed-two form on the open stratum $L_n(\Sigma, X)$ defined above extends to the boundary. From now on, we assume that every polarized vortex is stable. The symplectic cutting procedure of (11) gives a compact moduli space $\overline{L}_n(\Sigma, X)_{[\rho_1,\rho_2]}$. We suppose that the moduli spaces $\overline{L}_n(\Sigma, X)_{[\rho_1,\rho_2]}$ (at least for regular ρ_1, ρ_2) have virtual fundamental class with the usual properties. Define *polarized gauged Gromov-Witten invariants* as follows. Given a collection $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_G(X)^n$ and an interval $[\rho_1, \rho_2]$ such that ρ_1, ρ_2 are regular define

$$\langle \alpha; \beta \rangle_{d,[\rho_1,\rho_2]} = \int_{\overline{L}(\Sigma,X,d)_{[\rho_1,\rho_2]}} \mathrm{ev}^* \alpha \wedge f^*\beta \in \mathbb{Q}[\xi].$$

Here ξ is the equivariant parameter for the $U(1)$-action and the subscript $[\rho_1, \rho_2]$ denotes the symplectic cut as in (11).

The wall-crossing formulas of the first section yield

Conjecture 5.1.3. *The vortex invariants for ρ_1, ρ_2 are related by the formula*

$$\langle \alpha; \beta \rangle_{d,\rho_2} - \langle \alpha; \beta \rangle_{d,\rho_1} = \mathrm{Res}_\xi \left(\sum_F \int_F \iota_F^*(\mathrm{ev}^* \alpha \wedge f^*\beta) \wedge \mathrm{Eul}(\nu^F)^{-1} \right)$$

where F ranges over the fixed point components of $U(1)$ on $\overline{L}(\Sigma, X)$, with value of ϕ between ρ_1 and ρ_2.

5.2. **Fixed point structure.** The fixed points of $U(1)$ correspond to polarized vortices fixed by a one-parameter subgroup of $U(1) \times \mathcal{G}(P)$. Since $U(1)$ acts freely, if $(\tilde{A}, u)$ is $U(1) \times \mathcal{G}(P)$-fixed then (A, u) is fixed by a one-parameter subgroup $\zeta : U(1) \to \mathcal{G}(P)$. Let $\zeta(z) : U(1) \to \operatorname{Aut}(P_z)$ denote the evaluation of ζ at a point z and let G_ζ denote the centralizer of $\zeta(z)$. The structure group of A reduces to G_ζ, and u takes values in the fixed point set X^ζ of ζ. The bubbles of a fixed point need not take values in X^ζ, since the action of $U(1)$ can be cancelled by the action of reparametrizations. Let $P_\zeta \to \Sigma$ be a principal G_ζ-bundle.

Definition 5.2.1. An n-marked *extended vortex* is a datum $(\underline{\Sigma}, \underline{z}, A, \underline{u})$ where $\underline{\Sigma}$ is a rooted nodal curve with principal component $\Sigma_0 \cong \Sigma$ and genus zero bubble components $\Sigma_1, \ldots, \Sigma_m$, (A, u_0) is a symplectic vortex with values in the Hamiltonian G_ζ-manifold X^ζ, and for each component Σ_i of $\underline{\Sigma}$, $u_i : \Sigma_i \to X$ is a pseudoholomorphic map fixed up to reparametrization by G_ζ, satisfying matching conditions at the nodes.

Let $\overline{M}_n(\Sigma, X^\zeta, X)_\rho$ denote the moduli space of isomorphism classes of extended G_ζ-vortices with values in X^ζ and bubbles in X. $\overline{M}_n(\Sigma, X^\zeta, X)_\rho$ can be identified with a fiber product

$$\overline{M}_n(\Sigma, X^\zeta, X)_\rho = \bigcup_{r, I_1 \cup \ldots \cup I_r \subset \{1,\ldots,n\}} \left(M^{\mathrm{fr}}(\Sigma, X^\zeta)_\rho \times_{(X^\zeta)^m} \prod_{j=1}^r \overline{M}_{|I_j|+1}(X) \right) / G^r$$

by identifying each extended vortex with a vortex in X^ζ and a collection of bubble trees in X fixed (up to reparametrization) by the action of ζ. Hence our previous assumptions on existence of virtual fundamental classes imply that $\overline{M}_n(\Sigma, X^\zeta, X)_\rho$ has a virtual fundamental class. As before, one has *extended vortex invariants* defined by integration over $\overline{M}_n(\Sigma, X^\zeta, X)_\rho$.

The extended vortex invariants that appear in the wall-crossing formula involve further twists by Euler classes of index bundles. Let $\operatorname{Ind}(TX)$ denote the index bundle

$$\operatorname{Ind}(TX) \to \overline{M}_n(P_\zeta, X^\zeta, X)_\rho, \quad \operatorname{Ind}(TX)_{[\underline{\Sigma}, A, \underline{u}]} = \ker(D_{A,\underline{u}}) \ominus \operatorname{coker}(D_{A,\underline{u}})$$

and $\operatorname{Ind}(TX)^\zeta$ its ζ-invariant part

$$\operatorname{Ind}(TX)^\zeta \to \overline{M}_n(\Sigma, X^\zeta, X)_\rho, \quad \operatorname{Ind}(TX)^\zeta_{[\underline{\Sigma}, A, \underline{u}]} \cong \ker(D_{A,\underline{u}})^\zeta \ominus \operatorname{coker}(D_{A,\underline{u}})^\zeta$$

Consider the quotient as a K-class on $\overline{M}_n(\Sigma, X^\zeta, X)_\rho$,

$$\operatorname{Ind}(TX) \ominus \operatorname{Ind}(TX^\zeta) \in K^0(\overline{M}_n(\Sigma, X^\zeta, X)_\rho).$$

Denote by

$$\operatorname{Ind}((\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}}) \in K^0(\overline{M}_n(\Sigma, X^\zeta, X)_\rho)$$

the index class of the representation $(\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}}$, whose fiber at any point $[\underline{\Sigma}, A, \underline{u}]$ is given by index of the operator

$$(\mathrm{d}_A \oplus \mathrm{d}_A^*)((\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}}) : \Omega^1(\Sigma, P((\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}})) \to (\Omega^0 \oplus \Omega^2)(\Sigma, P((\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}})).$$

Let $U(1)$ act on these index classes by scalar multiplication on the fibers. Then their Euler classes are well-defined in $H(\overline{M}_n(\Sigma, X^\zeta, X)_\rho) \otimes \mathbb{Q}[\zeta, \zeta^{-1}]$, where ζ is the equivariant parameter for $U(1)$, as in [1]. For any d_ζ, define the *twisted extended vortex invariant*

$$
\begin{aligned}
(19)\quad & \langle \alpha; \beta \rangle^{twist}_{X^\zeta, X, G_\zeta, d_\zeta, \rho} \\
&= \int_{\overline{M}_n(\Sigma, X^\zeta, X, d_\zeta)_\rho} \operatorname{ev}^* \alpha \wedge f^* \beta \wedge \operatorname{Eul}(\operatorname{Ind}((\mathfrak{g}/\mathfrak{g}_\zeta)^{\mathbb{C}}) \oplus \operatorname{Ind}(TX) \ominus \operatorname{Ind}(TX)^\zeta)^{-1}.
\end{aligned}
$$

We can now state the wall-crossing formula. Let $W_\zeta \subset W$ denote the Weyl group of G_ζ. Inclusion of $\mathfrak{g}_\zeta$ in $\mathfrak{g}$ induces a map $\overline{M}_n(\Sigma, X^\zeta, X)_\rho \to \overline{L}_n(\Sigma, X)^{U(1)}$ which is locally surjective, and has fiber isomorphic to W/W_ζ. Under pull-back, the action of $U(1)$ on the normal bundle is isomorphic to the action of the one-parameter subgroup generated by ζ. Denote by

$$\varphi^{G_\zeta}_G : H^{G_\zeta}_\bullet(X) \to H^G_\bullet(X)$$

the restriction map induced by the inclusion $G_\zeta \to G$ and by $\varphi^G_{G_\zeta}$ the dual map in cohomology. Write $d_\zeta \mapsto d$ if $d \in H^G_2(X)$ is the image of $d_\zeta \in H^{G_\zeta}_2(X)$.

Conjecture 5.2.2 (Wall-crossing for gauged Gromov-Witten invariants)**.** *Given $J \in \mathcal{J}(X)^G$ and values ρ_1, ρ_2 such that every polystable ρ_j vortex is stable, $j = 1, 2$, the gauged Gromov-Witten invariants for ρ_1, ρ_2 are related by*

$$(20)\quad \langle \alpha; \beta \rangle_{X,G,d,\rho_2} - \langle \alpha; \beta \rangle_{X,G,d,\rho_1} = \sum_\zeta \sum_{d_\zeta \mapsto d} \sum_{\rho \in (\rho_1, \rho_2)} \operatorname{Res}_\zeta \frac{\# W_\zeta}{\# W} \langle \varphi^G_{G_\xi} \alpha; \beta \rangle^{twist}_{X^\zeta, X, G_\zeta, d_\zeta, \rho}.$$

The first sum is over equivalence classes of one-parameter subgroups generated by $\zeta \in \mathfrak{g}$.

6. Quantum Kirwan morphism

6.1. **Morphisms of CohFT's.** Recall from [28] that $\overline{M}_{n,1}(\mathbb{C})$ is the moduli space of *stable scaled lines*. This is a compactification of the moduli space of n distinct points on $\mathbb{C}$, modulo translation only. By an *affine line*, we mean a variety Σ with a free transitive action of $\mathbb{C}$. Any such variety is isomorphic to $\mathbb{C}$, canonically up to translation.

Definition 6.1.1. A *scaling* of an affine line Σ is a translation-invariant, non-zero one form $\alpha \in \Omega^{1,0}(\Sigma, \mathbb{C})^{\mathbb{C}}$. A *scaled affine line* is an affine line equipped with a scaling. An *n-marking* of an affine line is a collection $\underline{z} = (z_1, \ldots, z_n)$ of distinct points. An *isomorphism* of scaled n-marked affine lines is an affine isomorphism $\psi : \Sigma_0 \to \Sigma_1$, such that $\psi^* \alpha_1 = \alpha_0$ and $\psi(z_{0,i}) = z_{1,i}, i = 1, \ldots, n$.

Let $M_{n,1}(\mathbb{C})$ denote the moduli space of scaled n-marked affine lines. If Σ is a scaled affine line then the group of automorphisms of Σ preserving the scaling is the group $\mathbb{C}$ acting on Σ by translation. Thus the moduli space $M_{n,1}(\mathbb{C})$ may be identified with the configuration space $\operatorname{Conf}_n(\mathbb{C})$ of n-tuples of distinct points on $\mathbb{C}$ up to the action of $\mathbb{C}$ by translation, $M_{n,1}(\mathbb{C}) \cong \operatorname{Conf}_n(\mathbb{C})/\mathbb{C}$.

From the point of view of symplectic vortices it is natural to view this moduli space in a slightly different way: Any scaling α gives rise to a real area form $\omega := \alpha \wedge \overline{\alpha}$ on Σ. Replacing α with ω amounts to forgetting a phase; thus, one can view $M_{n,1}(\mathbb{C})$ as the moduli space of data $(z_1, \ldots, z_n, \omega, v)$ where $z_1, \ldots, z_n \in \mathbb{C}$ are distinct points, $\omega \in \Omega^2(\mathbb{C}, \mathbb{R})$ is a translationally-invariant area form, and $v \in \operatorname{Vect}(\mathbb{C})^{\mathbb{C}}/\mathbb{R}_+$ is a translationally-invariant vector field up to scalar multiplication. This is the point of view taken in the second author's thesis [41].

The moduli space $M_{n,1}(\mathbb{C})$ has a natural compactification obtained by allowing bubbles with degenerate scalings.

Definition 6.1.2. A *degenerate scaling* on an affine line Σ is an element of the set

$$\overline{\Omega}^{0,1}(\Sigma, \mathbb{C})^{\mathbb{C}} = \{0\} \cup \Omega^{0,1}(\Sigma, \mathbb{C})^{\mathbb{C}} \cup \{\infty\}.$$

The action of the group of automorphisms $\operatorname{Aut}(\Sigma)$ on $\Omega^{0,1}(\Sigma, \mathbb{C})^{\mathbb{C}}$ by pull-back extends naturally to an action on $\overline{\Omega}^{0,1}(\Sigma, \mathbb{C})$, with fixed points $\{0\}, \{\infty\}$.

Definition 6.1.3. A *nodal marked scaled affine line* consists of

(a) A nodal curve $\underline{\Sigma} = (\Sigma_0, \ldots, \Sigma_n)$ of arithemetic genus zero,
(b) A collection $\underline{z} = (z_0, \ldots, z_n)$ of distinct, smooth points, such that $z_0 \in \Sigma_0$;
(c) A (possibly zero or infinite) scaling on Σ_i', where Σ_i' denotes the affine line obtained by removing the node connection Σ_i to the root vertex, or z_0 if $i = 0$;

such that

(a) on any path from the root vertex Σ_0, to the component containing a marking, there is exactly one *colored component* with finite scaling;
(b) the components before (resp. after) this component have infinite (resp. zero scaling).

A nodal marked scaled affine line is *stable* if each component with finite scaling has at least two special points, and each component with degenerate scaling has at least three special points. An *isomorphism* of nodal marked scaled affine lines is an isomorphism of nodal curves, intertwining the (possibly degenerate) scalings and markings.

This compactification has the structure of a projective variety with toric singularities. The possible combinatorial types of nodal marked scaled affine lines are parametrized by *bicolored rooted trees*:

Definition 6.1.4. A *bicolored tree* is a rooted tree Γ equipped with a subset of *colored vertices* $\operatorname{Vert}^{col}(\Gamma)$ of the vertex set $\operatorname{Vert}(\Gamma)$ such that the unique non-self-crossing path between the root edge and any other seminfinite edge crosses exactly one colored vertex.

One can think of Γ as a tree divided into "two parts", the parts above and below the colored vertices. The local structure of $\overline{M}_{n,1}(\mathbb{C})$ near the stratum $M_{n,1,\Gamma}(\mathbb{C})$ of nodal lines with combinatorial type Γ may be described as follows. For any bicolored tree Γ, define a toric variety $X(\Gamma)$ as follow: $X(\Gamma)$ is the set of maps $\delta : \operatorname{Edge}^-(\Gamma) \to \mathbb{C}$ such that for any path γ connecting two colored vertices, the product

$$\prod_{e \in \gamma} \gamma(e)^{\pm} = 1$$

where the sign is positive if e is pointing towards the root vertex and negative otherwise. For the example in Figure 1, the relations are $\delta_3 = \delta_4, \delta_1\delta_3 = \delta_2\delta_5, \delta_5 = \delta_6$.

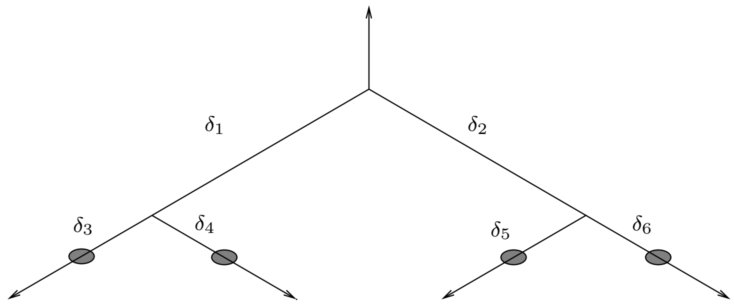


FIGURE 1. An example of a bicolored tree

Proposition 6.1.5. [28] *There exists a neighborhood of* $M_{n,1,\Gamma}(\mathbb{C})$ *in* $\overline{M}_{n,1}(\mathbb{C})$ *isomorphic to* $X(\Gamma) \times M_{n,1,\Gamma}(\mathbb{C})$.

The most important point here is that the codimension of a stratum corresponding to a colored tree is *not* the number of finite edges, but rather the number of finite edges plus one minus the number of colored vertices. Thus there are two types of boundary divisors: First, for any $I \subset \{1, \ldots, n\}$ of order at least two we have a divisor

$$\iota_I : D_I \to \overline{M}_{n,1}(\mathbb{C})$$

corresponding to the formation of a single bubble containing the markings I. There is an isomorphism

$$D_I \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,1}(\mathbb{C}). \tag{21}$$

Call these divisors of *type I.* Second, for any partition $\{I_1, \ldots, I_r\}$, $I_1 \cup \ldots \cup I_r = \{1, \ldots n\}$ of order at least two we have a divisor $D_{I_1 \cup \ldots \cup I_r}$ corresponding to the formation of r bubbles with markings $I_1, \ldots, I_r$, attached to a remaining component with infinite area form. We have a homeomorphism

$$D_{I_1 \cup \ldots \cup I_r} \cong \left(\prod_{i=1}^{r} \overline{M}_{|I_i|,1}(\mathbb{C}) \right) \times \overline{M}_{0,r}. \tag{22}$$

Call these *divisors of type II.*

The moduli space $\overline{M}_{n,1}(\mathbb{C})$ has a "positive real locus" that appears in Stasheff's description of A_∞ morphisms [35]. Namely, the anti-holomorphic involution on $\mathbb{C}$ induces an anti-holomorphic involution of $\overline{M}_{n,1}(\mathbb{C})$. We denote by $\overline{M}_{n,1}(\mathbb{R})$ the fixed point locus, in which all markings are on the real line. The symmetric group S_n acts on $\overline{M}_{n,1}(\mathbb{C})$. The action of S_n on $\overline{M}_{n,1}(\mathbb{R})$ has fundamental domain $\overline{M}_{n,1}(\mathbb{R})^+$ given as the closure of the subset $M_{n,1}(\mathbb{R})^+$ where $z_1 < z_2 < \ldots < z_n$. This space is homeomorphic to Stasheff's multiplihedron [28].

Morphisms of CohFT's are defined via divisors on $\overline{M}_{n,1}(\mathbb{C})$. $\overline{M}_{n,1}(\mathbb{C})$ is not smooth, and so not every Weil divisor is Cartier. In particular, given a divisor

$$D = \sum_I n_I [D_I] + \sum_{I_1 \cup \ldots \cup I_r = \{1, \ldots, n\}} n_{I_1 \cup \ldots \cup I_r} [D_{I_1 \cup \ldots \cup I_r}] \tag{23}$$

there may or may not exist a class $\delta \in H^2(\overline{M}_{n,1}(\mathbb{C}))$ that satisfies

$$\langle \beta, [D] \rangle = \langle \beta \wedge \delta, [\overline{M}_{n,1}(\mathbb{C})] \rangle.$$

Let V and W be cohomological field theories.

Definition 6.1.6. A *morphism of cohomological field theories* is a collection of maps

$$\phi^n : V^n \times H^\bullet(\overline{M}_{n,1}(\mathbb{C})) \to W, \quad n \geq 1$$

such that for any divisor D of the form (23) with dual class $\delta \in H^2(\overline{M}_{n,1}(\mathbb{C}))$ we have

$$\begin{aligned}(24)\quad \phi^n(\alpha, \beta \wedge \delta) &= \sum_I n_I \phi^{n-|I|}(\mu_V^{|I|}(\alpha_i, i \in I; \cdot), \alpha_j, j \notin I; \cdot)(\iota_I^* \beta) \\ &+ \sum_{I_1 \cup \ldots \cup I_r} n_{I_1 \cup \ldots \cup I_r} \mu_W^r(\phi^{I_1}(\alpha_i, i \in I_r; \cdot), \ldots, \phi^{I_r}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota^*_{I_1 \cup \ldots \cup I_r} \beta)\end{aligned}$$

where $\cdot$ indicates insertion of the Kunneth components of $\iota_I^* \beta$, $\iota^*_{I_1 \cup \ldots \cup I_r} \beta$, using the homeomorphisms (21), (22). A *weak morphism* of CohFT's is defined similarly but with the additional data of an element $\phi^0 \in W$ and the splitting axiom allowing arbitrary numbers of insertions of ϕ^0, that is,

$$\begin{aligned}(25)\quad \phi^n(\alpha, \beta \wedge \delta) &= \sum_I n_I \phi^{n-|I|}(\mu_V^{|I|}(\alpha_i, i \in I; \cdot), \alpha_j, j \notin I; \cdot)(\iota_I^* \beta) \\ + \sum_{r \leq s, I_1 \cup \ldots \cup I_r} & n_{I_1 \cup \ldots \cup I_r} \mu_W^s(\phi^{I_1}(\alpha_i, i \in I_r; \cdot), \ldots, \phi^{I_r}(\alpha_i, i \in I_r; \cdot), \phi^0, \ldots, \phi^0; \cdot)(\iota^*_{I_1 \cup \ldots \cup I_r} \beta)/(s-r)!.\end{aligned}$$

The definition of weak morphism of CohFT's is analogous to the definition of weak morphism of A_∞ algebras in [9]. We consider here only the even case; the general case involves additional signs. For example, in the case $n = 1$ we have $\overline{M}_{2,1}(\mathbb{C})$ a point and so we obtain a map $\phi^1 : V \to W$. The following proposition describes the properties of this map:

Proposition 6.1.7. *If $\phi^\bullet$ is a morphism of CohFT's from V to W, then ϕ^1 is a ring homomorphism from V to W:*

$$\phi^1 \circ \mu_V^2 = \mu_W^2 \circ (\phi^1 \times \phi^1).$$

If $\phi^\bullet$ is a weak morphism of CohFT's, we have instead that

$$(26) \qquad (\phi^1 \circ \mu_V^2)(v_1, v_2) = \sum_{s \geq 2} \mu_W^s(\phi^1(v_1), \phi^1(v_2), \phi^0, \ldots, \phi^0)/(s-2)!$$

for all $v_1, v_2 \in V$.

Proof. The relation $\gamma_{\{1,2\}} = \gamma_{\{1\},\{2\}}$ in $H^2(\overline{M}_{3,1}(\mathbb{C})) \cong \mathbb{Q}$ (both are point classes) gives the claim relation using (24),(25). □

For any CohFT's U, V, we denote by $\text{Hom}(U, V)$ the space of morphisms from U to V. Composition of morphisms of CohFT's is studied in Section 7.1. A *diagram of cohomological field theories* is a graph whose nodes are CohFT's or (Novikov) rings, and whose arrows are either morphisms of CohFT's (if the head and tail are CohFT's) or traces (if the tail is a CohFT and the head is a (Novikov) ring).

6.2. **Vortices on the affine line.** Let X be a compact Hamiltonian G-manifold. Just as the moduli space of stable maps $\overline{M}_{0,n}(X,d)$ to a projective variety X lives over $\overline{M}_{0,n}$, there is a natural moduli space of *vortices on the affine line* living over $\overline{M}_{n,1}(\mathbb{C})$. Following Ziltener [41]

Definition 6.2.1. An *n-marked symplectic vortex* from $\mathbb{C}$ to X is a datum $(A, u, \underline{z})$, where $A \in \Omega^1(\mathbb{C}, \mathfrak{g})$ is a connection on the trivial bundle, $u : \mathbb{C} \to X$ is a J_A-holomorphic map, $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is a collection of distinct points, and

$$F_A + u^*\Phi\, \omega_{\mathbb{C}} = 0.$$

Here $\omega_{\mathbb{C}}$ is the standard area form on $\mathbb{C}$. An *isomorphism* of marked symplectic vortices $(A_j, u_j, \underline{z}_j), j = 0,1$ is an automorphism of the trivial bundle $\phi : \mathbb{C} \times G$ such that $\phi^* A_1 = A_0$, $\phi^* u_1 = u_0$, such that ϕ covers a translation on the base, that is, there exists a $\lambda \in \mathbb{C}$ such that $\pi \circ \phi(z,g) = (z + \lambda, g)$ for all $z, g \in \mathbb{C} \times G$, and $z_{i,1} = z_{i,0} + \lambda$ for $i = 1, \ldots, n$.

Let $M_{n,1}(\mathbb{C}, X)$ denote the moduli space of isomorphism classes of finite energy n-marked vortices on $\mathbb{C}$ (the additional marking at infinity) with values in X. Similarly, let $M^{\mathrm{fr}}_{n,1}(\mathbb{C}, X)$ denote the moduli space of isomorphism classes of n-marked vortices on $\mathbb{C}$ with values in X and framings at the markings. The group G^n acts on $M^{\mathrm{fr}}_{n,1}(\mathbb{C}, X)$ by changing the framings at the markings. If the action is free, the projection

$$M^{\mathrm{fr}}_{n,1}(\mathbb{C}, X) \to M_{n,1}(\mathbb{C}, X) \tag{27}$$

has the structure of a smooth principal G^n bundle over the regular locus, by the existence of slices.

As explained in the second author's thesis [41], the moduli space $M_{n,1}(\mathbb{C}, X)$ and its framed analog $M^{\mathrm{fr}}_{n,1}(\mathbb{C}, X)$ admit a partial compactification allowing nodal scaled lines as the domain.

Definition 6.2.2. A *nodal scaled vortex* on $\mathbb{C}$ with values in X is a nodal scaled affine line $\underline{\Sigma}$ with principal component Σ, equipped with

(a) for each component Σ_i with zero scaling, a finite energy holomorphic map $\Sigma_i \to X$
(b) for each component Σ_i with non-degenerate scaling, a vortex on Σ_i with respect to the given area form;
(c) for each component Σ_i with infinite scaling, a finite energy holomorphic map $\Sigma_i \to X/\!\!/G$

satisfying matching conditions at the nodes. A nodal scaled vortex is *polystable* if each component with non-degenerate scaling and trivial vortex has at least two special points, and each component with degenerate scaling and trivial map has at least three special points, and *stable* if it has finite automorphism group An *isomorphism* of nodal scaled vortices is an automorphism of the underlying nodal scaled marked affine lines, together with a gauge transformation on the principal component, that intertwines the connections and maps. A *framed nodal scaled vortex* is a nodal scaled vortex equipped with framings at the marked points as in Definition 4.2.2.

Let $\overline{M}_{n,1}(\mathbb{C},X)$ resp. $\overline{M}^{\mathrm{fr}}_{n,1}(\mathbb{C},X)$ denote the moduli space of nodal scaled vortices from $\mathbb{C}$ to X, resp. the moduli space of framed nodal scaled vortices from $\mathbb{C}$ to X. $\overline{M}^{\mathrm{fr}}_{n,1}(\mathbb{C},X)$ admits an evaluation maps at the markings, and, as explained in [41] an additional evaluation map at infinity to $X/\!\!/G$:

$$\mathrm{ev}^{\mathrm{fr}} \times \mathrm{ev}_\infty : \overline{M}^{\mathrm{fr}}_{n,1}(\mathbb{C},X) \to X^n \times X/\!\!/G.$$

If G^n acts freely, combining this map with a classifying map for the projection (27) gives a map

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{M}_{n,1}(\mathbb{C},X) \to X^n_G \times X/\!\!/G.$$

For $n > 0$, there is a forgetful morphism to the moduli space of scaled lines,

$$f : \overline{M}_{n,1}(\mathbb{C},X) \to \overline{M}_{n,1}(\mathbb{C}).$$

Using Poincaré duality we obtain a map

$$H_G(X,\mathbb{Z})^n \otimes H(\overline{M}_{n,1}(\mathbb{C},X),\mathbb{Z}) \to H(X/\!\!/G,\mathbb{Z}).$$

More generally, if the G action is only locally free then the evaluation map exists after passing to the classifying space $B\overline{M}_{n,1}(\mathbb{C},X)$ and we obtain a map in rational cohomology

$$H_G(X,\mathbb{Q})^n \otimes H(\overline{M}_{n,1}(\mathbb{C},X),\mathbb{Q}) \to H(X/\!\!/G,\mathbb{Q}).$$

Suppose that the action of G on X extends to a symplectic action of a group K containing G as a normal subgroup. Let $\overline{M}^{\mathrm{fr},G,i}_{n,1}(\mathbb{C},X)$ denote the moduli space of G-vortices with framing at the i-th marked point. K acts on $\overline{M}^{\mathrm{fr},G,i}_{n,1}(\mathbb{C},X)$ by constant gauge transformations. The evaluation map

$$\mathrm{ev}^{\mathrm{fr}}_i : \overline{M}^{\mathrm{fr},G,i}_{n,1}(\mathbb{C},X) \to X.$$

is K-equivariant and so defines a map

$$\overline{M}^{\mathrm{fr},G,i}_{n,1}(\mathbb{C},X)_K \to X_K.$$

If the G-action is free, we have a homotopy equivalence

$$\overline{M}^{\mathrm{fr},G,i}_{n,1}(\mathbb{C},X)_K \to \overline{M}_{n,1}(\mathbb{C},X)_{K/G} \tag{28}$$

which induces a map

$$\mathrm{ev}_i : \overline{M}_{n,1}(\mathbb{C},X)_{K/G} \to X_K.$$

Pull-back in cohomology gives a map

$$\mathrm{ev}_i^* : H_K(X,\mathbb{Z})^n \to H_{K/G}(\overline{M}_{n,1}(\mathbb{C},X),\mathbb{Z}).$$

More generally, if G acts with finite stabilizers then the map (28) has fibers with torsion cohomology (being classifying spaces for the stabilizers), and we obtain a map in rational cohomology

$$\mathrm{ev}^* : H_K(X,\mathbb{Q}) \to H_{K/G}(\overline{M}^G_{n,1}(\mathbb{C},X),\mathbb{Q}). \tag{29}$$

Similarly for the evaluation map at infinity we have a G/K-equivariant evaluation map

$$\mathrm{ev}_\infty : \overline{M}^{\mathrm{fr},\infty}_{n,1}(\mathbb{C},X) \to X/\!\!/G.$$

Pull-back in equivariant cohomology gives a map

$$\mathrm{ev}_\infty^* : H_{K/G}(X/\!\!/G,\mathbb{Q}) \to H_{G/K}(\overline{M}^G_{n,1}(\mathbb{C},X),\mathbb{Q}).$$

6.3. **Quantum Kirwan morphism.** Suppose from now on the usual existence and axiomatic properties of virtual fundamental classes. In this section, we assume that $GW(X/\!\!/G)$ is the Gromov-Witten theory defined over the larger Novikov ring Λ_X^G. In particular, $QH(X/\!\!/G) = H(X/\!\!/G, \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_X^G$. For each $n \geq 0$ define a map

$$Q\kappa_G^n : QH_G(X)^n \times H(\overline{M}_{n,1}(\mathbb{C})) \to QH(X/\!\!/G)$$

as follows. For $\beta \in H^*(\overline{M}_{n,1}(\mathbb{C}))$

$$(Q\kappa_G^n(\alpha, \beta), \alpha_\infty) = \sum_{d \in H_2^G(X)} e^d \int_{\overline{M}_{n,1}(\mathbb{C}, X, d)} \operatorname{ev}^* \alpha \wedge f^* \beta \wedge \operatorname{ev}_\infty^* \alpha_\infty$$

using Poincaré duality; the pairing on the left is given by wedge product and integration over $X/\!\!/G$. We say that X is G-equivariantly monotone if the G-equivariant first Chern class is a positive multiple of the class of the equivariant symplectic form.

Conjecture 6.3.1. *$Q\kappa_G^\bullet$ is a weak morphism of CohFT's from $GW_G(X)$ to $GW(X/\!\!/G)$. If X is equivariantly monotone then $Q\kappa_G^\bullet$ is a morphism of CohFT's.*

More generally, if the action of G extends to an action of a group K containing G as a normal subgroup, we have a map

$$QH_K(X, \mathbb{Q})^n \times H(\overline{M}_{n,1}(\mathbb{C}, X), \mathbb{Q}) \to QH_{G/K}(X/\!\!/G, \mathbb{Q})$$

defined by the same formula, using the evaluation map (29). After extending the coefficient ring of $GW_{K/G}(X/\!\!/G)$ from $\Lambda_{X/\!\!/G}$ to $\Lambda_{X/\!\!/G}^{K/G}$ one expects this to define a morphism of CohFT's

$$Q\kappa_{K,G} : GW_K(X) \to GW_{K/G}(X/\!\!/G) \tag{30}$$

generalizing (5).

7. Quantum reduction in stages

7.1. **Composition of morphisms of CohFT's.** In this section we discuss *composition* of morphisms of CohFT's. This will make CohFT's into something like a category, but with additional information necessary for composition, as in Street's notion of higher categories, see for example [36]. The cohomological datum needed for the composition maps is associated to a moduli space of s-scaled n-marked lines as follows.

Definition 7.1.1. An *s-scaled, n-marked affine line* is an affine line Σ equipped with s scalings $\phi_1, \ldots, \phi_s : T\Sigma \to \Sigma \times \mathbb{C}$ as in Definition 6.1.1 and n distinct points $z_1, \ldots, z_n$. An *isomorphism* of s-scaled, n-marked affine lines Σ_0, Σ_1 is an isomorphism $\psi : \Sigma_0 \to \Sigma_1$ preserving the scalings and markings: $\phi_i^1 \circ D\psi = (\psi \times 1) \circ \phi_i^0$ for $i = 1, \ldots, s$ and $\psi(z_j^0) = z_j^1$ for $j = 1, \ldots, n$.

Let $M_{n,s}(\mathbb{C})$ denote the moduli space of s-scaled, n-marked affine lines up to automorphism. In particular, $M_{n,1}(\mathbb{C})$ is homeomorphic to the moduli space of n points on the affine line up to translation. $M_{n,s}(\mathbb{C})$ has a natural compactification obtained by allowing bubbles on which a proper subset of the scalings have gone to infinity or zero.

Definition 7.1.2. A *s-scaled n-marked nodal curve* $(\underline{\Sigma}, \underline{z}, \underline{\phi})$ consists of

(a) A nodal curve $\underline{\Sigma} = \Sigma_0, \ldots, \Sigma_k$ where the *root component* Σ_0 is an affine line and $\Sigma_1, \ldots, \Sigma_k$ are projective lines.
(b) A collection of distinct smooth points $\underline{z} = (z_1, \ldots, z_n) \in \underline{\Sigma}$.
(c) a collection of possibly degenerate scalings $\phi_i^j : T\Sigma_i' \to \Sigma_i' \times \mathbb{C}$, where Σ_i' is the complement of the node connecting Σ_i to the root component;

such that

(a) for any path from Σ_0 to a terminal component and for any index i, there is exactly one component $m(i)$ on which the i-th scaling is non-degenerate
(b) the components previous to $m(i)$ have zero i-th scaling and the components after that have infinite i-th scaling.
(c) For any indices i, j, the ratio between the i-th and j-th scaling is independent of the choice of component.

A s-scaled, n-marked line $\underline{\Sigma}$ is *stable* if each component with at least one non-degenerate scaling has at least one marked or nodal point, and each component with all degenerate scalings has at least two marked or nodal points. The *combinatorial type* of an s-scaled, n-marked affine line is an $s+1$-colored tree, defined as follows. Let Γ be the tree defined by $\underline{\Sigma}$. For each i, the i-colored vertices are those on which the i-th scaling is finite.

Let $\overline{M}_{n,s}(\mathbb{C})$ denote the moduli space of s-scaled, n-marked lines. The boundary of $\overline{M}_{n,s}(\mathbb{C})$ can be described as follows.

(a) For any subset $I \subset \{1, \ldots n\}$ of order at least two there is a divisor
$$\iota_I : D_I \to \overline{M}_{n,s}(\mathbb{C})$$
and an isomorphism
$$\varphi_I : D_I \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,s}(\mathbb{C})$$
corresponding to the formation of a bubble containing the markings $z_i, i \in I$ with zero scaling on that bubble.
(b) For any partition $I_1 \cup \ldots \cup I_r$ of $\{1, \ldots, n\}$ of order at least two and subset $J \subset \{1, \ldots, s\}$ there is a divisor
$$\iota_{I_1 \cup \ldots \cup I_r, J} : D_{I_1 \cup \ldots \cup I_r, J} \to \overline{M}_{n,s}(\mathbb{C})$$
with an isomorphism
$$\varphi_{I_1 \cup \ldots \cup I_r, J} : D_{I_1 \cup \ldots \cup I_r, J} \to \overline{M}_{r+1, s-|J|}(\mathbb{C}) \times \prod_{i=1}^{r} \overline{M}_{|I_i|+1,|J|}(\mathbb{C})$$
corresponding to the formation of r bubbles containing markings $I_j, j = 1, \ldots, r$ with the scalings $j \in J$ becoming infinite on those bubbles.

The union of these divisors is the boundary of $M_{n,s}(\mathbb{C})$:
$$\partial M_{n,s}(\mathbb{C}) = \bigcup_{I \subset \{1, \ldots n\}} D_I \cup \bigcup_{I_1 \cup \ldots \cup I_r = \{1, \ldots, n\}} D_{I_1 \cup \ldots \cup I_r}.$$

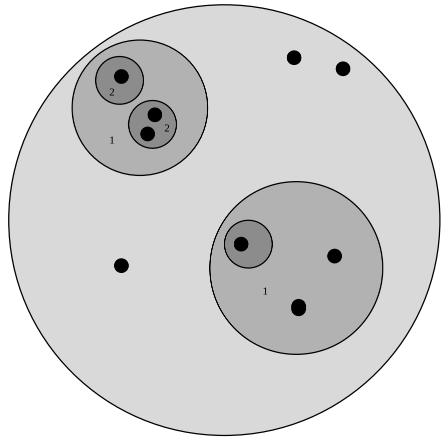


FIGURE 2. A scaled curve

For any partition $J_1 \cup \ldots \cup J_r$ of the scaling indices $\{1, \ldots, s\}$ we have a canonical embedding

$$\iota_{J_1 \cup \ldots \cup J_r} : \overline{M}_{n+1,r}(\mathbb{C}) \to \overline{M}_{n,s}(\mathbb{C})$$

corresponding to the locus where the scalings $j \in J_i$ are equal. In particular the inclusion

$$\iota_{\{1,\ldots,d\}} : \overline{M}_{n,1}(\mathbb{C}) \to \overline{M}_{n,s}(\mathbb{C})$$

is defined by setting all scalings equal.

The group $S_n \times S_d$ acts by permuting the labelling of the markings and scalings. The action of $S_n \times S_d$ on the real locus $\overline{M}^{\mathbb{R}}_{n+1,d}$ has fundamental domain $\overline{M}^{\mathbb{R},\geq 0}_{n,s}$ the closure of the set $M^{\mathbb{R},>0}_{n,s}$ where the markings $z_1 < z_2 < \ldots < z_n$ are real, the scalings $\phi_i : T\mathbb{C} \to \mathbb{C} \times \mathbb{C}$ are positive real scalars, and $\phi_1 < \phi_2 < \ldots < \phi_s$.

Consider the inclusion $\iota_{\{1,\ldots,s\}}$ given by setting all scalings equal. Denote the induced map in cohomology

$$\iota^*_{\{1,\ldots s\}} : H(\overline{M}_{n,1}(\mathbb{C})) \to H(\overline{M}_{n,s}(\mathbb{C}))$$

We denote by $\mathrm{Rinv}(\iota^*_{\{1,\ldots,s\}})$ the space of right inverses

$$\varphi : H^\bullet(\overline{M}_{n,s}(\mathbb{C})) \to H^\bullet(\overline{M}_{n,1}(\mathbb{C})), \quad \iota^\bullet_{\{1,\ldots,s\}} \circ \varphi = \mathrm{Id}\,.$$

Let

$$\mathrm{Rinv}(\iota^*_{\cdot,\{1,\ldots,s\}}) = \bigoplus_{n\geq 0} \mathrm{Rinv}(\iota^*_{n,\{1,\ldots,s\}}).$$

Let U, V, W be CohFT's.

Definition 7.1.3. Define the *composition of CohFT's*

$$\mu^2 : \mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) \times \mathrm{Rinv}(\iota^*_{\cdot,\{1,2\}}) \to \mathrm{Hom}(U, W)$$

for

$$\phi \in \mathrm{Hom}(U, V), \quad \psi \in \mathrm{Hom}(V, W), \quad \zeta \in \mathrm{Rinv}(\iota^*_{\cdot,\{1,2\}})$$

by

$$\mu^2(\phi, \psi, \zeta)_n : U^n \times H(\overline{M}_{n,1}(\mathbb{C})) \to W$$

where $(\mu^2(\phi, \psi, \zeta))(\alpha_1, \ldots, \alpha_n; \beta)$ is equal to

$$\sum_{I_1 \cup I_2 \cup \ldots \cup I_r = \{1,\ldots,n\}} \phi^r(\psi^{|I_1|}(\alpha_i, i \in I_1; \cdot), \ldots, \psi^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota^*_{I_1 \cup \ldots \cup I_r} \zeta(\beta)). \tag{31}$$

Here the dots indicate insertion of the Kunneth components of $\zeta(\beta)$ with respect to the decompositions (21), (22).

The definition of composition of weak morphisms of CohFT's is similar, except that one allows the subsets $I_1, \ldots, I_r$ in the partition to be empty. More generally, given a collection $U_1, \ldots, U_n$ of CohFT's, one can define a composition map

$$\mu^n : \operatorname{Hom}(U_0, U_1) \times \ldots \times \operatorname{Hom}(U_{s-1}, U_s) \times \operatorname{Rinv}(\iota^*_{\cdot,\{1,\ldots,s\}}) \to \operatorname{Hom}(U_1, U_s)$$

by a similar formula that we omit.

There is a notion of *commutative triangle of CohFT's* which does not require the choice of right-inverse. Namely, given morphisms of CohFT's ϕ, ψ define

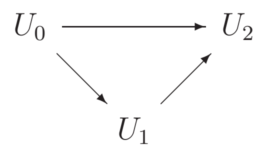
$$\phi \circ \psi : U_0^n \times H(\overline{M}_{n,2}(\mathbb{C})) \to U_2$$

by inserting $\iota^*_{1,2}\beta$ instead of $\zeta(\beta)$ in the expression (31). Let

$$\phi_{01} : U_0 \to U_1, \quad \phi_{12} : U_1 \to U_2, \quad \phi_{02} : U_0 \to U_2$$

be morphisms of CohFT's.

Definition 7.1.4. The diagram of CohFT's



commutes iff $\phi^n_{02}(\alpha; \iota^n_{\{1,2\}}\beta) = (\phi_{01} \circ \phi_{12})(\beta)$ for all $(\alpha, \beta) \in U_0^n \times H(\overline{M}_{n,2}(\mathbb{C}))$

One should think of the class β as labelling the interior of the triangle. One might formalize the above notion of composition as follows.

Definition 7.1.5. A *cohomological category* consists of

(a) a class of objects $\operatorname{Obj}(C)$,
(b) for each pair of objects V_0, V_1 a vector space of morphisms $\operatorname{Hom}(V_0, V_1)$, and
(c) for each collection of objects $V_0, \ldots, V_s$ a map

$$\operatorname{Hom}(V_0, V_1) \times \ldots \times \operatorname{Hom}(V_{s-1}, V_s) \times \operatorname{Rinv}(\iota^*_{\{1,\ldots,d\}}) \to \operatorname{Hom}(V_0, V_s) \tag{32}$$

satisfying *cohomological strict associativity*: the expression

$$\mu^{s-j+1}(\phi_1, \ldots, \phi_{i-1}, \mu^j(\phi_i, \ldots, \phi_{i+j-1}), \phi_{i+j}; \cdot), \phi_{i+j+1}, \ldots \phi_n; \cdot)$$

(where the dots denote the insertion of Kunneth components) is independent of the choice of i, j.

The following is a conjecture since the construction of the moduli spaces $\overline{M}_{n,s}(\mathbb{C})$ has not yet been carried out in detail.

Conjecture 7.1.6. *CohFT's form a cohomological category.*

7.2. **Multiply scaled vortices.** We now define a moduli space of *multiply scaled vortices* that "lives above" $\overline{M}_{n,s}(\mathbb{C})$. Consider a chain of normal subgroups

$$G = G_0 \supset G_1 \supset \ldots \supset G_s$$

Let X be a compact Hamiltonian G-manifold equipped with a G-invariant compatible almost structure $J \in \mathcal{J}(X)^G$.

Definition 7.2.1. An *s-scaled, n-marked nodal vortex* on the affine line $\mathbb{C}$ with values in X is a s-scaled, n-marked nodal curve $\underline{\Sigma}$ equipped with

(a) for each component Σ_i with all area forms non-zero, a vortex for G with values in X
(b) for each component Σ_i on which the area forms $1, \ldots, k$ are zero and area forms $l+1, \ldots, s$ are infinite, a G/G_k-equivariant G_l/G_k vortex with values in $X/\!\!/G_k$.

An s-scaled nodal vortex is *polystable* if each component with some non-degenerate scalings has at least two special points, and each bubble with all degenerate scalings has at least three special points. A multiply scaled nodal vortex is *stable* if it has finite automorphism group.

Let $\overline{M}_{n,s}(\mathbb{C}, X)$ denote the moduli space of s-scaled, n-marked nodal vortices on $\mathbb{C}$ with values in X. The boundary of $\overline{M}_{n,s}(\mathbb{C}, X)$ can be described as follows. First, for any subset $I \subset \{1, \ldots n\}$ there is a divisor

$$\iota_I : D_I \to \overline{M}_{n,s}(\mathbb{C}, X)$$

and an isomorphism

$$\varphi_I : D_I \to \overline{M}_{0,|I|+1}(X) \times_X \overline{M}_{n-|I|+1,s}(\mathbb{C}, X)$$

corresponding to the formation of a sphere bubble containing the markings $z_i, i \in I$ with zero scaling on that bubble. Second, for any partition $I_1 \cup \ldots \cup I_r$ (here the sets I_j may be empty) of $\{1, \ldots, n\}$ and subset $J \subset \{1, \ldots, s\}$ there is a divisor

$$\iota_{I_1 \cup \ldots \cup I_r, J} : D_{I_1 \cup \ldots \cup I_r, J} \to \overline{M}_{n,s}(\mathbb{C}, X)$$

with an isomorphism

$$\varphi_{I_1 \cup \ldots \cup I_r, J} : D_{I_1 \cup \ldots \cup I_r, J} \to \overline{M}_{r,s-|J|}(\mathbb{C}, X) \times \prod_{i=1}^{r} \overline{M}_{|I_i|+1,|J|}(\mathbb{C}, X)$$

corresponding to the formation of r bubbles containing markings $I_j, j = 1, \ldots, r$ with the scalings $j \in J$ becoming infinite on those bubbles. For any partition $J_1 \cup \ldots \cup J_r$ of the scaling indices $\{1, \ldots, s\}$ we also have a canonical divisor

$$\iota_{J_1 \cup \ldots \cup J_s} : \overline{M}_{n,r}(\mathbb{C}, X) \to \overline{M}_{n,s}(\mathbb{C}, X)$$

corresponding to the locus where the scalings $j \in J_s$ are equal.

In particular, $\overline{M}_{2,1}(\mathbb{C})$ is a projective line, and the linear equivalence between $D_{\{1,2\}}$ and $D_{\{1\},\{2\}}$ induces an equivalence in homology in $\overline{M}_{n,2}(\mathbb{C}, X)$ between $\overline{M}_{n,1}(\mathbb{C}, X)$ (embedded as the subspace where the area forms are equal) and the union of the divisors $D_{I_1 \cup \ldots \cup I_r, \{1\}}$.

7.3. **Quantum reduction in stages.** Suppose that G is a normal subgroup of K, and consider the equivariant quantum Kirwan morphisms

$$Q\kappa_{K,G} : GW_K(X) \to GW_{K/G}(X /\!\!/ G), \quad Q\kappa_{K/G} : GW_{K/G}(X /\!\!/ G) \to GW(X /\!\!/ K)$$

defined in (30). The linear equivalence mentioned at the end of the previous section leads naturally to the

Conjecture 7.3.1. *Suppose that X, K, G are as above, and the symplectic quotients by H and G are free. Then there is a commutative triangle of CohFT's*

$$\begin{array}{ccc} GW_K(X) & \xrightarrow{\hspace{3em}} & GW(X /\!\!/ K) \\ & \searrow \quad \nearrow & \\ & GW_{K/G}(X /\!\!/ G) & \end{array}$$

More generally, given a chain $G = G_0 \supset G_1 \supset \ldots G_s$ as above one should have a *commutative simplex* of CohFT's. We leave it to the reader to formulate the precise conjecture.

8. Quantum non-abelian localization

8.1. **Composition of traces with morphisms of CohFT's.** Let Σ be a compact Riemann surface, and $\omega_\Sigma \in \Omega^2(\Sigma)$ an area form with unit area. By Moser's theorem, any two such area forms are related by a diffeomorphism. By a *scaling* of Σ, we mean a choice of multiple $\rho^{-1}\omega_\Sigma$; the parameter ρ is called the *scaling parameter*. An *n-marking* of Σ is a choice of distinct points $z_1, \ldots, z_n$. Let $M_{n,1}(\Sigma)$ denote the space of n-marked 1-scaled curves with underlying curve Σ; we do not quotient by automorphisms of Σ. The space $M_{n,1}(\Sigma)$ admits a compactification by allowing *stable scaled curves* allowing bubbles with zero area form and some subset of the markings, or allowing the area form on Σ to degenerate to zero, in which case one also has bubbles with non-zero area form. Any marking is necessarily contained on such a bubble.

Definition 8.1.1. A *stable scaled Σ-rooted curve* consists of

(a) a nodal curve $\underline{\Sigma} = (\Sigma_0, \ldots, \Sigma_m)$
(b) an isomorphism of Σ_0 with Σ
(c) an area form $\rho^{-1}\omega_\Sigma$ on Σ
(d) a possibly degenerate affine scaling (in the sense of Definition 6.1.1) on each of the curves Σ_i' obtained by removing from Σ_i the node connecting Σ_i with the principal component

such that

(a) each bubble component $\Sigma_i, i > 0$ with degenerate scaling has at least three special points; and
(b) each bubble component $\Sigma_i, i > 0$ with non-degenerate scaling has at least two special points;

(c) if at least one scaling is non-zero then on any path from Σ_0 to a terminal component there is exactly one non-degenerate scaling; the scalings before that component on the path are infinite and those after that component on the path are zero.

In other words, a stable scaled curve is either a scaled curve with a collection of bubble trees attached (if the scaling on the principal component is zero or finite) or a curve with infinite scaling and a collection of stable scaled affine lines attached (if the scaling on the principal component is infinite).

Let $\overline{M}_{n,1}(\Sigma)$ denote the moduli space of n-marked, scaled curves with principal component Σ. The boundary structure of $\overline{M}_{n,1}(\Sigma)$ is described as follows. For any subset $I \subset \{1,\ldots,n\}$ of order at least two we have a divisor

$$\iota_I : D_I \to \overline{M}_{n,1}(\Sigma)$$

and an isomorphism

$$\varphi_I : D_i \to \overline{M}_{0,|I|+1} \times \overline{M}_{n-|I|+1,1}(\Sigma)$$

corresponding to the formation of a bubble with markings $z_i, i \in I$ with zero area form. For any partition $I_1 \cup \ldots \cup I_r$ of $\{1,\ldots,n\}$ we have a divisor

$$\iota_{I_1,\ldots,I_r} : D_{I_1,\ldots,I_r} \to \overline{M}_{n,1}(\Sigma)$$

and an isomorphism

$$\varphi_{I_1\cup\ldots\cup I_r} : D_{I_1\cup\ldots\cup I_r} \to \overline{M}_r(\Sigma) \times \prod_{j=1}^{r} \overline{M}_{|I_j|,1}(\mathbb{C})$$

corresponding to degeneration of the area form to infinity, and bubbles with non-degenerate area form containing the markings. For finite ρ there is an inclusion

$$\iota_\rho : \overline{M}_n(\Sigma) \to \overline{M}_{n,1}(\Sigma)$$

by choosing any scaling $\rho^{-1}\omega_\Sigma$. The induced map in cohomology

$$\iota^* := \iota_\rho^* : H(\overline{M}_{n,1}(\Sigma)) \to H(\overline{M}_n(\Sigma))$$

is independent of the choice of ρ. We denote by $\mathrm{Rinv}(\iota^*)$ the space of right inverses to ι^*.

Suppose that V, W are (even, genus zero) CohFTs, with structure maps

$$\mu_V^n : V^n \otimes H(\overline{M}_{0,n+1}) \to V, \quad \mu_W^n : W^n \otimes H(\overline{M}_{0,n+1}) \to W.$$

Definition 8.1.2. The *composition of morphism of CohFT's with traces* is the map

$$\circ : \mathrm{Hom}(V,W) \times \mathrm{Tr}(W) \times \mathrm{Rinv}(\iota^*) \to \mathrm{Tr}(V)$$

given by

$$(33)\quad (\circ(\phi,\tau,\zeta))(\alpha_i, i \in \{1,\ldots n\}, \beta) = \sum_{I_1\cup\ldots\cup I_r=\{1,\ldots,n\}} \tau^r(\phi^{|I_1|}(\alpha_i, i\in I_1;\cdot),\ldots,\phi^{|I_r|}(\alpha_i, i \in I_r;\cdot);\cdot)(\iota_{I_1,\ldots,I_r}(\zeta(\beta))).$$

The definition for composition of trace and weak morphism of CohFT's is similar, but allowing the subsets $I_1,\ldots,I_r$ in the partition to be empty.

There is also a notion of *commutative triangle of a morphism of CohFT's and two traces* that does not involve choice of right inverses. Namely, let τ_V, τ_W be Σ-traces on V, W with values in some Novikov ring Λ, and $\phi : V \to W$ a morphisms of CohFT's.

Definition 8.1.3. The diagram of CohFT's/traces

$$\begin{array}{ccc} V & \longrightarrow & W \\ & \searrow \quad \swarrow & \\ & \Lambda & \end{array}$$

commutes if for all $\beta \in H(\overline{M}_{n,1}(\Sigma))$ the formula

$$(34) \quad \tau_V(\alpha_1, \ldots, \alpha_n; \iota^* \beta) = \sum_{I_1 \cup \ldots \cup I_r = \{1, \ldots, n\}} \tau_W^r(\phi^{|I_1|}(\alpha_i, i \in I_1; \cdot), \ldots, \phi^{|I_r|}(\alpha_i, i \in I_r; \cdot); \cdot)(\iota^*_{I_1 \cup \ldots \cup I_r} \beta)$$

holds.

8.2. **Scaled vortices on the projective line.** Let X be a Hamiltonian G-manifold equipped with an invariant almost complex structure $J \in J(X)^G$.

Definition 8.2.1. A *nodal scaled vortex* on $P \to \Sigma$ with values in X is a scaled curve with principal component Σ, equipped with

(a) for each component Σ_i with zero scaling, a holomorphic map $\Sigma_i \to X$
(b) for each component Σ_i with non-degenerate scaling, a vortex on the trivial bundle over $\Sigma_i - w_i$, if $i > 0$, or on $P \to \Sigma_0 = \Sigma$ if $i = 0$
(c) for each component Σ_i with infinite scaling, a holomorphic map $\Sigma_i \to X /\!\!/ G$.

A nodal scaled vortex is *polystable* if each component with non-degenerate scaling and trivial vortex has at least two special points, and each component with degenerate scaling and trivial map has at least three special points, and *stable* if it has finite automorphism group.

Let $\overline{M}_{n,1}(P, X)$ denote the moduli space of polystable stable n-marked scaled vortices on P with values in X, and $\overline{M}_{n,1}(\Sigma, X)$ the union over types of bundles P. This moduli space certainly will not in general have a virtual fundamental class with good properties (at least in homology), because of the singularities caused by reducible vortices. However,

Theorem 8.2.2. *Suppose that the symplectic quotients $X /\!\!/ G$ and $\overline{M}_{n,1}(\Sigma, X)_\infty$ are free. Then for any $d \in H_2(X, \mathbb{Z})$, the set of vortex parameters ρ such that $\overline{M}_{n,1}(\Sigma, X, d)_\rho$ contains reducible vortices is finite.*

The limit of the symplectic vortex equations in the large area limit is studied in Gaio-Salamon [10]. Incorporating the various kinds of bubbbles described in their work yields the following:

Conjecture 8.2.3 (Large area conjecture). *Suppose that $Q\kappa^\bullet$ is a morphism of CohFT's. There exists an $\rho_0 > 0$ such that for $\rho < \rho_0$ and any $\alpha \in H_G(X)^n$ and $d \in H^2_G(X, \mathbb{Z})$,* *the ρ-vortex invariants are related to the Gromov-Witten invariants of the quotient by*

$$\langle \alpha; \iota_\rho^* \beta \rangle_\rho = \sum_{I_1 \cup \ldots \cup I_r = \{1,\ldots n\}} \langle Q\kappa^{|I_1|}(\alpha_{I_1}; \cdot), \ldots, Q\kappa^{|I_r|}(\alpha_{I_r}; \cdot); \cdot \rangle_{X/\!\!/G}(\iota^*_{I_1,\ldots,I_r}\beta)$$

where α_{I_j} denote the wedge product of the classes $\alpha_i, i \in I_j$, and the $\cdot$ denotes insertion of the Kunneth components of $\iota^*_{I_1,\ldots,I_r}\beta$.

If $Q\kappa^\bullet$ is only a weak morphism of CohFT then the formula in the conjecture is the similar, but with additional insertions of $Q\kappa^0$:

$$\langle \alpha; \iota_\rho^* \beta \rangle_\rho = \sum_{I_1 \cup \ldots \cup I_r = \{1,\ldots n\}, s} \langle Q\kappa^{|I_1|}(\alpha_{I_1}; \cdot), \ldots, Q\kappa^{|I_r|}(\alpha_{I_r}; \cdot), Q\kappa^0, \ldots, Q\kappa^0; \cdot \rangle_{X/\!\!/G}(\iota^*_{I_1,\ldots,I_r}\beta)/s!$$

where s is the number of appearances of $Q\kappa^0$.

In particular, if β is a point class then

$$\langle \alpha; pt \rangle_\rho = \langle Q\kappa^1(\alpha); pt \rangle_{X/\!\!/G}.$$

The right hand side involves a mixing of degrees, coming from the map $Q\kappa$. In other words, the triangle

$$\begin{array}{ccc} GW_G(X) & \longrightarrow & GW(X/\!\!/G) \\ & \searrow \qquad \swarrow & \\ & \Lambda_X^G & \end{array}$$

where the left arrow is the trace defined by the vortex invariants for large area, commutes. We remark that this conjecture seems consistent with the "Birkhoff factorization" philosophy explained in Coates-Givental [6]: in the case of a torus action on a vector space, the left-hand side gives the correlators of the gauged linear sigma model while the right-hand side gives the correlators for the sigma model for the Grassmannian.

8.3. **Quantum non-abelian localization.** Putting together the small area Conjecture 4.5.6, the large area Conjecture 8.2.3, and the wall-crossing formula 5.2.2 gives the following in the case that $Q\kappa$ is a morphism of CohFT's; we leave the case of weak morphism to the reader.

Conjecture 8.3.1. *The Gromov-Witten invariants of a symplectic quotient $X/\!\!/G$ are related to the invariant part of the equivariant Gromov-Witten invariants of X by the formula*

$$\sum_{I_1 \cup \ldots \cup I_r = \{1,\ldots n\}} \langle Q\kappa^{|I_1|}(\alpha_{I_1}; \cdot), \ldots, Q\kappa^{|I_r|}(\alpha_{I_r}; \cdot); \cdot \rangle_{d, X/\!\!/G}(\iota^*_{I_1 \cup \ldots \cup I_r}\beta) = \langle \alpha; \iota_\infty^* \beta \rangle_{X,d}^G + \sum_{\zeta \in \mathfrak{g}} \sum_{d_\zeta \mapsto d} \sum_{\rho \in (0,\infty)} \mathrm{Res}_\zeta \frac{\# W_\zeta}{\# W} \langle \varphi^G_{G_\zeta} \alpha; \iota_\rho^* \beta \rangle^{twist}_{X^\zeta, G_\zeta, d_\zeta, \rho} \tag{35}$$

where $\alpha \in H_G(X)^n, \beta \in H(\overline{M}_{n,1}(\mathbb{P}^1))$, the first term is defined to be zero unless d lies in the image of $H_2(X) \to H_2^G(X)$, and the invariants of the quotient are defined to be *zero unless d is in the image of $H_2(X/\!\!/G) \to H_2^G(X)$. In particular, from the class β corresponding to fixing all markings we obtain*

$$\langle Q\kappa_G^1(\alpha); \mathrm{pt}\rangle_{X/\!\!/G,d} = \langle \alpha; \mathrm{pt}\rangle_{X,d}^G + \sum_{\zeta\in\mathfrak{g}} \sum_{d_\zeta \mapsto d} \sum_{\rho\in(0,\infty)} \mathrm{Res}_\zeta \frac{\# W_\zeta}{\# W} \langle \varphi_{G_\zeta}^G \alpha; pt\rangle_{X^\zeta, G_\zeta, d_\zeta, \rho}^{twist}. \tag{36}$$

In other words, the triangle

$$\begin{array}{ccc} GW_G(X) & \longrightarrow & GW(X/\!\!/G) \\ & \searrow \quad \swarrow & \\ & \Lambda_X^G & \end{array}$$

fails to commute by an explicit sum of wall-crossing terms involving extended vortices of groups with smaller dimension.

9. Quantum abelianization

9.1. **The quantum Martin conjecture.** The *abelianization* or *quantum Martin conjecture* of Bertram, Ciocan-Fontanine and Kim [2] relate the Gromov-Witten invariants of a symplectic or equivalently geometric invariant theory quotient with the twisted Gromov-Witten invariants of the quotient by the maximal torus T. Consider the canonical map $H_2^T(X,\mathbb{Z}) \to H_2^G(X,\mathbb{Z})$. Any element $d_T \in H_2^G(X,\mathbb{Z})$ in the pre-image of $d_G \in H_2^G(X,\mathbb{Z})$ is called a *lift* of d_G. In particular, given a class $d_G \in H_2(X/\!\!/G)$, we may embed $H_2(X/\!\!/G) \to H_2^G(X)$ and say that $d_T \in H_2(X/\!\!/T)$ is a lift of d_G if its image is a lift of the image of d_G. If so, we write $d_T \mapsto d_G$. We denote by φ_T^G the restriction map $H_G(X) \to H_T(X)$.

The quantum Martin conjecture of Bertram et al involves *twisted Gromov-Witten invariants*, given by integrating pull-back classes together with Euler classes of index bundles defined as follows. The moduli spaces $\overline{M}_{0,n}(X,d)$ (with $n \geq 4$ if $d =$) are equipped with *forgetful morphisms*

$$f_j : \overline{M}_{0,n}(X,d) \to \overline{M}_{0,n-1}(X,d)$$

which forget the j-th marked point and collapses the unstable components. For any vector bundle $E \to X$, let

$$\mathrm{Ind}(E) = Rf_{n+1,*}(\mathrm{ev}_{n+1}^* E)$$

denote the derived push-forward of E, considered as the index class of E in the K-theory of orbifold vector bundles on $\overline{M}_{0,n}(X,d)$. Let $\mathrm{Eul}(\mathrm{Ind}(E))$ denote its Euler class, defined formally using the $\mathbb{C}^*$-action by scalar multiplication on the fibers of $\mathrm{Ind}(E)$. Define the *E-twisted Gromov-Witten invariant*

$$\langle \alpha\rangle_{X,E,d} = \int_{[\overline{M}_{0,n}(X,d)]} \mathrm{ev}^* \alpha \wedge \mathrm{Eul}(\mathrm{Ind}(E)).$$

For X a $G_\mathbb{C}$-variety, and V a T-representation let $V/\!\!/T := \Phi_T^{-1}(0) \times_T V$ denote the associated vector bundle over $X/\!\!/T$. In particular, $(\mathfrak{g}_\mathbb{C}/\mathfrak{t}_\mathbb{C})/\!\!/T$ is the sum of the line bundles associated to the roots. We denote the $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}})/\!\!/T$-twisted invariants

$$\langle \alpha, \beta \rangle^{twist}_{X/\!\!/T} = \langle \alpha, \beta \rangle_{X/\!\!/T, (\mathfrak{g}/\mathfrak{t})^{\mathbb{C}}/\!\!/T}.$$

Motivated by a conjecture of Hori and Vafa [20, Appendix] relating Gromov-Witten invariants of the Grassmannian with those of products of projective spaces, Bertram-Ciocan-Fontanine-Kim [2] conjectured the following

Conjecture 9.1.1 (Quantum Martin formula)**.** *For any classes $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_G(X)^n$, the Gromov-Witten invariants for $X/\!\!/T$ and $X/\!\!/G$ are related by*

$$\langle \kappa_G(\alpha); \beta \rangle_{X/\!\!/G, d_G} = (\#W)^{-1} \sum_{d_T \to d_G} \langle \kappa_T(\varphi^G_T \alpha); \beta \rangle^{twist}_{X/\!\!/T, d_T}.$$

We propose that a version of this formula holds for symplectic vortex invariants for any positive value $\rho \in (0, \infty)$ of the vortex parameter:

Conjecture 9.1.2. *For any vortex parameter $\rho \in (0, \infty)$ such that the moduli spaces $\overline{M}(\Sigma, X, G)_\rho$ and $\overline{M}(\Sigma, X, T)_\rho$ consist entirely of stable vortices, and $\alpha \in H_G(X)^n$, and for any degree $d_G \in H^G_2(X)$,*

$$\langle \alpha; \beta \rangle_{G, d_G, \rho} = (\#W)^{-1} \sum_{d_T \mapsto d_G} \langle \varphi^G_T \alpha; \beta \rangle^{twist}_{T, d_T, \rho}.$$

A possible proof starts with the fact that abelianization holds at zero-area, because Martin's original argument holds, and then compares the wall-crossing terms. In the case $\rho = 0$, we conjecture the original abelianization conjecture needs to be modified, by inserting the higher corrections from the quantum Kirwan morphism, as follows. (We continue with the assumption that $Q\kappa$ is a morphism, not a weak morphism of CohFT's, to simplify the formulas.)

Conjecture 9.1.3 (Modified quantum Martin conjecture)**.** *Let X be a compact Hamiltonian G-manifold, such that the quotients $X/\!\!/G$ and $X/\!\!/T$ are smooth in the sense that G resp. T acts freely on the zero level set. For any classes $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_G(X)^n$ and $\beta \in \overline{M}_{n,1}(\mathbb{P}^1)$ the Gromov-Witten invariants for $X/\!\!/T$ and $X/\!\!/G$ are related by*

$$\begin{aligned} (37) \quad & \sum_{I_1 \cup \ldots \cup I_r} \langle Q\kappa^{|I_1|}_G(\alpha_{I_1}; \cdot), \ldots, Q\kappa^{|I_r|}_G(\alpha_{I_r}; \cdot); \cdot \rangle_{X/\!\!/G, d_G} (\iota^*_{I_1 \cup \ldots \cup I_r} \beta) \\ & = \frac{1}{|W|} \sum_{I_1 \cup \ldots \cup I_s} \sum_{d_T \to d_G} \langle Q\kappa^{|I_1|}_T(\alpha_{I_1}; \cdot), \ldots, Q\kappa^{|I_r|}_T(\alpha_{I_r}; \cdot); \cdot \rangle^{twist}_{X/\!\!/T, d_T} (\iota^*_{I_1 \cup \ldots \cup I_r} \beta) \end{aligned}$$

This can be viewed as a commutative diagram of CohFT's/traces as follows. First, the restriction map

$$\varphi^G_T : H_G(X) \to H_T(X)$$

defines a morphism of CohFT's from $GW_G(X) \to GW_T(X)$ with all higher maps zero. Composing the restriction map with the quantum Kirwan morphism for T gives a morphism of CohFT's $GW_G(X) \to GW(X/\!\!/T)$. We then have a diagram of CohFT's/traces

$$\begin{array}{ccccc} & & GW_G(X) & & \\ & \swarrow & & \searrow & \\ GW(X/\!\!/G) & & & & GW(X/\!\!/T) \\ & \searrow & & \swarrow & \\ & & \Lambda_X^G & & \end{array}$$

where the map $GW(X/\!\!/T)$ is obtained by composing the twisted correlator map $GW(X/\!\!/T) \to \Lambda_X^T$ with the push-forward (sum over lifts) $\Lambda_X^T \to \Lambda_X^G$. Each triangle in the diagram (formed by adding an arrow from the first to last node) is not expected to commute; the claim is that the wall-crossing terms for the triangles cancel.

Conjecture 9.1.4. *If the minimal Chern number of $X/\!\!/G$ and $X/\!\!/T$ is sufficiently large as in* [10] *(such as for Grassmannians, as studied by Bertram et al.) then the quantum Kirwan corrections do not appear, and that the original quantum Martin conjecture holds.*

10. The singular limit

We describe a conjectural Lagrangian limit of the Kirwan morphism, similar to the way that Lagrangian Floer limit of instanton homology conjecturally appears as the metric on a three-manifold is degenerated in the Atiyah-Floer conjecture. The resulting picture is closely related to one introduced by M. Schwarz, and K. Wehrheim and the first author in [37]. Consider the moduli space $\overline{M}_{1,1}(\mathbb{C}, X)$ defining the first Kirwan map $Q\kappa^1$. Identifying

$$S^1 \times \mathbb{R} \to \mathbb{C} - \{0\}, \quad (\theta, r) \mapsto e^{i\theta + r}$$

the vortex equation becomes

$$F_A + e^{2r} \mathrm{d}r \mathrm{d}\theta u^* \Phi = 0.$$

One can imagine here using an arbitrary function

$$f \in \mathbb{C}^\infty(\mathbb{R}), \quad \lim_{r \to \pm\infty} = e^{\pm\infty}$$

instead of e^{2r}. In particular, consider a family $f_s \in C^\infty(\mathbb{R})$ with

$$\lim_{s \to \infty} f_s(r) = \infty, \ s > 0, \lim_{s \to -\infty} f_s(r) = 0, \ s > 0.$$

Let (A_s, u_s) be a one-parameter sequence of vortices with area form $f_s \mathrm{d}r \wedge \mathrm{d}\theta$. One might conjecture that in good cases, (A_s, u_s) converges to a pair (A_∞, u_∞), with $F_{A_\infty} = 0$ on the left side of the cylinder, and $u_\infty^* \Phi = 0$ on the right side. Since the limiting holonomy of the connection A_∞ is trivial, the two parts (u_-, u_+) of u_∞ consist of maps to X and $X/\!\!/G$ respectively, with the condition at $r = 0$ given by

$$p(u_+(0, \theta)) = \iota(u_-(0, \theta)), \quad p : \Phi^{-1}(0) \to X/\!\!/G, \ \ \iota : \Phi^{-1}(0) \to X.$$

This fits into the framework of Lagrangian correspondences as in [37].

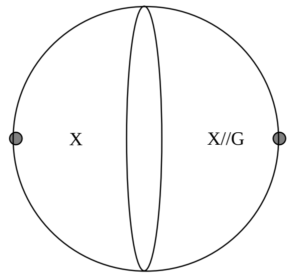


FIGURE 3. Quilted sphere with Lagrangian seam condition

Let $\overline{M}(\Phi^{-1}(0))$ denote the moduli space of such cylinders. If one prefers, using removal of singularities one can pass to spheres divided into two halves. That is, $\overline{M}_{2,1}(\Phi^{-1}(0))$ is the moduli space of pairs $u_- : D \to X, u_+ : D \to X/\!\!/G$, with

$$u_-(z) \in \Phi^{-1}(0), \quad \mathfrak{u}_+(z) = p(u_-(1/z)), \quad \forall z \in \partial D.$$

It admits evaluation maps

$$\overline{M}_{2,1}(\Phi^{-1}(0)) \to X \times X/\!\!/G$$

by evaluation at the midpoints of the disks. Suppose that $\overline{M}_{2,1}(\Phi^{-1}(0))$ admits a virtual fundamental class with the usual properties. Using Poincaré duality we define

$$Q\kappa_G^{1,\mathrm{Lag}} : H_G(X) \to H(X/\!\!/G)$$

by pull-push over the moduli space $\overline{M}_{2,1}(\Phi^{-1}(0))$.

One can ask, independent of the conjectural relation to vortices, whether counting such cylinders should give a ring homomorphism. The relevant pictures are shown below: the homomorphism property would follow from a cobordism between the moduli spaces of surfaces pictured on the left and right; interpolating between the two pictures gives a moduli space of surfaces with a "figure eight" mapping to the zero level set of the moment map. Understanding the analysis relevant for this situation is a program of M. Schwarz via "pseudoholomorphic curves with Lagrangian corners".

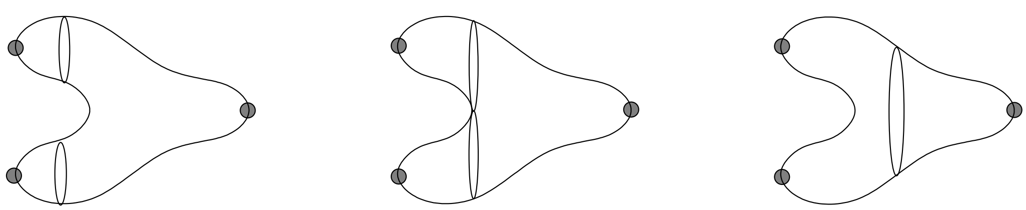


FIGURE 4. Strategy for proving $Q\kappa_G^{1,\mathrm{Lag}}$ is a homomorphism

One might more boldly conjecture that any Lagrangian correspondence which is *breakable* in the sense shown in the diagram (that is the counts of holomorphic quilts on both sides of the picture are equal) leads to a morphism of CohFT's. However, we know of no examples. One might also hope to prove the reduction in stages theorem in the Lagrangian framework. Namely, suppose that $H \subset G$ is a normal subgroup. One can now count half-cylinders with meridian on $\Phi_G^{-1}(0), \Phi_H^{-1}(0)$ and $\Phi_{G/H}^{-1}(0)$; reduction in stages would follow from a cobordism of the moduli spaces shown below: The first steps

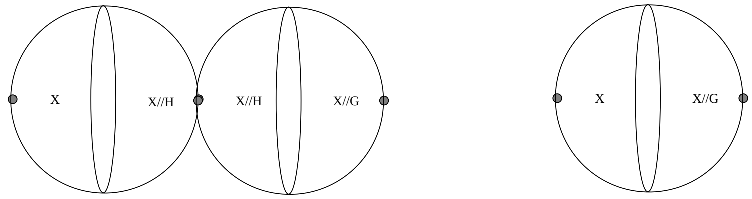


FIGURE 5. Reduction in stages, Lagrangian version

towards understanding the analysis in this case in [37]. The results there imply that if $X, X/\!\!/H, X/\!\!/G$ are all monotone, then there is a cobordism between the zero dimensional components of the relevant moduli space. (The zero level set $\Phi_G^{-1}(0)$, considered as a Lagrangian correspondence, is the composition of $\Phi_H^{-1}(0)$ and $\Phi_{G/H}^{-1}(0)$, and this composition is transversal and embedded.)

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